Liouville's Theorem on Integration in Terms of Elementary Functions

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The notes should be regarded as an elementary introduction to differential algebra.

Contents

In these notes we present a purely algebraic proof, due to M. Rosenlicht, of an 1835 theorem of Liouville on the existence of "elementary" integrals of "elementary" functions (the precise meaning of elementary will be specified). As an application we prove that the indefinite integral $\int e^{x^2}\,dx$ cannot be expressed in terms of elementary functions.

Unless specifically stated to the contrary "ring" means "commutative ring with (multiplicative) *identity". That identity is denoted by* 1 *when the ring should be clear from context; by* 1_R *when this may not be the case and the ring is denoted* R*.*

1. Basic (Ordinary) Differential Algebra

Throughout the section R *is a ring. The product* rs *of elements* $r, s \in R$ *is occasionally written* $r \cdot s$*.*

An additive group homomorphism $\delta : r \in R \mapsto r' \in R$ is a *derivation* if the *Leibniz rule*

$$(rs)' = r \cdot s' + r' \cdot s \tag{1.1}$$

holds for all $r, s \in R$; when δ is understood one simply refers to "the derivation $r \mapsto r'$ (on R)". Using (1.1) and induction one sees that

$$(r^n)' = nr^{n-1} \cdot r', \qquad 1 \leq n \in \mathbb{Z}. \tag{1.2}$$

A *differential ring* consists of a ring R and a derivation[1] $\delta_R : R \to R$. When (R, δ_R) and (S, δ_S) are such a ring homomorphism $\varphi : R \to S$ is a *morphism of differential rings* when φ commutes with the derivations, i.e., when

$$\delta_S \circ \varphi = \varphi \circ \delta_R\,. \tag{1.3}$$

In particular, the collection of differential rings constitutes the objects of a category; morphisms of differential rings form the morphisms thereof. An ideal $\mathfrak{i}$ of a differential ring R is a *differential ideal* if $\mathfrak{i}$ is "closed under the derivation", i.e., if $r \in \mathfrak{i}$ implies $r' \in \mathfrak{i}$.

Examples 1.4 :

(a) The usual derivative d/dx gives the polynomial ring $\mathbb{R}[x]$ the structure of a differential ring. More generally, when $R[x]$ is the polynomial algebra over R in a single variable x the mapping $d/dx : \sum_j r_j x^j \mapsto \sum_j j r_j x^{j-1}$ is a derivation on $R[x]$, and thereby endows $R[x]$ with the structure of a differential ring.

[1] More generally one can consider rings with many derivations, in which case what we have called a differential ring would be called an "ordinary differential ring". We have no need for the added generality.

(b) The mapping $r \in R \mapsto 0 \in R$ is a derivation on R; this is the *trivial derivation*.

(c) The kernel of any morphism $\varphi : R \to S$ of differential rings, i.e., the kernel of the underlying ring homomorphism, is a differential ideal of R.

(d) When $\mathfrak{i}$ is a differential ideal of a differential ring R the quotient $R/\mathfrak{i}$ becomes a differential ring in the expected way, and when this structure is assumed the quotient mapping $R \to R/\mathfrak{i}$ becomes a morphism of differential rings.

(e) The only derivation on a finite field is the trivial derivation. Indeed, when K is a finite-field of characteristic $p > 0$ the Frobenius mapping $k \in K \mapsto k^p \in K$ is an isomorphism, and as a result any $k \in K$ has the form $k = \ell^p$ for some $\ell \in K$. By (1.2) we then have $k' = (\ell^p)' = p\ell^{p-1}\ell' = 0$, and the assertion follows.

Suppose R is

- a differential ring with derivation δ,
- a subring of a differential ring S with derivation δ_S, and
- $\delta_S | R = \delta_R$;

then R is a *differential subring* of S, $S \supset R$ is a *differential ring extension*, the derivation δ_R on R is said to *extend* to (the derivation δ_S on) S, and δ_S is said to be an *extension* of δ_R. Of course in the first two definitions "ring" is replaced by "integral domain" or "field" when R and S have that structure.

Examples 1.5 :

(a) The kernel R_C of a derivation on R is a differential subring of R, and is a differential subfield when R is a field. Moreover, R_C contains the prime ring of R, i.e., the image of $\mathbb{Z}$ under the mapping $n \in \mathbb{Z} \mapsto n \cdot 1_R \in R$. The verifications of these assertions are elementary. R_C is the *ring* (resp. *field*) *of constants* of R.

(b) Suppose R is an integral domain and Q_R is the associated quotient field. Then any derivation $\delta : r \to r'$ on R extends to Q_R via the quotient rule $(r/s)' := (sr' - rs')/s^2$, and this is the unique extension of δ to Q_R. The verifications (that this extension is well-defined and unique) are again elementary.

Proposition 1.6 : *Suppose $K \subset L$ is a differential extension of fields of characteristic zero and $\ell \in L \backslash K$ satisfies $0 \neq \ell' \in K$. Then ℓ is transcendental over K.*

Proof : Otherwise ℓ is algebraic over K. Assume this is the case and write the corresponding irreducible polynomial $p(t) \in K[t]$ as $t^n + \sum_{j=0}^m c_j t^j \in K[t]$, where $n > 1$, $0 \leq m < n$, and $c_m \neq 0$. ($c_m = 0$ would imply $\ell = 0$, resulting in the contradiction $\ell \in K$.) Now set $b := \ell' \in K$, recall by hypothesis that $b \neq 0$, and define $q(t) \in K[t]$ by

$$\begin{aligned} q(t) &= nbt^{n-1} + \textstyle\sum_{j=0}^m c_j' t^j + \sum_{j=1}^m jc_j bt^{j-1} \\ &= nbt^{n-1} + c_m' t^m + mbc_m t^{m-1} + \cdots \ , \end{aligned}$$

where the omitted terms represent a polynomial in $K[t]$ of degree at most $m-2$ (which we understand to mean the zero polynomial if $m < 2$).

We claim that $q(t) \neq 0$. Indeed, if $n - 1 > m$ the polynomial $q(t)$ has leading coefficient nb, which is non-zero by the characteristic zero hypothesis. Alternatively, if $m = n - 1$ the leading coefficient is either $nb + c_m'$, assuming this entity is not zero, or else is mbc_m, which is again non-zero by the characteristic assumption. The claim follows.

Now check that differentiating

$$p(\ell) = \ell^n + c_m \ell^m + \cdots + c_0 = 0$$

gives

$$\begin{aligned} 0 &= n\ell^{n-1}\ell' + \textstyle\sum_{j=0}^m c'\ell^j + \sum_{j=1}^m jc_j \ell^{j-1}\ell' \\ &= nb\ell^{n-1} + \textstyle\sum_{j=0}^m c'\ell^j + \sum_{j=1}^m jc_j b\ell^{j-1} \\ &= nb\ell^{n-1} + c_m'\ell^m + mbc_m\ell^{m-1} + \cdots \\ &= q(\ell). \end{aligned}$$

Since $0 \neq q[t] \in K[t]$ we can divide by the leading coefficient to obtain a monic polynomial in $K[t]$ of degree less than n satisfied by ℓ, and this contradicts the minimality of n. **q.e.d.**

The following immediate consequence of this last result is well-known from elementary calculus, but one seldom sees a proof.

Corollary 1.7 : *The real and complex natural logarithm functions are transcendental over the rational function fields $\mathbb{R}(x)$ and $\mathbb{C}(x)$ respectively, and the real arctangent function is transcendental over $\mathbb{R}(x)$.*

In the subject of differential algebra the characteristic of field under discussion plays a very important role. The following result suggests why this is the case.

Proposition 1.8 : *Suppose K is a differential field with a non-trivial derivation.*

(a) *When K has characteristic 0 every element of $K \backslash K_C$ is transcendental over K_C. In particular, the differential field extension $K_C \subset K$ is not algebraic.*

(b) *When K has characteristic $p > 0$ every element of $K \backslash K_C$ is algebraic over K_C. In particular, the field extension $K_C \subset K$ is algebraic.*

Proof :

(a) Apply Proposition 1.6 to the differential field extension $K_C \subset K$.

(b) From $(k^p)' = pk^{p-1}k' = 0$ one sees that $k^p \in K_C$ for any $k \in K \backslash K_C$.

q.e.d.

2. Differential Ring Extensions with No New Constants

In this section $S \subset R$ is an extension of differential rings; the derivations on both rings are written $t \mapsto t'$.

Note that $R_C \subset S_C$.

Proposition 2.1 : *When $R \subset S$ is an extension of differential rings the following statements are equivalent:*

(a) ("no new constants") $R_C = S_C$*;*

(b) *if* $r \in R$ (already) *admits a primitive in* R *then* r *does not admit a primitive in* $S \backslash R$*; and*

(c) *if* $s \in S \backslash R$ *satisfies* $s' \in R$ *then* s' *has no primitive in* R*.*

Ring (or field) extensions satisfying any (and therefore all) of these conditions are called *no new constant* extensions. They should be regarded as "economical": they do not introduce antiderivatives for elements of R which can already be integrated in R.

Proof :
(a) $\Rightarrow$ (b): When $r \in R$ admits a primitive $t \in R$ as well as a primitive $s \in S \backslash R$ the element $s - t \in S \backslash R$ is a constant, thereby contradicting (a).

(b) $\Rightarrow$ (a): When (a) fails there is a constant $s \in S \backslash R$, i.e., a primitive for $0 \in R$. Since $0 \in R$ is also a primitive for 0 this contradicts (b).

The equivalence of (b) and (c) is clear.

q.e.d.

Example 2.2 : For an example of a differential field extension in which the no new constant hypothesis fails consider $\mathbb{R}[x] \subset L$, where $\mathbb{R}[x]$ is the ring of (real-valued) polynomial functions on $\mathbb{R}$ and L is any field of complex-valued differentiable functions (in the standard sense) of the real variable x containing $\exp ix$. Here $\mathbb{R}[x]_C = \mathbb{R}$, and from $i = (\exp ix)'/\exp ix \in L \backslash \mathbb{R}[x]$ we conclude that $L_C = \mathbb{C}$ is a proper extension of $\mathbb{R}[x]_C$.

Proposition 2.3 : *Suppose $K \subset L$ is a no new constant differential extension of fields of characteristic* 0 *and $\ell \in L \backslash K$ satisfies $\ell' \in K$. Then:*

(a) *ℓ is transcendental over K; and*

(b) *the derivative $(p(\ell))'$ of any polynomial $p(\ell) = \sum_{j=0}^{n} k_j \ell^j \in K[\ell]$ of degree $n > 0$ is a polynomial of degree n if and only if $k_n \notin K_C$, and is otherwise of degree $n-1$.*

Proof :

(a) By Proposition 1.6.

(b) The initial assertion regarding k_n is seen immediately by writing

$$(p(\ell))' = k_n' \ell^n + n k_n \ell^{n-1} \ell' + k_{n-1}' \ell^{n-1} + \cdots + k_0'$$

in the form

$$k_n' \ell^n + (k_{n-1}' + n b k_n) \ell^{n-1} + \cdots + k_0' = 0 \,, \qquad b := \ell' .$$

If $k_n \in K_C$ and the final assertion fails then $0 = k_{n-1}' + n b k_n = (k_{n-1} + n k_n \ell)'$, forcing $k_{n-1} + n k_n \ell \in K_C \subset K$. But from the characteristic 0 assumption this implies $\ell \in K$, contradicting transcendency.

q.e.d.

Proposition 2.4 : *Suppose $K \subset L$ is a no new constant differential extension of fields of characteristic* 0 *and $\ell \in L \backslash K$ satisfies $\ell'/\ell \in K$. Then:*

(a) *ℓ is algebraic over K if and only if $\ell^n \in K$ for some integer $n > 1$; and*

(b) *when* (a) *is not the case the derivative $(p(\ell))'$ of any polynomial $p(\ell) = \sum_{j=0}^{n} k_j \ell^j \in K[\ell]$ of degree $n > 0$ is again a polynomial of degree n, and is a multiple of $p(\ell)$ if and only if $p(\ell)$ is a monomial.*

Proof : Let $b := \ell'/\ell \in K$ and note from the no new constant hypothesis that $b \neq 0$.

(a) Assuming ℓ is algebraic over K let $t^n + c_m t^m + \cdots + c_0 \in K[t]$ be the corresponding irreducible polynomial, where $n \geq 1$ and $0 \leq m < n$. If all c_j vanish then $\ell = 0$, resulting in the contraction $\ell \in K$, and we may therefore assume $c_m \neq 0$. Differentiating

(i) $$\ell^n + c_m \ell^m + \cdots + c_0 = 0$$

now gives

(ii) $$bn\ell^n + (c_m' + bmc_m)\ell^m + \cdots + c_0' = 0\,.$$

Multiplying (i) by bn and subtracting from (ii) results in a lower degree polynomial relation for ℓ unless $c_m' + mbc_m = bnc_m$, which in turn implies $c_m'/c_m = (n-m)b$. It follows that

$$\begin{aligned}\frac{(c_m\ell^{m-n})'}{c_m\ell^{m-n}} &= \frac{(m-n)c_m\ell^{n-m-1}b\ell + c_m'\ell^{n-m}}{c_m\ell^{m-n}}\\ &= \frac{(m-n)bc_m\ell^{m-n} + c_m'\ell^{m-n}}{c_m\ell^{m-n}}\\ &= (m-n)b + c_m'/c_m\\ &= 0\,.\end{aligned}$$

This gives $c_m\ell^{m-n} \in L_C = K_C \subset K$, hence $\ell^{n-m} \in K$, and from the minimality of n we conclude that $\ell^n \in K$.

The converse is obvious.

(b) Write $(p(\ell))'$ in the form

$$(k_n' + bnk_n)\ell^n + \cdots + kpr_0 = 0\,.$$

If $0 = k_n' + bnk_n = (k_n\ell^n)'$ then $k_n\ell^n \in K_C \in K$, and here the transcendency contradiction is $\ell^n \in K$. The assertion on the degree of $(p(\ell))'$ is thereby established.

By virtually the same argument we see that $(k\ell^n)' = \hat{k}\ell^n$, where $0 \neq \hat{k} := k' + nbk \in K$, showing that $(p(\ell))'$ is a multiple of $p(\ell)$ when $p(\ell)$ is a monomial.

Conversely, suppose $(p(\ell))' = q(\ell)p(\ell)$. Then then equality of the degrees of $p(\ell)$ and $(p(\ell))'$ implies $c := q(\ell) \in K$. If $p(\ell)$ is not a monomial let $k_n\ell^n$ and $k_m\ell^m$ be two distinct nonzero terms and note from $(p(\ell))' = cp(\ell)$ that

$$k_j' + jk_jb = ck_j \quad \text{for} \quad j = n, m\,.$$

This implies
$$\frac{k_n' + nk_nb}{k_n} = \frac{k_m' + mk_mb}{k_m}\,,$$

which in turn reduces to

$$a := (n-m)k_nk_mb + k_mk_n' - k_nk_m' = 0\,.$$

But direct calculation then gives

$$\left(\frac{k_n\ell^n}{k_m\ell^m}\right)' = \frac{a\,\ell^{n+m}}{(k_m\ell^m)^2} = 0\,,$$

hence $\frac{k_n\ell^n}{k_m\ell^m} \in K_C \subset K$, and once again we have a contradiction to the transcendency assumption on ℓ. We conclude that $p(\ell)$ must be a monomial when $(p(\ell))'$ is a multiple of $p(\ell)$, and the proof is complete.

q.e.d.

Corollary 2.5 : *For any non zero rational function* $g(x) \in \mathbb{R}(x)$ *the composition* $\exp g(x)$ *is transcendental over the rational function field* $\mathbb{R}(x)$.

Proof : This is immediate from Proposition 2.4(a) since no non zero integer power of $\exp g(x)$ is contained in $\mathbb{R}(x)$. **q.e.d.**

3. Extending Derivations

Throughout the section $K \subset L$ *is an extension of fields and* $\delta : k \mapsto k'$ *is a derivation on* K.

We will be concerned with extending δ to a derivation on L and to this end it first proves useful to generalize the definition of a derivation. Specially, let R be a ring, let $\mathcal{A}$ be an R-algebra (by which we mean a left and right R-algebra), and let M be an R-module (by which we always mean a left and right R-module). An R-linear mapping $\delta : \mathcal{A} \to M$ is an *R-derivation* (*of* $\mathcal{A}$ *into* M) if the *Leibniz rule*

$$\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b \tag{3.1}$$

holds for all $a, b \in \mathcal{A}$. For example, any derivation $\delta : R \to R$ can be regarded as an R_C-derivation of R into the R-module R; additional examples can be seen from the discussion of CASE I below. The abbreviations δa and a' are also used in this context, and extensions of such mappings have the obvious meaning.

Now choose any element $\ell \in L\backslash K$. We first extend $\delta : K \to K$ to a K_C-derivation $\delta : K(\ell) \to L$ by examining the transcendental and algebraic cases separately.

CASE I: ℓ *is transcendental over* K.

In this instance the collection $\{\ell^n\}_{n\geq 0}$ is a K-space basis for the K-algebra $K[\ell]$, and as result any $a \in K[\ell]$ has a unique representation as a finite K-linear combination $a = \sum_j a_j\ell^j$. Given a derivation $k \mapsto k'$ from K into L and an arbitrary element $m \in L$ define a K-linear mapping $\delta : K[\ell] \to L$ by

$$\delta : a = \sum_j a_j\ell^j \mapsto \sum_j a_j'\ell^j + \sum_j j a_j \ell^{j-1} m\,.$$

Note, in particular, that

$$\delta(\ell) = m\,.$$

We claim that δ is a $K[\ell]$-derivation of $K[\ell]$ into L extending the original derivation $\delta : K \to L$. Indeed, that δ is a K-linear function extending the given derivation is clear; what requires verification is the Leibnitz rule. To this end write $a = \sum_i a_i\ell^i$ and choose any element $b = \sum_j b_j\ell^j \in L[\ell]$.

As preliminary observations note that

$$\begin{aligned}(\textstyle\sum_i a_i\ell^i)(\sum_j j b_j\ell^{j-1}) &= \textstyle\sum_{ij} j a_i b_j \ell^{i+j-1} \\ &= \textstyle\sum_k(\sum_i (k-i)a_i b_{k-i})\ell^{k-1} \\ &= \textstyle\sum_k k(\sum_i a_i b_{k-i})\ell^{k-1} - \sum_k(\sum_i i a_i b_{k-i})\ell^{k-1}\end{aligned}$$

and that

$$\begin{aligned}(\textstyle\sum_k i a_i\ell^{i-1})(\sum_j b_j\ell^j) &= \textstyle\sum_{ij} i a_i b_j \ell^{i+j-1} \\ &= \textstyle\sum_k(\sum_i i a_i b_{k-i})\ell^{k-1}\,,\end{aligned}$$

and as a consequence we have

$$\text{(i)} \qquad \left\{ \begin{aligned} &\textstyle\sum_k k(\sum_i a_i b_{k-i})\ell^{k-1} \\ &\qquad = (\textstyle\sum_i a_i\ell^i)(\sum_j j b_j \ell^{j-1}) + (\sum_k i a_i \ell^{i-1})(\sum_j b_j\ell^j)\,. \end{aligned}\right.$$

With the aid of (i) we then see that

$$
\begin{aligned}
(ab)' &= \Big(\textstyle\sum_k(\sum_{i=0}^k(a_ib_{k-i}))\ell^k\Big)' \\
&= \textstyle\sum_k(\sum_{i=0}^k(a_ib_{k-i})')\ell^k + \sum_k k(\sum_{i=0}^k(a_ib_{k-i}))\ell^{k-1}m \\
&= \textstyle\sum_k(\sum_{i=0}^k(a_ib'_{k-i} + a'_ib_{k-i}))\ell^k \\
&\qquad + (\textstyle\sum_i a_i\ell^i)(\sum_j jb_j\ell^{j-1}m) + (\sum_k ia_i\ell^{i-1}m)(\sum_j b_j\ell^j) \\
&= (\textstyle\sum_i a_i\ell^i)(\sum_j b'_j\ell^j) + (\sum_i a'_i\ell^i)(\sum_j b_j\ell^j) \\
&\qquad + (\textstyle\sum_i a_i\ell^i)(\sum_j jb_j\ell^{j-1}m) + (\sum_k ia_i\ell^{i-1}m)(\sum_j b_j\ell^j) \\
&= (\textstyle\sum_i a_i\ell_i)(\sum_j b'_j\ell^j + \sum_j jb_j\ell^{j-1}m) \\
&\qquad + (\textstyle\sum_i a'_i\ell^i + \sum_i ia_i\ell^{i-1}m)(\sum_j b_j\ell^j) \\
&= ab' + a'b\,,
\end{aligned}
$$

and the claim is thereby established.

In summary, *when $\ell \in L\backslash K$ is transcendental over K any derivation from K into L extends to a $K[\ell]$-derivation of $K[\ell]$ into L. Moreover, for any $m \in L$ there is an extension satisfying $\ell' = m$.*

CASE II: *ℓ is algebraic over K.*

First recall (from elementary field theory that) this implies

(i) $$K[\ell] = K(\ell).$$

Assume in addition that ℓ is separable over K and let $p(t) = t^n + \sum_{j=0}^{n-1} k_jt^j \in K[t]$ denote the associated monic irreducible polynomial. If a given derivation $k \to k'$ on K can be extended to a K_0-derivation of $K[\ell]$ into L then from $p(\ell) = 0$ we see that

$$
\begin{aligned}
0 &= (p(\ell))' \\
&= n\ell^{n-1}\ell' + \textstyle\sum jk_j\ell^{j-1}\ell' + \sum k'_j\ell^j \\
&= \ell'(n\ell^{n-1} + \textstyle\sum jk_j\ell^{j-1}) + \sum k'_j\ell^j \\
&= \ell'(p'(\ell)) + \textstyle\sum k'_j\ell^j\,.
\end{aligned}
$$

From the separability hypothesis we have $p'(t) \neq 0$, and since $p(t)$ has minimal degree w.r.t. $p(\ell) = 0$ it follows that $p'(\ell) \neq 0$. The calculation thus implies

(ii) $$\ell' = \frac{-\sum_{j=0}^{m} k_j' \ell^j}{p'(\ell)}\,.$$

We conclude that *there is at most one extension of the given derivation on K to a K_C-derivation of $K[\ell]$ into L, and if such an extension exists* (ii) *must hold and* (as a result) *the image must be contained in $K[\ell]$*. This is in stark contrast to the situation studied in CASE I, wherein the extensions were parameterized by the elements $m \in L$.

To verify that an extension does exist for each derivation $k \mapsto k'$ on K it proves convenient to define $\hat{D}q(t) \in K[t]$, for any polynomial $q(t) = \sum_j a_j t^y \in K[t]$, by $\hat{D}q(t) = \sum_j a_j' t^j \in K[t]$. Notice this enables us to write (ii) as

(iii) $$\alpha' = \frac{-\hat{D}p(\ell)}{p'(\ell)}\,.$$

In fact $\hat{D} : q(t) \mapsto \hat{D}q(t)$ is a K_C-derivation of $K[t]$ into $K[t]$: it is the extension of $k \mapsto k'$ obtained from the choice $m = 0$ in CASE I.

Now note from (i) that we can find a polynomial $s(t) \in K[t]$ such that

(iv) $$s(\alpha) = \frac{-\hat{D}p(\alpha)}{p'(\alpha)} \in K[\alpha]\,;$$

we define a mapping $\check{D} : K[t] \to K[t]$ (read $\check{D}$ as "D check") by

$$\check{D} : q(t) \mapsto \hat{D}q(t) + s(t)q'(t)\,.$$

It is clear that $\check{D}$ is K_C-linear, and from the derivation properties of $q(t) \mapsto \hat{D}q(t)$ and $q(t) \mapsto q'(t)$ we see that for any $r(t) \in K[t]$ we have

$$\begin{aligned}
\check{D}(q(t)r(t)) &= \hat{D}(q(t)r(t)) + s(t)(q(t)r(t))' \\
&= q(t)\hat{D}r(t) + \hat{D}q(t)r(t) + s(t)(q(t)r'(t) + q'(t)r(t)) \\
&= q(t)\left(\hat{D}r(t) + s(t)r'(t)\right) + \left(\hat{D}q(t) + s(t)q'(t)\right)r(t) \\
&= q(t)\check{D}r(t) + \check{D}q(t)r(t)\,.
\end{aligned}$$

We conclude that $\check{D}$ is a K_0-derivation of $K[t]$ into $K[t]$.

Now let $\eta : K[t] \to L$ denote the "substitution" homomorphism $q(t) \in K[t] \mapsto q(\ell) \in L$, set $\mathcal{I} := ker(\eta)$, and note that $\mathcal{I} \subset K[t]$ can also be described as the principal ideal generated by $p(t)$. Any $q(t) \in \mathcal{I}$ therefore has the form $q(t) = p(t)r(t)$ for some $r(t) \in K[t]$, and by substituting $p(t)$ for $q(t)$ in the previous calculation we see that

$$\begin{aligned} \check{D}q(t) &= \check{D}(p(t)r(t)) \\ &= p(t)\hat{D}r(t) + \hat{D}p(t)r(t) + s(t)(p(t)r'(t) + p'(t)r(t))\,. \end{aligned}$$

Evaluating t at ℓ and using (iv) then gives

$$\begin{aligned} \check{D}q(\ell) &= p(\ell)\hat{D}r(\ell) + \hat{D}p(\ell)r(\ell) + s(\ell)\,(p(\ell)r'(\ell) + p'(\ell)r(\ell)) \\ &= 0 \cdot \hat{D}r(\ell) + \hat{D}p(\ell)r(\ell) + s(\ell) \cdot 0 \cdot r'(\ell) + s(\ell)p'(\ell)r(\ell) \\ &= \Big(\hat{D}p(\ell) + s(\ell)p'(\ell)\Big)\, r(\ell) \\ &= \Big(\hat{D}p(\ell) + \tfrac{-\hat{D}p(\ell)}{p'(\ell)}p'(\ell)\Big)\, r(\ell) \\ &= 0 \end{aligned}$$

from which we see that $\check{D}(\mathcal{I}) \subset \mathcal{I}$. It follows immediately that $\check{D}$ induces a K_C-linear mapping $D : K[\ell] \to K[\alpha]$, i.e., that a K_C-linear mapping $D : K[\ell] \to K[\ell]$ is well-defined by

$$Dq(\ell) := \eta(\check{D}q(t))\,, \quad q(\ell) \in K[\ell]\,.$$

Using the derivation properties of $\check{D}$ and the ring homomorphism properties of η we observe that for any $q(\ell), r(\ell) \in K[\ell]$ we have

$$\begin{aligned} D(q(\ell)r(\ell)) &= \eta(\check{D}(q(t)r(t))) \\ &= \eta(\, q(t)\check{D}r(t) + \check{D}q(t)r(t)\,) \\ &= \eta(q(t))\eta(\check{D}r(t)) + \eta(\check{D}q(t))\eta(r(t)) \\ &= q(\ell)Dr(\ell) + Dq(\ell)r(\ell)\,. \end{aligned}$$

We conclude that $D : K[\ell] \to K[\ell]$ is a K_0-derivation, obviously extending $k \to k'$.

In summary: *when $\ell \in L \backslash K$ is separable algebraic over K any derivation $k \to k'$ on K has a unique extension to the field $K(\ell) = K[\ell]$.*

Moreover, the derivative of ℓ *within this extension is given by* (ii), *wherein* $p(t) = t^n + \sum_{j=0}^{n-1} k_j t^j \in K[t]$ *denotes the monic irreducible polynomial* of ℓ.

Theorem 3.2 : *When* $K \subset L$ *is an extension of fields of characteristic zero and* $\delta : K \to K$ *is a derivation the following statements hold.*

(a) δ *extends to a derivation* $\delta_L : L \to L$.

(b) *When* $\ell \in L \backslash K$ *is transcendental over* K *and* $m \in L$ *is arbitrary one can choose the extension* δ_L *so as to satisfy* $\delta_L(\ell) = m$.

(c) *When* $K \subset L$ *is algebraic the extension* $\delta_L : L \to L$ *of* (a) *is unique.*

(d) *When* $K \subset L$ *is algebraic the extension* $\delta_L : L \to L$ *of* (a) *commutes with every automorphism of* L *over* K (i.e., with every automorphism of L which fixes K pointwise).

Proof :

(a) When $K \subset L$ is a simple extension this is immediate from the preceding discussion. (The separability condition required in CASE II is immediate from the characteristic zero assumption.) The remainder of the argument is a routine application of Zorn's lemma.

(b) Immediate from the discussion of CASE I.

(c) Immediate from the discussion of CASE II.

(d) When $\sigma : L \to L$ is an automorphism over K one sees easily that $\sigma \circ \delta_L \circ \sigma^{-1}$ is a derivation of L extending δ, and therefore coincides with δ_L by (c).

q.e.d.

4. Integration in Finite Terms

Throughout the section K denotes a differential field of characteristic 0.

In this section we formulate and prove[2] the result of J. Liouville mentioned in the introduction and as an application show that one cannot integrate $\exp(x^2)$ in terms of "elementary functions". A precise definition of such entities is the first order of business.

Let K be a differential field. An element $\ell \in K$ is a *logarithm* of an element $k \in K$, and k an *exponential* of ℓ, if $\ell' = k'/k$. When this is the case it is customary to write ℓ as $ln\,k$ and/or k as e^{ℓ}; one then has the expected formulas

$$(\ln k)' = k'/k \quad \text{and} \quad (e^{\ell})' = e^{\ell}\,\ell'\,. \tag{4.1}$$

For any $k \in K$ one refers to the element $k'/k \in K$ as the *logarithmic derivative* of k. These definitions are obvious generalizations of concepts from elementary calculus, and examples are therefore omitted. Notice by induction and the Leibniz rule that for any nonzero $k_1, \ldots, k_n \in K$ and any (not necessarily positive) integers $m_1, \ldots, m_n$ one has the *logarithmic derivative identity*

$$\frac{(\Pi_{j=1}^{n} k_j^{m_j})'}{\Pi_{j=1}^{n} k_j^{m_j}} = \sum_{j=1}^{n} m_j \frac{k_j'}{k_j}\,. \tag{4.2}$$

Now let $K \subset K(\ell)$ be a non trivial simple differential field extension. One says that $K(\ell)$ is obtained from K by

(a) the *adjunction of an algebraic element over* K when ℓ algebraic over K, by

(b) the *adjunction of a logarithm of an element of* K when $\ell = \ln k$ for some $k \in K$, or by

(c) the *adjunction of an exponential of an element of* K when $\ell = e^k$ for some $k \in K$.

A differential field extension $K \subset L$ is *elementary* if there is a finite sequence of intermediate differential field extensions $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ such that

[2] Our argument is from [Ros_2], which is adapted from [Ros_1]. A generalization of Liouville's Theorem 4.3 can be found in [Ros_3]; for the original formulation see [Liouville]. Our application is also found in [Mead].

each $K_j \subset K_{j+1}$ has one of these three forms, and when this is the case any $\ell \in L$ is said to be *elementary* over K. By an *elementary function* we mean an element of an elementary differential field extension $R(x) \subset L$ wherein $R = \mathbb{R}$ or $\mathbb{C}$, the elements of L are functions (in the sense of elementary calculus), and the standard derivative is assumed.

Theorem 4.3 (Liouville) : *Let K be a differential field of characteristic 0 and suppose $\alpha \in K$. has no primitive in K. Then α has a primitive within an elementary no new constant differential field extension of K if and only if there is an integer $m \geq 1$, a collection of constants $c_1, \ldots, c_m \in K_C$, and elements $\beta_1, \ldots, \beta_m, \gamma \in K$ such that*

$$\text{(i)} \qquad \alpha = \sum\nolimits_{j=1}^{m} c_j \, \frac{\beta_j'}{\beta_j} + \gamma' \, .$$

Proof (M. Rosenlicht) **:**

$\Rightarrow$ By assumption there is a tower $K = K_0 \subset K_1 \subset \cdots \subset K_n$ of differential field extensions such that each K_j with $j \geq 1$ is obtained from K_{j-1} by the adjunction of an algebraic element over K_{j-1}, a logarithm of an element of K, or the exponential of an element in K. Moreover, there is an element $\rho \in K_n$ such that $\rho' = \alpha$.

We argue by induction on the "length" n of an elementary extension, first noting that when $n = 0$ the desired equality holds by taking $m = 1$, $c_1 = 0$, and $\gamma := \rho$. If $n > 0$ and the result holds for $n-1$ we can view α as an element of K_1 and thereby apply the induction hypothesis to the elementary extension $K_1 \subset K_n$, obtaining a non-negative integer m, constants $c_1, \ldots, c_m$, and elements $\beta_1, \ldots, \beta_m, \gamma \in K_1$ such that the displayed equation of the theorem statement holds. We are thereby reduced to proving the following result: *If $\alpha \in K$ can be written as displayed above, with γ and all β_j in $K(\ell) = K_1$, then it can also be expressed in this form, although possibly with a different m, with the corresponding γ and β_j now contained in K.*

Case (a) : *ℓ is algebraic over K.*

First note that in this case we have $K(\ell) = K[\ell]$.

Choose an algebraic closure K^a of K containing ℓ and let $\sigma_i : K[\ell] \to K^a$, $i = 1, \ldots, s$, denote the distinct embeddings over K, where w.l.o.g. σ_1 is inclusion. Then the monic irreducible polynomial $p(x)$ of ℓ must factor in K^a as $p(x) = \Pi_{i=1}^{s}(x - \ell_i)$. Moreover, for any $q(\ell) \in K[\ell]$ we obviously have $\sigma_i(q(\ell)) = q(\ell_i)$, and from Theorem 3.2(d) we see that $\sigma_i((q(\ell))') = (q(\ell_i))'$ holds as well, $i = 1, \ldots, s$.

Now choose polynomials $q_1, \ldots, q_n, r \in K[x]$ such that

$$\beta_j = q_j(\ell), \quad j = 1, \ldots, n, \quad \text{and } \gamma = r(\ell)$$

and write (i) accordingly, i.e., as

$$\alpha = \sum_j \frac{(q_j(\ell))'}{q_j(\ell)} + (r(\ell))'\,.$$

Applying σ_i then gives

$$\begin{aligned}\alpha &= \sum_j c_j \frac{\sigma_i(\beta_j')}{\sigma_i(\beta_j)} + \sigma_i(\gamma') \\ &= \sum_j c_j \frac{(\sigma_i(\beta_j))'}{\sigma_i(\beta_j)} + (\sigma_i(\gamma))' \\ &= \sum_j c_j \frac{(q_j(\ell_i))'}{q_j(\ell_i)} + (r(\ell_i))'\,,\end{aligned}$$

whereupon summing over i, dividing by s (which requires the characteristic 0 hypothesis), and appealing to (4.2) yields

$$\alpha = \sum_{j=1}^{n} \frac{c_j}{s} \frac{(\Pi_{i=1}^{s} q_j(\ell_i))'}{\Pi_{i=1}^{s} q_j(\ell_i)} + \left(\frac{\sum_{i=1}^{s} r(\ell_i)}{s}\right)'\,.$$

By construction the two terms on the right-hand-side of this equality are fixed by all the embeddings $\sigma_j : K(\alpha) \to K^a$, which by the separability of $K(\ell) \supset K$ (guaranteed by the characteristic zero hypothesis) implies they belong to K. In particular, this last expression for α has the required form.

Having established Case (a) we may assume for the remainder of the proof that ℓ is transcendental over K. In this case we can find $q_j(\ell)$, $r(\ell)$ such that $\beta_j = q_j(\ell)$ and $\gamma = r(\ell)$, and thereby write

(ii) $$\alpha = \sum_{j=1}^{n} c_j \frac{(q_j(\ell))'}{q_j(\ell)} + (r(\ell))'\,,$$

although initially it seems we must assume that $r(\ell)$ and the $q_j(\ell)$ belong to $K(\ell)$ rather than to $K[\ell]$. But each $q_j(\ell)$ can be written in the form $k_j \Pi_{i=1}^{n_j} (q_{ji}(\ell))^{n_{ji}}$, where $k_j \in K$, $q_{ji}(\ell) \in K[\ell]$ is monic and irreducible, and the n_j and n_{ji} are integers with $n_j > 0$ (no such restriction occurs for the n_{ij}). The logarithmic identity (4.2) then allows us to assume the $q_j(\ell)$ appearing in (ii) are either non-constant monic irreducible polynomials in $K[\ell]$ or elements of K.

Case (b): ℓ *is a logarithm of an element of* K, i.e., $\ell' = k'/k$ for some $k \in K$.

Let $p(\ell) \in K[\ell]$ be non-constant, monic and irreducible. Then $(p(\ell))' \in K[\ell]$ is easily seen to have degree strictly less than that of $p(\ell)$, guaranteeing that $p(\ell)$ cannot divide $(p(\ell))'$. If in (ii) we have $q_j(\ell) = p(\ell)$ for some j then the quotient $(q_j(\ell))'/q_j(\ell)$ already appears in lowest terms. In particular, if the polynomial $p(\ell)$ appears as a denominator in the sum $\sum_{j=1}^n c_j \frac{(q_j(\ell))'}{q_j(\ell)}$ within (ii) then it will not appear to a power greater than 1.

Now suppose $p(\ell)$ occurs as a denominator in the partial fraction expansion of $r(\ell)$. Each such occurrence has the form $f(\ell)/(p(\ell))^m$, where the degree of $f(\ell)$ is less than that of $p(\ell)$. Let $d \geq 1$ denote the maximal such m. The corresponding terms of the partial fraction expansion of $(r(\ell))'$ then consist of $(f(\ell)(1/p(\ell))^d)' = -d \cdot f(\ell)(p(\ell))'/(p(\ell))^{d+1}$ together with at most d terms having lower powers of $p(\ell)$ as denominators. Moreover, from the preceding paragraph we see that the remaining terms on the right hand side of (ii) cannot contribute a denominator $p(\ell)$ to any power greater than 1, hence cannot cancel $-d \cdot f(\ell)(p(\ell))'/(p(\ell))^{d+1}$.

These considerations lead to the following conclusion. If in (ii) we have $q_j(\ell) = p(\ell)$ for some j, and/or if $p(\ell)$ occurs as a denominator in the partial fraction expansion of $r(\ell)$, then $p(\ell)$ will occur as a denominator in the partial fraction expansion of α. But that partial fraction expansion is unique and is obviously given by $\alpha = \alpha$. The $q_j = q_j(\ell)$ occurring in (ii) are therefore in K (as required), and $r(\ell)$ in that expression is a polynomial.

Now observe from $(r(\ell))' \in K$ and Proposition 2.3(b) (which requires the no new constant hypothesis) that $r(\ell)$ must have the form $r(\ell) = c\ell + \hat{c}$, where $c \in K_C$ and $, \hat{c} \in K$. Equality (ii) is thereby reduced to

$$\alpha = \sum_{j=1}^{n} c_j \frac{q_j'}{q_j} + c\,\frac{k'}{k} + \hat{c}\,',$$

precisely as desired.

Case (c): ℓ *is an exponential of an element of* K, i.e., $\ell'/\ell = k'$ for some $k \in K$.

As in Case (b) let $p(\ell) \in K[\ell]$ be non-constant, monic, and irreducible. From Proposition 2.4 we see that for $p(\ell) \neq \ell$ we have $(p(\ell))' \in K[\ell]$ and that $p(\ell)$ does not divide $(p(\ell))'$; we can then argue as in the previous case to conclude that the $q_j = q_j(\ell)$ in (ii) are in K, with $q_j(\ell) = \ell$ as a possible exception, and that $r(\ell)$ in that expression can be written in the form $r(\ell) = \sum_{-t}^{t} k_j \ell^j$, where $t > 0$ is an integer and the coefficients k_j are in K.

Since each quotient $(q_j(\ell))'/q_j(\ell)$ is in K the same holds for $(r(\ell))'$, whereupon a second appeal to Proposition 2.4 gives $r := r(\ell) \in K$. If $q_j(\ell) \neq \ell$ for all j we are done, so assume w.l.o.g. that $q_1(\ell) = \ell$. We can then write

$$\alpha = c_1 \frac{k'}{k} + \sum_{j=2}^{n} c_j \frac{q_j'}{q_j} + r' = \sum_{j=2}^{n} c_j \frac{q_j'}{q_j} + (c_1 k + r)',$$

which achieves the required form.

$\Leftarrow$ This is clear.

q.e.d.

Corollary 4.4 (Liouville) : *Suppose $E \subset K = E(e^g)$ is a no new constant differential extension of fields of characteristic 0 obtained by adjoining the exponential of an element $g \in E$. Suppose in addition that e^g is transcendental over E and let $f \in E$ be arbitrary. Then $fe^g \in K$ has a primitive within some elementary no new constant differential field extension of K if and only if there is an element $a \in E$ such that*

(i) $$f = a' + ag'$$

or, equivalently, such that

(ii) $$fe^g = (ae^g)'.$$

Proof : To ease notation write e^g as ℓ.

$\Rightarrow$ By Theorem 4.3 the element $f\ell \in K$ has a primitive as stated if and only if there are elements $c_j \in K_C = E_C$ and elements γ and $\beta_j \in K$, $j = 1, \ldots, n$, such that

(iii) $$f\ell = \sum_{j=1}^{n} c_j \frac{\beta_j'}{\beta} + \gamma'.$$

Now write γ as $r(\ell)$ and β_j as $g_j(\ell)$ and assume, as in the discussion surrounding equation (ii) of the proof of Theorem 4.3, that the $q_j(\ell)$ are either non-constant monic irreducible polynomials in $K[\ell]$ or elements of E. Arguing as in Case (c) of that proof (again with K replaced by E) we can then conclude that ℓ is both the only possible monic irreducible factor in a denominator of the partial fraction expansion of $r(\ell)$ as well as the only possibility for a monic irreducible $g_j(\ell) \in K[\ell]\backslash E$. But this gives $(g_j(\ell))'/g_j(\ell) \in E$ for all j as well as $r(\ell) = \sum_{j=-t}^{t} k_j \ell^j$, where $t > 0$ is an integer and the coefficients k_j are in E. In particular, (iii) can now be written

$$f\ell = c + \sum_{j=-t}^{t} k_j' \ell^j + g' \sum_{j=-t}^{t} j k_j \ell^j$$

and upon comparing coefficients of equal powers of ℓ we conclude that $f\ell = k_1' + k_1 g'$. Equation (i) then follows by taking $a = k_1$.

$\Leftarrow$ When (i) holds $ae^g \in K$ is a primitive of fe^g.

q.e.d.

Corollary 4.5 : *For $R = \mathbb{R}$ or $\mathbb{C}$ the function $x \in R \mapsto e^{x^2} \in R$ has no elementary primitive.*

By an *elementary primitive* we mean a primitive within some no new constant differential field extension of $R(x)(e^{x^2})$.

Proof : By Corollaries 1.7 and 4.4 the given function has an elementary primitive if and only if there is a function $a \in R(x)$ such that $1 = a' + 2ax$.

There is no such function. To see this assume $a = p/q \in R(x)$ satisfies this equation, where w.l.o.g. $p, q \in R[x]$ are relatively prime. Then $1 = a' + 2ax \Rightarrow 1 = \frac{qp'-q'p}{q^2} + 2 \cdot \frac{px}{q} \Rightarrow q^2 = qp' - q'p + 2xqp \Rightarrow q(q - 2px - p') = -q'p \Rightarrow q|q'p \Rightarrow q|q' \Rightarrow q' = 0$, and q is therefore a constant polynomial, i.e., w.l.o.g. $a = p$. Comparing the degrees in x on the two sides of $1 = a' + 2ax$ now results in a contradiction. **q.e.d.**

Remarks 4.6 :

(a) Additional examples of meromorphic functions without elementary primitives, including $\sin z/z$, are discussed in [Mead] and [Ros_2, pp. 971-2]. The arguments are easy (but not always obvious) modifications of the proof of Corollary 4.5.

(b) One can (easily) produce an elementary differential field extension of the rational function field $\mathbb{R}(x)$ containing $\arctan x$ (we are assuming the standard derivative), but not one with no new constants. Indeed, it is not hard to show that $1/(x^2 + 1)$ cannot be written in the form (i) of the statement Theorem 4.3 (see p. 598 of [Ros_2]). This indicates the importance of the no new constant hypothesis in the statement of Liouville's Theorem.

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