The classical case is given by taking E to be the Eilenberg-MacLane spectrum $K(Z_p)$. The analogue of A_* in the generalised case is therefore $E_*(E) = \pi_*(E \wedge E)$, the homology of E with coefficients in E. The analogue of Z_p is $E_*(S^0) = \pi_*(E)$. Since E is a ring-spectrum, we have various products. More precisely, suppose given a pairing $\mu: E \wedge F \longrightarrow G$ of spectra. Then we shall have to consider three products, which appear in the following commutative diagram.

$$\begin{array}{ccc}
\pi_*(E \wedge X) \otimes \pi_*(F \wedge Y) & \xrightarrow{\nu} & \pi_*(G \wedge X \wedge Y) \\
\uparrow \tau_* \otimes 1 & & \uparrow (\tau \wedge 1)_* \\
\pi_*(X \wedge E) \otimes \pi_*(F \wedge Y) & \xrightarrow{m} & \pi_*(X \wedge G \wedge Y) \\
\downarrow 1 \otimes \tau_* & & \downarrow (1 \wedge \tau)_* \\
\pi_*(X \wedge E) \otimes \pi_*(Y \wedge F) & \xrightarrow{\nu'} & \pi_*(X \wedge Y \wedge G)
\end{array}$$

Here the product ν is the usual external homology product, as used (for example) in Lecture 1, Note 7. The product ν' is a back-to-front version of ν. The product m is defined as follows. Suppose given maps

$$f: S^p \longrightarrow X \wedge E, \qquad g: S^q \longrightarrow F \wedge Y.$$

Then $m(f \otimes g)$ is the following composite.

$$S^p \wedge S^q \xrightarrow{f \wedge g} X \wedge E \wedge F \wedge Y \xrightarrow{1 \wedge \mu \wedge 1} X \wedge G \wedge Y .$$

Since it is important for us in this lecture to keep factors in their correct order, we will use m as our basic product. By taking $X = S^0$ or $Y = S^0$, we obtain the following special cases.

$$m: \pi_p(E) \otimes \pi_q(F \wedge Y) \longrightarrow \pi_{p+q}(G \wedge Y)$$

$$m: \pi_p(X \wedge E) \otimes \pi_q(F) \longrightarrow \pi_{p+q}(X \wedge G)$$

$$m: \pi_p(E) \otimes \pi_q(F) \longrightarrow \pi_{p+q}(G) .$$

In particular, $\pi_*(E)$ is an anticommutative ring with unit. For any Y, $\pi_*(E \wedge Y)$ is a left module over $\pi_*(E)$; the product map

$$m: \pi_*(E) \otimes \pi_*(E \wedge Y) \longrightarrow \pi_*(E \wedge Y)$$

is the usual one, and coincides with the map μ considered in UCT 2 (see Lecture 1, Note 2). For any X, $\pi_*(X \wedge E)$ is a right module over $\pi_*(E)$. The product

$$m: \pi_*(X \wedge E) \otimes \pi_*(E \wedge Y) \longrightarrow \pi_*(X \wedge E \wedge Y)$$

factors to give a map

$$\pi_*(X \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge Y) \longrightarrow \pi_*(X \wedge E \wedge Y),$$

which we also call m.

We have product maps

$$m: \pi_*(E) \otimes \pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E)$$

$$m: \pi_*(E \wedge E) \otimes \pi_*(E) \longrightarrow \pi_*(E \wedge E),$$

and thus $\pi_*(E \wedge E)$ becomes a bimodule over $\pi_*(E)$. It should be noted that the two actions of $\pi_*(E)$ on $\pi_*(E \wedge E)$

are in general quite distinct; this is the main difference between the generalised case and the classical case, in which we have only one action of Z_p on A_*. The presence of these two actions means that the generalised case demands a little more care than the classical case.

We now assume that $\pi_*(E \wedge E)$ is flat as a right module over $\pi_*(E)$ (using the right action). By using the switch map

$$\tau: E \wedge E \longrightarrow E \wedge E$$

to interchange the two factors, we check that it is equivalent to assume that $\pi_*(E \wedge E)$ is flat as a left module over $\pi_*(E)$ (using the left action). This hypothesis is somewhat restrictive, but it is satisfied in many important cases, notably the cases

$$E = \underline{BO}, \underline{BU}, MO, MU, MSp, S \text{ and } K(Z_p)$$

(see Lecture 1, Lemma 28).

With this hypothesis, we will see that $\pi_*(E \wedge E)$ is a Hopf algebra in a fully satisfactory sense, and that for any spectrum X, $\pi_*(E \wedge X)$ is a comodule over the coalgebra $\pi_*(E \wedge E)$. We will now make this more precise by listing the structure maps we shall introduce, and giving their principal properties.

The structure maps comprise a product map

$$\phi: \pi_*(E \wedge E) \otimes \pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E),$$

two "unit" maps

$$\eta_L: \pi_*(E) \longrightarrow \pi_*(E \wedge E)$$

$$\eta_R: \pi_*(E) \longrightarrow \pi_*(E \wedge E)$$

a counit map

$$\varepsilon: \pi_*(E \wedge E) \longrightarrow \pi_*(E)$$

a canonical anti-automorphism

$$c: \pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E)$$

a diagonal map

$$\psi = \psi_E: \pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E)$$

and for each spectrum X, a coaction map

$$\psi = \psi_X: \pi_*(E \wedge X) \longrightarrow \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X) .$$

(The diagonal map ψ_E is obtained by specialising the coaction map ψ_X to the case $X = E$.)

It is important to note that in the tensor-product $\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X)$, the action of $\pi_*(E)$ on the left-hand factor $\pi_*(E \wedge E)$ is the right action. (The action of $\pi_*(E)$ on the right-hand factor $\pi_*(E \wedge X)$ is the usual left action.) This is exactly what we need to use the tensor-product notation in a systematic way.

The tensor-product $\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X)$ can be considered as a left module over $\pi_*(E)$, by using the left action of $\pi_*(E)$ on $\pi_*(E \wedge E)$; that is,

$$\lambda(e \otimes x) = (\lambda e) \otimes x$$

$(\lambda \in \pi_*(E),\ e \in \pi_*(E \wedge E),\ x \in \pi_*(E \wedge X))$.

The coaction map ψ_X is a map of left modules over $\pi_*(E)$.

In particular, the previous two paragraphs apply to the case $X = E$. Here the tensor-product $\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E)$ can also be considered as a right module over $\pi_*(E)$, by using the right action of $\pi_*(E)$ on the right-hand factor. The diagonal map ψ_E is a map of bimodules over $\pi_*(E)$.

The behaviour of the other structure maps with respect to the actions of $\pi_*(E)$ will emerge from the properties given below. The tensor-product on which the product map ϕ is defined can be taken over the integers.

The principal properties of these structure maps are as follows. The product map ϕ is associative, anticommutative and has a unit element 1. The maps η_L, η_R, ε and c are homomorphisms of graded rings with unit. The left action of $\pi_*(E)$ on $\pi_*(E \wedge E)$ is given by

$$\lambda e = \phi((\eta_L \lambda) \otimes e) \qquad (\lambda \in \pi_*(E),\ e \in \pi_*(E \wedge E)) .$$

Similarly, the right action of $\pi_*(E)$ on $\pi_*(E \wedge E)$ is given by

$$e\lambda = \phi(e \otimes (\eta_R \lambda)) \qquad (e \in \pi_*(E \wedge E),\ \lambda \in \pi_*(E)) .$$

We have

$$\varepsilon\eta_L = 1,\ \varepsilon\eta_R = 1,\ c\eta_L = \eta_R,\ c\eta_R = \eta_L,$$

$$\varepsilon c = \varepsilon,\ c^2 = 1 .$$

These properties determine the behaviour of ϕ, η_L, η_R, ε and c with respect to the actions of $\pi_*(E)$. In particular, ε is a map of bimodules.

The coaction map is natural for maps of X. The coaction map is associative, in the sense that the following diagram is commutative.

$$\begin{array}{ccc}
\pi_*(E\wedge X) & \xrightarrow{\psi_X} & \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X) \\
\downarrow \psi_X & & \downarrow 1\otimes\psi_X \\
\pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X) & \xrightarrow{\psi_E\otimes 1} & \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X)
\end{array}$$

(Note that $1 \otimes \psi_X$ is defined because ψ_X is a map of left modules over $\pi_*(E)$, and $\psi_E \otimes 1$ is defined because ψ_E is a map of right modules over $\pi_*(E)$.) In particular, we can specialise this diagram to the case $X = E$, and we see that the diagonal map is associative.

The behaviour of the diagonal with respect to the product is given by the following commutative diagram.

$$\begin{array}{ccc}
\pi_*(E\wedge E) \otimes \pi_*(E\wedge E) & \xrightarrow{\phi} & \pi_*(E\wedge E) \\
\downarrow \psi_E\otimes\psi_E & & \downarrow \psi_E \\
[\pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge E)] \otimes [\pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge E)] & \xrightarrow{\Phi} & \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge E)
\end{array}$$

Here the map Φ is defined by

$$\Phi(e \otimes f \otimes g \otimes h) = (-1)^{pq}\phi(e \otimes g) \otimes \phi(f \otimes h)$$

where $f \in \pi_p(E \wedge E)$, $g \in \pi_q(E \wedge E)$. It has to be verified that this formula does give a well-defined map of the product of tensor products over $\pi_*(E)$, but this can be done using the facts stated above.

The behaviour of the diagonal map on the unit is given by $\psi_E(1) = 1 \otimes 1$. It follows that we have

$$\psi_E \eta_L \lambda = (\eta_L \lambda) \otimes 1 , \; \psi_E \eta_R \lambda = 1 \otimes (\eta_R \lambda) \quad (\lambda \in \pi_*(E)) .$$

The behaviour of the diagonal map with respect to the counit is given by the following commutative diagram.

$$\begin{array}{ccc}
\pi_*(E \wedge X) & \xrightarrow{\psi_X} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varepsilon \otimes 1} \\
\pi_*(E \wedge X) & \xleftarrow{\cong} & \pi_*(E) \otimes_{\pi_*(E)} \pi_*(E \wedge X)
\end{array}$$

Here the bottom arrow is given by the usual left action of $\pi_*(E)$ on $\pi_*(E \wedge X)$. The map $\varepsilon \otimes 1$ is defined because ε is a map of right modules over $\pi_*(E)$. Similarly, we have the following commutative diagram.

$$\begin{array}{ccc}
\pi_*(E \wedge E) & \xrightarrow{\psi_E} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1 \otimes \varepsilon} \\
\pi_*(E \wedge E) & \xleftarrow{\cong} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E)
\end{array}$$

Here the bottom arrow is given by the right action of $\pi_*(E)$ on $\pi_*(E \wedge E)$. The map $1 \otimes \varepsilon$ is defined because ε is a map of left modules over $\pi_*(E)$.

The behaviour of the diagonal with respect to the canonical anti-automorphism c is given by the following commutative diagram.

$$\begin{array}{ccc}
\pi_*(E \wedge E) & \xrightarrow{\psi_E} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E) \\
\Big\downarrow c & & \Big\downarrow C \\
\pi_*(E \wedge E) & \xrightarrow{\psi_E} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E)
\end{array}$$

Here the map C is defined by

$$C(e \otimes f) = (-1)^{pq} cf \otimes ce$$

$(e \in \pi_p(E \wedge E),\ f \in \pi_q(E \wedge E))$.

It has to be verified that this formula does give a well-defined map of the tensor product over $\pi_*(E)$, but this can be done using the facts stated above.

The following commutative diagrams express that property of the canonical anti-automorphism which in the classical case is taken as its definition.

$$\begin{array}{ccc}
\pi_*(E \wedge E) & \xrightarrow{\qquad\varepsilon\qquad} & \pi_*(E) \\
\Big\downarrow \psi_E & & \Big\downarrow \eta_L \\
\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E) & \xrightarrow{\phi(1 \otimes c)} & \pi_*(E \wedge E)
\end{array}$$

$$\begin{array}{ccc}
\pi_*(E \wedge E) & \xrightarrow{\ \varepsilon\ } & \pi_*(E) \\
\downarrow{\scriptstyle \psi_E} & & \downarrow{\scriptstyle \eta_R} \\
\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E) & \xrightarrow{\ \phi(c\otimes 1)\ } & \pi_*(E \wedge E)
\end{array}$$

It has to be verified that $\phi(1\otimes c)$ and $\phi(c\otimes 1)$ do give well-defined maps of the tensor product over $\pi_*(E)$, but this can be done using the facts stated above.

This completes the list of properties of our structure maps. We also require one further formal property in order to show that certain comodules $E_*(X)$ are extended (see Lectures 1, 2). Let F be a left module-spectrum over the ring-spectrum E; for example, we might have $F = E \wedge Y$. Then the following diagram is commutative.

$$\begin{array}{ccc}
\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(F) & \xrightarrow{\ m\ } & \pi_*(E \wedge F) \\
\downarrow{\scriptstyle \psi_E \otimes 1} & & \downarrow{\scriptstyle \psi_F} \\
\pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(F) & \xrightarrow{\ 1\otimes m\ } & \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge F)
\end{array}$$

The map $1 \otimes m$ is defined because m is a map of left modules over $\pi_*(E)$.

We now give the definition of our structure maps. The product ϕ is given by either way of chasing round the following commutative square.

$$\begin{array}{ccc}
\pi_*(E \wedge E) \otimes \pi_*(E \wedge E) & \xrightarrow{\nu'} & \pi_*(E \wedge E \wedge E) \\
\downarrow \nu & & \downarrow (\mu \wedge 1)_* \\
\pi_*(E \wedge E \wedge E) & \xrightarrow{(1 \wedge \mu)_*} & \pi_*(E \wedge E)
\end{array}$$

(For ν and ν', see the discussion of products at the beginning of this lecture.) In other words, suppose given

$$f: S^p \longrightarrow E \wedge E, \ g: S^q \longrightarrow E \wedge E;$$

then $\phi(f \otimes g)$ is the following composite.

$$S^p \wedge S^q \xrightarrow{f \wedge g} E \wedge E \wedge E \wedge E \xrightarrow{1 \wedge \tau \wedge 1} E \wedge E \wedge E \wedge E \xrightarrow{\mu \wedge \mu} E \wedge E.$$

We have maps

$$E \simeq E \wedge S^0 \xrightarrow{1 \wedge i} E \wedge E$$

$$E \simeq S^0 \wedge E \xrightarrow{i \wedge 1} E \wedge E$$

which map E into $E \wedge E$ as the left and right factors. We define η_L and η_R to be the corresponding induced homomorphisms. We define ε and c to be the homomorphisms induced by

$$\mu: E \wedge E \longrightarrow E$$

and

$$\tau: E \wedge E \longrightarrow E \wedge E .$$

It only remains to define ψ_X.

Lemma 1

If $\pi_*(X \wedge E)$ is flat as a right module over $\pi_*(E)$, then $m: \pi_*(X \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge Y) \longrightarrow \pi_*(X \wedge E \wedge Y)$ is iso.

Proof. This is essentially the trivial case of KT 1 (see Lecture 1, Note 12). The map m is a natural transformation between homology functors of Y which is iso for $Y = S^0$; therefore it is iso for any finite complex Y. Pass to direct limits.

We now define

$$h: \pi_*(X \wedge Y) \longrightarrow \pi_*(X \wedge E \wedge Y)$$

to be the homomorphism induced by

$$X \wedge Y \simeq X \wedge S^0 \wedge Y \xrightarrow{1\wedge i\wedge 1} X \wedge E \wedge Y .$$

The map h is essentially the Hurewicz homomorphism in E-homology.

If $\pi_*(X \wedge E)$ is flat, we can consider the following composite.

$$\pi_*(X \wedge Y) \xrightarrow{h} \pi_*(X \wedge E \wedge Y) \xrightarrow{m^{-1}} \pi_*(X \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge Y) .$$

We define $\psi = m^{-1}h$. In particular, since we are assuming that $\pi_*(E \wedge E)$ is flat, we can specialise to the case $X = E$; we take the resulting map ψ for our coaction map ψ_Y. This completes the definition of the structure maps.

The proofs of all the formal properties are by diagram-chasing. In proving any property of ψ_X, of course we have to make our diagram up out of two subdiagrams, one for h and one for m. For example, in proving that the coaction map is associative, we first prove two more elementary results; ψ_X is natural for maps of X, and $\psi_F m = (1 \otimes m)(\psi_E \otimes 1)$ (which is the diagram required to prove that $E_*(F)$ is an extended comodule). We now set up the following diagram.

$$\begin{array}{ccc}
\pi_*(E \wedge X) & \xrightarrow{\psi_X} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X) \\
\Big\downarrow h & & \Big\downarrow 1 \otimes h \\
\pi_*(E \wedge E \wedge X) & \xrightarrow{\psi_{E \wedge X}} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E \wedge X) \\
\Big\uparrow m & & \Big\uparrow 1 \otimes m \\
\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X) & \xrightarrow{\psi_E \otimes 1} & \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X)
\end{array}$$

Here the top square is commutative because h is induced by a map

$$X \simeq S^0 \wedge X \xrightarrow{i \wedge 1} E \wedge X ,$$

and ψ_X is natural for maps of X. Similarly, the bottom square is commutative by the second result mentioned, taking $F = E \wedge X$. This gives the required result. The two subsidiary results are proved in the same way.

In proving the behaviour of the diagonal with respect to the product, it is convenient to prove a slightly more general result first. Suppose that $\pi_*(A \wedge E)$, $\pi_*(B \wedge E)$ and $\pi_*(A \wedge B \wedge E)$ are all flat; then the following diagram is commutative.

$$\begin{array}{ccc}
\pi_*(A\wedge X) \otimes \pi_*(B\wedge Y) & \longrightarrow & \pi_*(A\wedge B\wedge X\wedge Y) \\
\downarrow {\scriptstyle \psi \otimes \psi} & & \downarrow {\scriptstyle \psi} \\
[\pi_*(A\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X)] \otimes [\pi_*(B\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge Y)] & \longrightarrow & \pi_*(A\wedge B\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X\wedge Y)
\end{array}$$

Here the upper horizontal map is the obvious product, and the lower horizontal map sends $e \otimes f \otimes g \otimes h$ into $(-1)^{pq} \nu'(e \otimes g) \otimes \nu(f \otimes h)$ (see the discussion of products at the beginning of this lecture). This diagram is proved commutative in the same way as before - separate h and m.

Next observe that since the functor $\pi_*(E \wedge E) \otimes_{\pi_*(E)}$ preserves exactness, applying it twice preserves exactness; that is, the right module

$$\pi_*(E \wedge E \wedge E) \cong \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E)$$

is flat. So we may specialise to the case $A = B = E$. Now apply naturality to the map

$$A \wedge B = E \wedge E \xrightarrow{\mu} E ;$$

we see that the following diagram is commutative.

$$\begin{array}{ccc}
\pi_*(E\wedge X) \otimes \pi_*(E\wedge Y) & \xrightarrow{\ \nu\ } & \pi_*(E\wedge X\wedge Y) \\
\Big\downarrow{\scriptstyle \psi_X \otimes \psi_Y} & & \Big\downarrow{\scriptstyle \psi_{X\wedge Y}} \\
[\pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X)] \otimes [\pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge Y)] & \longrightarrow & \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X\wedge Y)
\end{array}$$

Here the lower horizontal map sends $e \otimes f \otimes g \otimes h$ into $(-1)^{pq}\phi(e \otimes g) \otimes \nu(f \otimes h)$. This diagram gives the behaviour of the coaction map with respect to the external homology product. Finally we specialise to the case $X = Y = E$ and apply naturality to the map

$$X \wedge Y = E \wedge E \xrightarrow{\mu} E .$$

We obtain the required commutative diagram.

The proof of the remaining formal properties does not call for any special comment.

We now turn to further formulae, involving cohomology, which will help to show that our definitions specialise correctly to the classical case. We recall that the cohomology groups of a spectrum X with coefficients in E are given by

$$E^{-n}(X) = [S^n \wedge X, E] .$$

We have a Kronecker product

$$E^{-p}(X) \otimes E_q(X) \longrightarrow \pi_{p+q}(E)$$

defined as follows. Suppose given maps

$$f\colon S^p \wedge X \longrightarrow E, \quad g\colon S^q \longrightarrow E \wedge X .$$

Then $\langle f,g \rangle$ is the following composite.

$$S^p \wedge S^q \xrightarrow{1\wedge g} S^p \wedge E \wedge X \xrightarrow{1\wedge \tau} S^p \wedge X \wedge E \xrightarrow{f\wedge 1} E \wedge E \xrightarrow{\mu} E .$$

In particular, we have the cohomology groups $E^*(E)$. Since these are defined in terms of maps from E to E (up to suspension), they act on the left on the homology and cohomology groups $E_*(X)$ and $E^*(X)$. The precise definitions are as follows. Suppose given maps

$$a\colon S^p \wedge E \longrightarrow E, \quad f\colon S^q \longrightarrow E \wedge X, \quad g\colon S^r \wedge X \longrightarrow E .$$

Then af is

$$S^p \wedge S^q \xrightarrow{1\wedge f} S^p \wedge E \wedge X \xrightarrow{a\wedge 1} E \wedge X ,$$

and ag is

$$S^p \wedge S^r \wedge X \xrightarrow{1\wedge g} S^p \wedge E \xrightarrow{a} E .$$

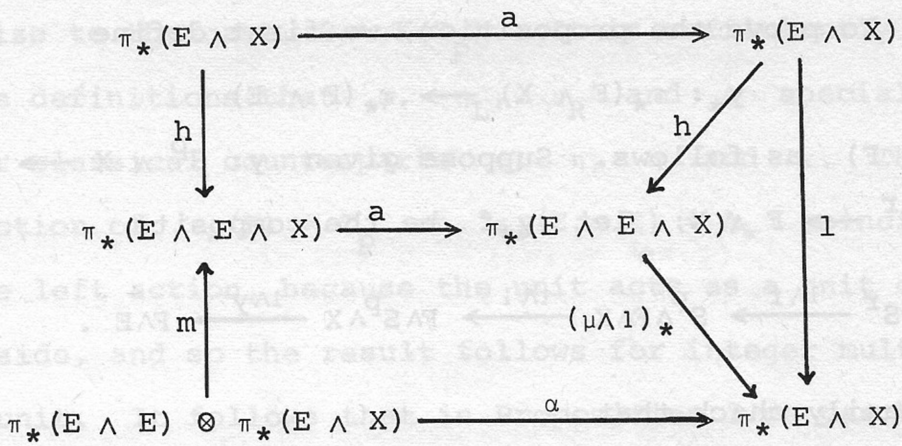
In this way $E^*(E)$ becomes a ring with unit, and $E_*(X)$, $E^*(X)$ become left modules over this ring.

We will show that the action of $E^*(E)$ on $E_*(X)$ is determined by the coaction map ψ_X. Suppose $a \in E^*(E)$, $x \in E_*(X)$ and $\psi_X x = \sum_i e_i \otimes x_i$, where $e_i \in E_*(E)$, $x_i \in E_*(X)$. Then we have:

<u>Proposition 2</u>

$$ax = \sum_i \langle a, ce_i \rangle x_i .$$

To prove this proposition, we set up the following diagram.



Here α is defined by

$$\alpha(e \otimes x) = \langle a, ce \rangle x \ .$$

It is easy to show that the diagram is commutative. This proves Proposition 2.

In the case when an element $z \in E^*(X)$ is determined by the values of $\langle z, x \rangle$ for all $x \in E_*(X)$, it is reasonable to ask for a calculation of the action of $E^*(E)$ on $E^*(X)$ in terms of ψ_X. There is a choice of formulae which answer this question; here I will give one which seems neater than that which I actually gave in Seattle. Suppose $a \in E^*(E)$, $y \in E^p(X)$, $x \in E_*(X)$ and $\psi_X x = \sum_i e_i \otimes x_i$, where $e_i \in E_{q(i)}(E)$, $x_i \in E_*(X)$. Then we have:

Proposition 3

$$\langle ay, x \rangle = \sum_i (-1)^{pq(i)} \langle a, e_i \langle y, x_i \rangle \rangle .$$

The formula on the right makes sense, because e_i lies in $\pi_*(E \wedge E)$, and $\langle y, x_i \rangle$ lies in $\pi_*(E)$, which acts on the right on $\pi_*(E \wedge E)$.

To prove the proposition, we first define

$$y_*: \pi_*(F \wedge X) \longrightarrow \pi_*(F \wedge E)$$

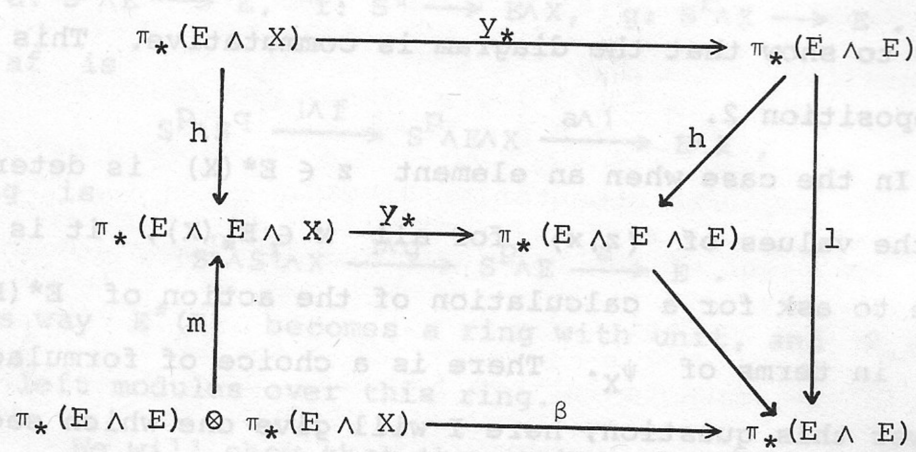
(for any F) as follows. Suppose given $y: S^p \wedge X \longrightarrow E$ and $f: S^r \longrightarrow F \wedge X$; let $y_* f$ be the composite

$$S^p \wedge S^r \xrightarrow{1 \wedge f} S^p \wedge F \wedge X \xrightarrow{\tau \wedge 1} F \wedge S^p \wedge X \xrightarrow{1 \wedge y} F \wedge E .$$

Then we easily check that

$$\langle ay, x \rangle = \langle a, y_* x \rangle .$$

We now set up the following diagram.



Here β is defined by

$$\beta(e \otimes x) = (-1)^{pq} e\langle y, x \rangle$$

for $e \in E_q(E)$. It is easy to show that this diagram is commutative. This shows that

$$y_* x = \sum_i (-1)^{pq(i)} e_i \langle y, x_i \rangle ,$$

and proves Proposition 3.

We will now discuss the way in which our constructions specialise to the case $E = K(Z_p)$. It is sufficiently clear from the definitions that ϕ, η_L, η_R and ε specialise to their classical counterparts ϕ, η, η and ε. The right action of $\pi_*(E) = Z_p$ on $\pi_*(E \wedge E) = A_*$ coincides with the left action, because the unit acts as a unit on either side, and so the result follows for integer multiples of the unit. It follows that in Proposition 3 we can bring the factor $\langle y, x_i \rangle$ to the left of e_i; and after that we can bring it outside the Kronecker product, so as to obtain the following formula.

$$\langle ay, x \rangle = \sum_i (-1)^{pq(i)} \langle a, e_i \rangle \langle y, x_i \rangle .$$

It follows that ψ_X is indeed the dual of the action map $A^* \otimes H^* \longrightarrow H^*$, and (specialising to the case $X = E$) that ψ_E is the dual of the composition map $A^* \otimes A^* \longrightarrow A^*$. Thus ψ_E and ψ_X specialise to their classical counterparts.

Since we have seen that

$$\phi(1 \otimes c)\psi_E = \eta_L \varepsilon$$

and

$$\phi(c \otimes 1)\psi_E = \eta_R \varepsilon ,$$

it now follows that c specialises to its classical counterpart.

It remains only to point out one difference between the classical case and the generalised case. In the generalised case we have introduced a left action of $E^*(E)$ on

$E_*(X)$. This does not specialise to the action of A^* on H_* which is usually considered in the classical case, since the latter is a right action, defined by

$$\langle y, xa \rangle = (-1)^{(p+q)r} \langle ay, x \rangle$$

$(y \in H^p,\ x \in H_q,\ a \in A^r)$.

The connection between the two actions may be read off from Proposition 2 and 3. We have

$$xa = (-1)^{qr}(ca)x \qquad (x \in H_q,\ a \in A^r)\ .$$

Thus the left and right actions differ by the canonical anti-automorphism, as one might expect.