APPLICATIONS OF ELLIPTIC OPERATORS AND THE ATIYAH SINGER INDEX THEOREM

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Contents

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1. Review of Differential Geometry

References for this material are [ST], [S] or any reasonable differential geometry text.

We first recall the definition of a locally trivial vector bundle, over a pointed space $\mathbf{X}$, $\xi : F \to E \to \mathbf{X}$. E is called the total space and $F = \pi^{-1}(\mathbf{x}) \approx F^n$,$F = \mathbb{R}$ or $\mathbb{C}$.

We assume $\mathbf{X} = \bigcup \mathcal{U}_i$ with $\xi|_{\mathcal{U}_i}$ is trivial. What this means is that there is a diagram

$$\begin{array}{ccc} E|_{\mathcal{U}_i} & \xrightarrow{T_i} & F^n \times \mathcal{U}_i \\ \downarrow & & \Big\downarrow \pi \\ \mathcal{U}_i & = & \mathcal{U}_i \end{array}$$

On the intersections there are the maps $\lambda_{ij} = T_i^{-1} \circ T_j : F^n \times \mathcal{U}_i \bigcap \mathcal{U}_j \to F^n \times \mathcal{U}_i \bigcap \mathcal{U}_j$ which induce maps (also denoted λ_{ij}) $\mathcal{U}_i \bigcap \mathcal{U}_j \xrightarrow{\lambda_{i,j}} F(n)$ ($F(n)$ is $O(n)$ if $F = \mathbb{R}$, $U(n)$if $F = \mathbb{C}$). The $\{\lambda_{ij}\}$ satisfy the following composition laws:

(1.1)

- (1) $\lambda_{ik} = \lambda_{ij} \circ \lambda_{jk}$ on $\mathcal{U}_i \bigcap \mathcal{U}_j \bigcap \mathcal{U}_k$
- (2)$\lambda_{ii} =$ identity
- (3)$\lambda_{ij} = \lambda_{ji}^{-1}$
- (4) $\lambda_{i,j}$ is a continuous map from $\mathcal{U}_i \bigcap \mathcal{U}_j$ to $F(n)$

Notation 1.2.

- $\mathbf{X}$ *will denote an* n *dimensional,* C^∞*, compact, closed, oriented manifold.*
- $r_j : \mathbb{R}^n \to \mathbb{R}$ *denotes the* j*-th coordinate function. In particular* $\frac{\partial}{\partial r_j}$ *is the usual partial with respect to the* j*-th coordinate.*
- $\mathbf{x}_j$ *denotes* $r_j \circ \varphi$ *where* $\varphi : \mathrm{U} \to \mathbb{R}^n$ *is a local chart,* $\mathrm{U} \subset \mathbf{X}$.
- $C(\mathbf{X}, \mathbf{x}, \mathbb{R})$ *denotes the* C^∞ *functions from some neighborhood of* $\mathbf{x} \in \mathbf{X}$ *to* $\mathbb{R}$*. Notice that* $\mathbf{x}_j$ *may be viewed as an element of* $C(\mathbf{X}, \mathbf{x}, \mathbb{R})$.
- $C(\mathbf{X}, \mathbb{R})$ *denotes the* C^∞ *global functions* $f : \mathbf{X} \to \mathbb{R}$*. More generally* $C(\mathbf{X}, \mathbf{Y})$ *denotes* C^∞ *functions from* $\mathbf{X}$ *to* $\mathbf{Y}$.

Definition 1.3. *A tangent vector* υ*, at* $\mathbf{x} \in \mathbf{X}$ *is a map* $\upsilon : C(\mathbf{X}, \mathbf{x}, \mathbb{R}) \to \mathbb{R}$ *such that*

(1) $\upsilon(f+g) = \upsilon(f) + \upsilon(g)$

(2) $\upsilon(\lambda f) = \lambda\upsilon(f)$

(3) $\upsilon(f \cdot g) = \upsilon(f)g(\mathbf{x}) + f(\mathbf{x})\upsilon(g)$

for $f, g \in C(\mathbf{X}, \mathbf{x}, \mathbb{R}), \lambda \in \mathbb{R}$

Locally $\upsilon(f) = \Sigma a_i \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi^{-1}(\mathbf{x})}$
We denote $\frac{\partial}{\partial r_i}(f \circ \phi^{-1})$ by $\frac{\partial}{\partial \mathbf{x}_i}$ and write $\upsilon = \Sigma a_i \frac{\partial}{\partial \mathbf{x}_i}$ (locally).
Let $T(\mathbf{X}, \mathbf{x})$ denote the vector space of all tangent vectors at $\mathbf{x}$. Then for $\psi \in C(\mathbf{X}, \mathbf{Y})$ there is a linear map

$$d\psi : T(\mathbf{X}, \mathbf{x}) \to T(\mathbf{Y}, \psi(\mathbf{x}))$$

defined as follows. For $\upsilon \in T(\mathbf{X}, \mathbf{x}), g \in C(\mathbf{Y}, \psi(\mathbf{x}), \mathbb{R})$ $d\psi(\upsilon)(g) = \upsilon(g \circ \psi)$

With

$$T(\mathbf{X}) = \bigcup_x T(\mathbf{X}, \mathbf{x})$$

we have a commutative diagram

$$\begin{array}{ccc} T(\mathbf{X}) & \xrightarrow{d\psi} & T(\mathbf{Y}) \\ \downarrow & & \downarrow \\ \mathbf{X} & \xrightarrow{\psi} & \mathbf{Y} \end{array}$$

Locally if $\upsilon = \Sigma a_i \frac{\partial}{\partial \mathbf{x}_i}$ then

$$d\psi(\upsilon) = \Sigma\upsilon(y_i \circ \psi)\frac{\partial}{\partial \mathbf{y}_i}$$

$\{\mathbf{y}_i\}$ local coordinates functions for a neighborhood of $\psi(\mathbf{x})$ in $\mathbf{Y}$.

Definition 1.4. *$T^*(\mathbf{X})$ is the dual of $T(\mathbf{X})$.*

Locally $T^*(\mathbf{X})$ can be identified with $\{df | f \in C(\mathbf{X}, \mathbf{x}, \mathbb{R})\}$. From the above we have $df : T(\mathbf{X}, \mathbf{x}) \to \mathbb{R}$ on a vector υ is defined by $df(\upsilon) = \upsilon(f) \in \mathbb{R}$. $\{d\mathbf{x}_i \in T^*(\mathbf{X}, \mathbf{x}) | i = 1, \cdots n\}$ span $T^*(\mathbf{X}, \mathbf{x})$. In fact $d\mathbf{x}_i$ is dual to $\frac{\partial}{\partial \mathbf{x}_i}$ because $d\mathbf{x}_i(\frac{\partial}{\partial \mathbf{x}_j}) \stackrel{\text{def}}{=} \frac{\partial}{\partial \mathbf{x}_i}(\mathbf{x}_j) = \delta_{ij}$.

Definition 1.5. *A smooth section, ω, to the bundle $T^*(\mathbf{X}) \xrightarrow{\pi} \mathbf{X}$ is called a 1-form.*

Locally $\omega = \Sigma a_i d\mathbf{x}_i$ where $a_i \in C(U, \mathbb{R}), (U \subseteq \mathbf{X})$. For example if $f \in C(\mathbf{X}, \mathbf{x}, \mathbb{R}), df = \Sigma \frac{\partial}{\partial \mathbf{x}_i}(f) d\mathbf{x}_i$ In fact $a_i = \Sigma a_j d\mathbf{x}_j (\frac{\partial}{\partial \mathbf{x}_i}) = df(\frac{\partial}{\partial \mathbf{x}_i}) = \frac{\partial}{\partial \mathbf{x}_i}(f)$.

Notation 1.6. *To generalize the above we define:*

$\bullet \Lambda^k(\mathbf{X}) = \bigcup_{\mathbf{x} \in \mathbf{X}} \Lambda^k(T^*(\mathbf{X}, \mathbf{x}))$

$\bullet \Lambda^*(\mathbf{X}) = \bigoplus_k \Lambda^k(\mathbf{X})$

•*A smooth section to the bundle* $\Lambda^k(\mathbf{X}) \to \mathbf{X}$ *is a* $k-$*form. The space of k-forms is denoted* $\Omega^k(\mathbf{X}, \mathbb{R})$.

•*A smooth section to the bundle* $\Lambda^*(\mathbf{X}) \to \mathbf{X}$ *is a differential form.The space of differential-forms is denoted* $\Omega^*(\mathbf{X}, \mathbb{R})$.

There is a unique map

(1.7)
$$d : \Omega^k(\mathbf{X}, \mathbb{R}) \to \Omega^{k+1}(\mathbf{X}, \mathbb{R})$$

such that:

- $d(f) = df (df$ as in the paragraph after 1.4) where $f \in C(\mathbf{X}, \mathbb{R})$ is thought of as a 0-form $(\mathbb{R} = \Omega^0)$.
- $d(\mu \wedge \tau) = d\mu \wedge \tau + (-1)^k \mu \wedge d\tau (\mu \in \Omega^k(\mathbf{X}, \mathbb{R}))$
- $d^2 = 0$

Locally if $\omega = \Sigma a_I d\mathbf{x}_I$ then

$$d\omega = \sum_{I,j} \frac{\partial}{\partial \mathbf{x}_j}(a_I) d\mathbf{x}_j \wedge d\mathbf{x}_I$$

(I a sequence of non-negative integers $i_1 < i_2 \cdots < i_k, d\mathbf{x}_I = d\mathbf{x}_1 \wedge d\mathbf{x}_2 \cdots \wedge d\mathbf{x}_k$)

Definition 1.8. *A volume element is a choice of basis for* $\Lambda^n(T^*(\mathbf{X}, \mathbf{x}))$. *For example* $d\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n$.

2. Definition of an Elliptic Operator

Notation 2.1.

- *Let $U \subset \mathbb{R}^n$ be an open set, $t = (t_1, \cdots t_n)$ an n-tuple of non-negative integers. $|t| = \Sigma t_i$. Define*

$$D^t = \frac{1}{i^{|t|}} \frac{\partial^{|t|}}{\partial \mathbf{x}_1^{t_1} \partial \mathbf{x}_2^{t_2} \cdots \partial \mathbf{x}_n^{t_n}}$$

- *If $v \in T^*(\mathbf{X}, \mathbf{x})$ with $v = \Sigma v_i d\mathbf{x}_i$ then $v^t = v_1^{t_1} v_2^{t_2} \cdots v_n^{t_n} \in \mathbb{R}$.*

Let A, B be finite dimensional complex vector spaces, $C(U, A), C(U, B)$ C^∞ functions, as before (with the compact open topology).

Definition 2.2. *A linear differential operator of order r is a linear map*

$$D : C(U, A) \to C(U, B)$$

such that

$$D(f) = \sum_{|t| \leq r} g_t D^t(f), f \in C(U, A)$$

where g_t is a C^∞ function, $g_t : U \to Hom(A, B)$ i.e. $g_t(u)$is a matrix.

Now suppose E, F are complex vector bundles over $\mathbf{X}$.

Definition 2.3. *A differential operator, D, is a linear map*

$$D : \Gamma(\mathbf{X}, E) \to \Gamma(\mathbf{X}, F)$$

(Γ is the space of C^∞ sections with the compact open topology) such that there exist open sets U_i which cover $\mathbf{X}$ where $E|_{U_i}, F|_{U_i}$ are trivial and there exist linear differential operators, D_i on U_i such that $D(f)|_{U_i} = D_i$

Let $\pi : T^*(\mathbf{X}) \to \mathbf{X}$ be the projection. For $q \in \mathbf{X}$ let$g : \mathbf{X} \to \mathbb{R}$ such that $v_q = dg_{(q)} \in T^*(\mathbf{X}, q)$ and $s \in \Gamma(E)$ such that $s(q) = e, \lambda \in \mathbb{R}$. Now write

$$e^{-i\lambda g} D(e^{i\lambda g} s) = \lambda^r p_r(s, g) + \cdots \lambda_1 p_1 + p_0 \in \Gamma(\mathbf{X}, F).$$

For fixed g, $s \to p_r(s, g)$ is a homomorphism of $E_q \to F_q$ which only depends on the derivative of g, i.e. v.

Definition 2.4. *The symbol $\sigma_D : \pi^*(E) \to \pi^*(F)$ of the operator, D is defined by*

$$\sigma_D(v,e) = \frac{1}{i^{|t|}} p_r(s,g)$$

for D and r-th order differential operator.

Example 2.5. *Let $D = \frac{\partial}{\partial \mathbf{x}_{i_1}} \frac{\partial}{\partial \mathbf{x}_{i_2}}$. Then*

$$e^{-\lambda g} \frac{\partial}{\partial \mathbf{x}_{i_1}} \frac{\partial}{\partial \mathbf{x}_{i_2}} (e^{i\lambda g} s) =$$

$$e^{-i\lambda g} \frac{\partial}{\partial \mathbf{x}_{i_1}} [i\lambda \frac{\partial g}{\partial \mathbf{x}_{i_2}} e^{i\lambda g} s + \text{ lower terms in } \lambda] =$$

$$i^2 \lambda^2 \frac{\partial g}{\partial \mathbf{x}_{i_1}} \frac{\partial g}{\partial \mathbf{x}_{i_2}} s + \text{ lower terms in } \lambda$$

In local coordinates we suppose $dg_{\mathbf{x}} = \Sigma \xi_i d\mathbf{x}_i, \xi_i = \frac{\partial g}{\partial \mathbf{x}_i}$ then

$$e^{-i\lambda g} \frac{\partial^{|t|}}{\partial \mathbf{x}_1^{t_1} \cdots \partial \mathbf{x}_n^{t_n}} (e^{i\lambda g} s)_q = i^{|t|} \lambda^{|t|} \xi^t s(q) + \text{lower terms in } \lambda$$

Finally

$$\sigma_D(v_q, e) = \sum_{|t|=r} g_t(\mathbf{x}) v^t e \in F$$

i.e. σ_D is the leading term of D with v^t substituted for D^t.

Definition 2.6. *D is an elliptic operator if $\sigma_D(v)$ is an isomorphism for all $v \neq 0$.*

i.e. $\sigma_D : \pi^*(E) \to \pi^*(F)$ is an isomorphism of the fibers away from the zero section of $T^*(\mathbf{X}) \to \mathbf{X}$.

3. Properties of Elliptic Operators

Reference for this section is [P].

Theorem 3.1. *If D is an elliptic differential operator then*

$$D : \Gamma(\mathbf{X}, E) \to \Gamma(\mathbf{X}, F)$$

has finite dimensional kernel and finite dimensional cokernel.

Proof : [P] page 178.

Definition 3.2. *The index of the elliptic operator D is defined to be $IndexD = dim_{\mathbb{C}}(\text{ kernel } D) - dim_{\mathbb{C}}(\text{ cokernel } D)$.*

Theorem 3.3. *(**Stability of the Index**) Index D is invariant under "deformations" through elliptic operators.*

Proof : [P] page 185.

Corollary 3.4. *The index depends only on the symbol.*

Proof : If D_1 and D_2 have the same symbol then $t(D_1) - (1-t)D_2$ is a deformation of D_1 to D_2 which is elliptic at each stage. **q.e.d.**

We now assume E and F have a Hermitian inner product.

Definition 3.5. *A formal adjoint for D is a differential operator, D^* such that for $s \in \Gamma(E), t \in \Gamma(F)$*

$$\int_{\mathbf{X}} \langle D(s), t \rangle_F = \int_{\mathbf{X}} \langle s, D^*(t) \rangle_E$$

Theorem 3.6. *For a fixed metric, there is a unique formal adjoint. Furthermore*

- $\sigma(D^*) = (\sigma(D))^*$
- *cokernel* $D =$ *kernel* D^*
- *Index* $D = dim_{\mathbb{C}} kernel D - dim_{\mathbb{C}} kernel D^*$
- *Index* $D = -$*Index* D^*

Proof : Locally the existence of D^* is integration by parts.

Finally we have

Proposition 3.7. *If D^1 is an ℓ-order differential operator*

$$D^1 : \Gamma(E) \to \Gamma(F)$$

and D^2 is a k-th order differential operator

$$D^2 : \Gamma(F) \to \Gamma(G)$$

then

$$\sigma_{k+\ell} = \sigma_\ell(D^2)\sigma_k(D^1)$$

4. Example of an Elliptic Operator

We start with slight changes in notation. For V a finite dimensional, real vector space:

•$\Lambda^*(V;\mathbb{R})$ denotes the Grassman algebra.
•$\Lambda^*(V) = \Lambda^*(V;\mathbb{R}) \otimes \mathbb{C}$.
•$\Lambda^*(\mathbf{X}) = \Lambda^*(\mathbf{X})$.
•$\Omega^*(\mathbf{X}) = \Gamma(\Lambda^*(\mathbf{X}))$ = Complex differential forms.

$$d : \Omega^k(\mathbf{X}) \to \Omega^{k+1}(\mathbf{X})$$

is the complexification of the map in (1.7).

Our first task is to determine $\sigma_d(v) : \Lambda^*(T^*(\mathbf{X},q) \to \Lambda^*(T^*(\mathbf{X},q)$.

Proposition 4.1. $\sigma_d(v) = iA_v$ *where* $A_v(\alpha) = v \wedge \alpha$.

Proof : For $\omega(q) = \alpha, dg = v$ we have

(4.2) $e^{-i\lambda g} d(e^{i\lambda g}\omega) =$

$$e^{-i\lambda g}[de^{i\lambda g} \wedge \omega(q) + \text{lower terms in } \lambda] =$$
$$e^{-i\lambda g}[i\lambda dge^{i\lambda g} \wedge \omega(q)] + \text{lower terms in } \lambda =$$
$$i\lambda v \wedge \alpha + \text{lower terms}$$

So $\sigma_d(v) = iv \wedge \alpha$. **q.e.d.**

Having chosen a Riemannian metric for $T^*(\mathbf{X})$ we can define a metric on $\Lambda^*(\mathbf{X};\mathbb{R})$ by

Definition 4.3.

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = det|\langle v_i, w_j \rangle|$$

$\Lambda^k(\mathbf{X};\mathbb{R})$ *is orthogonal to* $\Lambda^s(\mathbf{X},\mathbb{R})$ *if* $k \neq s$

This induces a well defined Hermitian metric on $\Lambda^*(\mathbf{X})$. Let $d^* : \Omega^{r+1}(\mathbf{X}) \to \Omega^r(\mathbf{X})$ be the formal adjoint of d with respect to this metric.

Theorem 4.4.

$$A_v^*(v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} \overline{\langle v, v_i \rangle}_{T^*(\mathbf{X},\mathbf{x})} v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p$$

where $\widehat{\ }$ *denotes deletion, and* $v_i \in T^*(\mathbf{X},\mathbf{x})$.

Proof : We need to show that $\langle A_v\alpha, \beta\rangle_{\Lambda^*} = \langle \alpha, A_v^*\beta\rangle_{\Lambda^*}$ where $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k \in \Lambda^k, \beta = \beta_1 \wedge \cdots \wedge \beta_{k+1} \in \Lambda^{k+1}, \alpha_i, b_i \in T^*(\mathbf{X}, \mathbf{x})$ and $\langle -, -\rangle_{\Lambda^*}$ given by (4.3). So we must show that

$$\langle v \wedge \alpha, \beta\rangle =$$
$$\langle \alpha, \Sigma(-1)^{i+1}\overline{\langle v, \beta_i\rangle}\beta_1\wedge\cdots\wedge\widehat{\beta_i}\cdots\wedge\beta_{k+1}\rangle = \Sigma(-1)^{i+1}\langle v, \beta_i\rangle\langle \alpha, \beta_1\wedge\cdots\wedge\widehat{\beta_i}\cdots\wedge\beta_{k+1}\rangle$$

But this is just the expansion of the determinant $\langle v\wedge\alpha, \beta\rangle$ by minors of he row containing v. **q.e.d.**

Corollary 4.5.
$$(A_v - A_v^*)^2(\alpha) = -||v||^2\alpha.$$

Proof : Let $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$.

$$(A_v - A_v^*)^2(\alpha) =$$
$$(A_v - A_v^*)[v \wedge \alpha - \Sigma(-1)^{i+1}\overline{\langle v, \alpha_i\rangle}\alpha_1 \wedge \cdots \widehat{\alpha_i} \cdots \wedge \alpha_k] =$$
$$-\langle v, v\rangle\alpha_1\wedge\cdots\wedge\alpha_k + \Sigma(-1)^{i+1}\overline{\langle v, \alpha_i\rangle}v\wedge\alpha_1\wedge\cdots\widehat{\alpha_i}\cdots\wedge\alpha_k - \Sigma(-1)^{i+1}\overline{\langle v, \alpha_i\rangle}v\wedge\alpha_1\wedge\cdots\widehat{\alpha_i}\cdots\wedge\alpha_k =$$
$$-||v||^2\alpha$$

(we have used $A_v^2 = (A_v^*)^2 = 0$.) **q.e.d.**

Now

(4.6)
$$\sigma_{d+d^*}(v) = i(A_v - A_v^*)$$

(the negative sign is a consequence of the fact that the inner product is Hermitian) and

$$\sigma_{d+d^*} \circ \sigma_{d+d^*}(v) = \sigma_{(d+d^*)^2}(v) = ||v||^2\alpha.$$

It follows that $d + d^*$ is elliptic.

Definition 4.7. *The Laplace operator,* Δ *is defined to be* $(d + d^*)^2 = dd^* + d^*d$.

Proposition 4.8. $ker(d + d^*) = ker(\Delta) = \{x|dx = d^*x = 0\}$

Proof : If $(d + d^*)x = 0$ then clearly $\Delta(x) = 0$. Suppose $\Delta(x) = 0$ then

$$0 = \int_{\mathbf{X}} < \Delta x, x >= \int < (d+d^*)x, (d+d^*x >= \int < dx, dx > + \int < d^*x, d^*x >$$

Since both terms are zero it follows that $dx = d^* = 0$. **q.e.d.**

Definition 4.9. $\ker \Delta =$ the *Harmonic forms.*

Using the sequence

$$\to \Omega^i \xrightarrow{d} \Omega^{i+1} \to$$

we define

Definition 4.10. $H^n(\mathbf{X}; \mathbb{C}) = ker\ d/image\ d.$

By proposition (4.8) a Harmonic form represents an element of $H^*(\mathbf{X}, \mathbb{C})$. We say $\alpha \in H^*(\mathbf{X}; \mathbb{C})$ has a *harmonic representative* if $\exists a \in \ker\Delta$ such that $[a] = \alpha$.

Theorem 4.11. ***(Hödge Theorem)*** *Every class in $H^*(\mathbf{X}; \mathbb{C})$ has a unique harmonic representative. i.e. $ker\Delta = H^*(\mathbf{X}; \mathbb{C})$.*

5. Example: The Euler Characteristic

Index$(d+d^*) = 0$ since $d+d^* : \Omega^* \to \Omega^*$ is self adjoint. Notice that $d+d^* : \Omega^{even} \to \Omega^{odd}$.

Definition 5.1. $D_\chi = (d+d^*)|_{\Omega^{even}}$

Theorem 5.2. *$Index(D_\chi) = \chi(\mathbf{X}) =$ the Euler characteristic of $\mathbf{X}$.*

Proof : $kerD_\chi = ker\Delta|_{\Omega^{even}} = dimH^{even}(\mathbf{X}, \mathbb{C})$.
$cokerD_\chi = kerD_\chi^* = ker(d+d^*)|_{\Omega^{odd}} = ker\Delta|_{\Omega^{odd}} = dimH^{odd}(\mathbf{X}; \mathbb{C})$
$\therefore IndexD_\chi = \chi(\mathbf{X})$. **q.e.d.**

Theorem 5.3. ***(Hopf)*** *If u is a nowhere vanishing vector field on $\mathbf{X}$ then $\chi(\mathbf{X}) = 0$.*

In order to prove (5.3) we give an alternate description of σ_{d+d^*}. For a real vector space V with inner product $\langle -, - \rangle$ let $e_1, \cdots, e_k$ be an orthonormal basis.

Definition 5.4. *Cliff(V) is the free associative algebra, with 1 generated by $\{e_i\}$ subject to the relations*
- $v \cdot v = -||v|| \cdot 1$, $1 \in$ *Cliff*(V), $v \in V$.
- $e_i \cdot e_j = -e_j \cdot e_i$

As a vector space Cliff$(V)$is generated by $e_{i_1} \cdots e_{i_\ell}, i_1 < i_2 < \cdots < i_\ell$. Hence as a vector space $\Lambda^*(V) \approx$ Cliff(V). via the map $e_{i_1} \wedge \cdots \wedge e_{i_\ell} \mapsto e_{i_1} \cdots e_{i_\ell}$. So we think of $\Lambda^*(V)$ as endowed with two ring structures, exterior and Clifford multiplication.

Definition 5.5. *L_v is defined by $L_v(\alpha) = v \cdot \alpha$ ($\cdot$ denotes Clifford multiplication).*

Note: Cliff(V)has a $\mathbb{Z}_2$-grading Cliff$(V)^0 \bigoplus$ Cliff$(V)^1$. Cliff$(V)^0$ =even grading, Cliff$(V)^1$ = odd grading.

Theorem 5.6. *$L_v(\alpha) = (A_v - A_v^*)\alpha$ for $v \in T^*(\mathbf{X}, \mathbf{x})$*

Proof : L_v and $A_v - A_v^* \in End(\Lambda^*)$ are linear in v. It therefore suffices to verify (5.6) for $v = e_i, \alpha = e_{i_1} \wedge \cdots \wedge e_{i_t}$.
Case I. $e_i \neq e_{i_\ell}$ any i_ℓ.. Then $L_v = e_i \cdot (e_{i_1} \cdot e_{i_t})$ and $(A_v - A_v^*)\alpha = e_i \wedge (e_{i_1} \wedge \cdots \wedge e_{i_t}) - \sum_r (-1)^{r+1} \langle e_i, e_{i_r} \rangle e_{i_1} \wedge \cdots \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_t} = e_i \wedge (e_{i_1} \wedge \cdots \wedge e_{i_t})$. Hence $L_v(\alpha) = (A_v - A_v^*)\alpha$ after identification.

Case II. $e_i = e_{i_r}$ for some i_r. then $L_v(\alpha) = (-1)^r e_{i_1} \cdots \widehat{e_{i_r}} \cdots e_{i_\ell}$.

$$(A_v - A_v^*)(\alpha) = \underbrace{e_{i_r} \wedge e_{i_1} \wedge \cdots \wedge e_{i_\ell}}_{=0} - \Sigma(-1)^{t+1} \langle e_{i_r}, e_{i_t} \rangle e_{i_1} \wedge \cdots \widehat{e_{i_t}} \wedge \cdots \wedge e_{i_\ell} = L_v(\alpha).$$

Corollary 5.7. $\sigma_{d+d^*} = iL_v$

Proof : This follows from (5.6) and (4.6). **q.e.d.**

Notice that (4.5) follows from the Clifford identities.

Now continuing with the proof of (5.3). We identify $T(\mathbf{X}, \mathbb{R})$ with $T^*(\mathbf{X}, \mathbb{R})$ via the metric. The vector field, u determines a non-vanishing 1-form, which we also denote by u. Define $B_u : \Omega^*(\mathbf{X}) \to \Omega^*(\mathbf{X})$ by $B_u(\alpha) = a \wedge u$. B_u^* is the adjoint and, as in (5.6) $R_u = B_u - B_u^*$ is right Clifford multiplication. $R_u^2 = -||u||^2 id$. Furthermore R_u sends Ω^{odd} to Ω^{even} and visa versa. So we have a diagram:

(5.8)
$$\begin{array}{ccc} \Omega^{\text{even}} & \xrightarrow{D_x} & \Omega^{\text{odd}} \\ \downarrow R_u & & \downarrow R_u \\ \Omega^{\text{odd}} & \xrightarrow{D_x} & \Omega^{\text{even}} \end{array}$$

We claim that (5.8) commutes on the symbol level.

Proof : On the level of symbols, using (3.7), and thinking of R_u as a zero order operator we have

$$\sigma_{R_u \circ D_x}(\nu) = (\nu\alpha)u, \quad \sigma_{D^* \circ R_u}(\nu) = \nu(\alpha u).$$

The claim now follows for the associative law in Cliff(V).

Hence by (3.4)

$$\text{Index } (R_u \circ D_x) = \text{Index } (D_x^* \circ R_u)$$

But R_u is an isomorphism (since $u \neq 0$). So

$$\text{Index } (R_u \circ D_x) = \text{Index } (D_x)$$
$$\text{Index } (D_x^* \circ R_u) = \text{Index } (D_x^*)$$

Hence by (3.6) Index $(D_x) = \chi(\mathbf{X}) = 0$.

6. Example: The Signature Invariant

Let V be a finite dimensional, real vector space. $\langle -,-\rangle$ a symmetric, non-degenerate , bilinear form on V. There is a basis $\{e_1, \cdots, e_m, f_1, \cdots, f_n\}$ of V such that $\langle e_i, e_j\rangle = \delta - i, j, \langle f_i, f_j\rangle = -\delta_{i,j}, \langle e_i, f_j\rangle = 0$.

Definition 6.1. *The signature of $\langle -,-\rangle$ is defined to be $m-n$.*

Now let $\mathbf{X}$ be a closed oriented manifold of dimension $n = 4k$. Then on the real vector space $H^{2k}(\mathbf{X};\mathbb{R})$ we have the bilinear form $\langle \alpha, \beta\rangle = \alpha \cup \beta[\mathbf{X}]$. This form is symmetric (because $n = 4k$) and non-singular (because of Poincare' duality).

Definition 6.2. *Sign($\mathbf{X}$) = signature($\langle -,-\rangle$).*

We now look for a differential operator whose index is $Sign(\mathbf{X})$.

Let $\omega \in \Lambda^n(\mathbf{X})$ be a volume form, i.e. $\omega_{\mathbf{x}} = e_1 \wedge \cdots \wedge e_n$ where $e_1, \cdots e_n$ is any orthonormal basis for $T^*(\mathbf{X}, \mathbf{x})$.

An operator $* : \Lambda^p(\mathbf{X}, \mathbb{R}) \to \Lambda^{n-p}(\mathbf{X}, \mathbb{R})$ is defined by the formula

$$[\lambda \wedge (*\mu)]_n = \langle \lambda, \mu\rangle_{\Lambda^*_{\mathbb{R}}} \omega_{\mathbf{x}} \tag{6.3}$$

for $\lambda, \mu \in \Lambda^*_{\mathbf{x}}$ and for $t \in \Lambda^*, [t]_n$ denotes the term with grading n.

By linearity there is induced

$$* : \Lambda^p(\mathbf{X}) \to \Lambda^{n-p}(\mathbf{X})$$

and

$$*; \Omega^p(\mathbf{X}) \to \Omega^{n-p}(\mathbf{X})$$

Locally, if $\overline{e_1}, \cdots, \overline{e_n}$ is a basis of $T^*(\mathbf{X}, \mathbf{x})$ (with the same orientation as $e_1, \cdots e_n$) then

$$*(\overline{e_1} \wedge \cdots \wedge \overline{e_p}) = \overline{e_{p+1}} \wedge \cdots \wedge \overline{e_n} \tag{6.4}$$

Properties of $*:\Omega^*\to\Omega^*$

Since $*$ is the complexification of a real operator we have

$$*(\overline{\beta}) = \overline{*\beta}. \tag{6.5}$$

Such an operator is called a real operator.

Note: If V is a complex vector space with a conjugation, $\overline{v}$, we define $V_{\mathbb{R}}$ to be the fixed space of conjugation and we have

$$V = V_{\mathbb{R}} \otimes \mathbb{C}. \tag{6.6}$$

If $L : V \to W$ is an operator which commutes with conjugation (i.e. a real operator) then L induces $L_{\mathbb{R}} : V_{\mathbb{R}} \to W_{\mathbb{R}}$ and $L = L_{\mathbb{R}} \otimes \mathbb{C}$.

Proposition 6.7. *If $\alpha \in \Lambda^i$ then $**\alpha = (-1)^i\alpha$*

Proof : Let $\alpha = e_1 \wedge \cdots e_i$ then $*\alpha = e_{i+1} \wedge \cdots \wedge e_n$

$$(\lambda \wedge (**\alpha)) = \langle \lambda, *\alpha\rangle \omega_{\mathbf{X}} \tag{6.8}$$

Let $\lambda = e_{i+1} \wedge \cdots \wedge e_n$. Then $**\alpha = \epsilon e_1 \wedge \cdots \wedge e_i$. From (6.8) we have

$$\epsilon(e_{i+1} \wedge \cdots \wedge e_n \wedge e_1 \wedge \cdots \wedge e_i) = e_1 \wedge \cdots \wedge e_n$$

Hence $\epsilon = (-1)^{i(n-i)} = (-1)^i$ **q.e.d.**

We also have, from (6.3)

$$\int_{\mathbf{X}} \langle \alpha, \beta\rangle_{\Lambda^*} = \int_{\mathbf{X}} (\alpha \wedge *\beta)_n \tag{6.9}$$

In fact the notation $\int_{\mathbf{X}}\langle\alpha,\beta\rangle$ stands for the integral $\int_{\mathbf{X}}\langle\alpha,\beta\rangle\omega$.

Proposition 6.10.

$$d^* = - * d*$$

Proof : Let $f \in \Omega^p, g \in \Omega^{p+1}$. We must show that

$$\int_{\mathbf{X}} \langle df, g\rangle = \int_{\mathbf{X}} \langle f, - * d*\rangle$$

Now $d(f \wedge (*g))_n = (df \wedge (*g))_n + (-1)^p(f \wedge d(*g))_n$, so we have

$$\int_{\mathbf{X}} d(f \wedge (*g))_n = \int_{\mathbf{X}} (df \wedge (*g))_n + (-1)^p \int_{\mathbf{X}} (f \wedge d * g))_n. \tag{6.11}$$

But $\mathbf{X}$ has no boundary, so by Stoke's theorem the left hand side of (6.11) is zero. Using (6.7) and (6.9) we obtain

$$0 = \int_{\mathbf{X}} \langle df, g\rangle + (-1)^p(-1)^{n-p} \int_{\mathbf{X}} \langle f, *d * g\rangle$$

and (6.10) follows. **q.e.d.**

Recall the dimension of $\mathbf{X}$ is $n = 4k$. Let $\alpha \in \Omega^p$

Definition 6.12. $\tau(\alpha) = (-1)^{\frac{p(p-1)}{2}+k} * (\alpha)$

By (6.7) $\tau^2(\alpha) = \alpha$.

On each fiber we have the eigenspaces $\Lambda_{\mathbf{x}^+}, \Lambda_{\mathbf{x}^-}$ of $\tau_{\mathbf{x}}$. We therefore have a decomposition $\Lambda_{\mathbf{x}} = \Lambda_{\mathbf{x}^+} \bigoplus \Lambda_{\mathbf{x}^-}$. Because $\tau : \Lambda^* \to \Lambda^*$ is equivariant under the action of $SO(n)$, we may glue $\Lambda_{\mathbf{x}^+}$and $\Lambda_{\mathbf{x}^-}$ together to obtain subbundles $\Lambda_+(\mathbf{X})$ and $\Lambda_-(\mathbf{X})$ such that $\Lambda^* = \Lambda^*_+(\mathbf{X}) \bigoplus \Lambda^*_-(\mathbf{X})$. Correspondingly we get a decomposition $\Omega^* = \Omega^*_+ \bigoplus \Omega^*_-$ and $\Omega^*(\mathbf{X}, \mathbb{R}) = \Omega^*_+(\mathbf{X}, \mathbb{R}) \bigoplus \Omega^*_-(\mathbf{X}, \mathbb{R})$. So if $\varphi \in \Omega^*_\pm, \tau(\varphi) = \pm\varphi$.

Proposition 6.13.

$$\tau(d + d^*) = -(d + d^*)\tau$$

Proof : Use $*^2 = (-1)^k$ on Ω^k, $d^* = - * d*$ and the definition of τ **q.e.d.**

So $d + d^*$ defines a homomorphism

$$D_S : \Omega_+ \to \Omega_-$$

The formal adjoint of D_S, $D_S^* : \Omega_- \to \Omega_+$ is the restriction of $d + d^*$.

Theorem 6.14. *Index* $(D_S) =$ *Signature*$(\mathbf{X})$

Proof : Let $\Delta = (d + d^*)^2, \quad \tau\Delta = \Delta\tau$. Identify ker Δ (=harmonic forms)with $H^*(\mathbf{X}; \mathbb{C})$. We have $\tau : H^*(\mathbf{X}; \mathbb{C}) \to H^*(\mathbf{X}; \mathbb{C})$ such that $\tau^2 = \mathrm{id}$ and a decomposition

$$H^*(\mathbf{X}; \mathbb{C}) = H^*_+(\mathbf{X}; \mathbb{C}) \bigoplus H^*_-(\mathbf{X}; \mathbb{C}).$$

So Index $(D_S) = \dim H^*_+(\mathbf{X}; \mathbb{C}) - \dim H^*_-(\mathbf{X}; \mathbb{C})$. We first look at $H^\ell(\mathbf{X}; \mathbb{C}) \bigoplus H^{n-\ell}(\mathbf{X}; \mathbb{C}), \ell < 2k$. Then τ leaves this invariant. Let

$$\widehat{H}^\ell_\pm = \begin{cases} = \{x \in H^\ell(\mathbf{X}; \mathbb{C}) \bigoplus H^{n-\ell}(\mathbf{X}; \mathbb{C}) | \tau(x) = \pm x\} & \ell < 2k, \\ H^{2k}_\pm & \ell = 2k \end{cases}$$

Then

$$H^*_{\pm}(\mathbf{X},\mathbb{C}) = \bigoplus_{\ell\le 2k} \widehat{H}^{\ell}_{\pm} \tag{6.15}$$

(To see this let $y \in H^*_{\pm}(\mathbf{X},\mathbb{C})$ be $y = y^0 + y^1 + \cdots y^n, y^i \in H^i$. Then apply τ to Σy^i and use the definition of $H^*_{\pm}$.) The argument in the parenthesis shows that for $\ell < 2k$, $H^{\ell}_{\pm} = \{x \pm \tau(x), x \in H^{\ell}(\mathbf{X};\mathbb{C})\}$. Using (6.15) we see that the map $x \mapsto x \pm \tau(x)$ gives an isomorphism

$$H^{\ell}(\mathbf{X};\mathbb{C}) \to H^{\ell}_{\pm}(\mathbf{X};\mathbb{C}), \ell < 2k.$$

So Index $(D_S) = \dim\, H^{2k}_+ - \dim\, H^{2k}_-$

On $H^{2k}(\mathbf{X};\mathbb{C})$ $\tau = *$ (because $(-1)^{\frac{2k(2k-1)}{2}+k} = 1$). Since τ is a real operator we have

$$H^{2k}_{\pm}(\mathbf{X};\mathbb{C}) = H^{2k}_{\pm}(\mathbf{X};\mathbb{R}) \otimes \mathbb{C}$$

(see (6.5)) and Index $(D_S) =$ Real dim $H^{2k}_+(\mathbf{X};\mathbb{R})-$Real dim $H^{2k}_-(\mathbf{X};\mathbb{R})$.

By the DeRham theorem the form $a \cup b$ is given by $\int_{\mathbf{X}} \alpha \wedge \beta$ on $H^{2k}(\mathbf{X};\mathbb{R})$. Now let $\alpha \in H^{2k}_+$. Then

$$\alpha \cup \alpha = \int \alpha \wedge \alpha = \int \alpha \wedge \tau(a) = \int \alpha \wedge *\alpha = \int \langle \alpha, \alpha \rangle > 0$$

If $\beta \in H^{2k}_-$ then

$$\beta \cup \beta = \int \beta \wedge \beta = \beta \wedge -\tau(\beta) = -\int \beta \wedge *\beta < o$$

We also have

$$\alpha \cup \beta = \int \alpha \wedge \beta = \int \alpha \wedge -\tau(\beta) = -\int a \wedge *\beta = -\int \langle \alpha, \beta \rangle_{\Lambda^*}$$

$$\beta \cup \alpha = \int \beta \wedge \alpha = \int \beta \wedge \tau(\alpha) = \int \beta \wedge *\alpha = \int \langle \beta, \alpha \rangle_{\Lambda^*}$$

But on H^{2k} $\langle \alpha, \beta \rangle_{\Lambda^*} = \langle \beta, \alpha \rangle_{\Lambda^*}, \quad \alpha \cup \beta = \beta \cup \alpha$. So $H^{2k}_+ \perp H^{2k}_-$.

q.e.d.

7. A Theorem of Atiyah, Frank and Mayer

Recall the definition of $Cliff(V)$ (see 5.4). For $V = \mathbb{R}^n, \langle -,-\rangle =$ the usual inner product, we denote $Cliff(\mathbb{R}^n)$ by $\mathcal{C}_n$.

Let M be a real vector space which is a $Cliff(\mathrm{V})_n$ module. We say M is a $\mathbb{Z}_2$ graded $Cliff(\mathrm{V})_n$ module if $M = M^0 \bigoplus M^1$ and $Cliff(\mathrm{V})_n^i \cdot M^j \subset M^{i+j}$, $(i,j \in \mathbb{Z}_2)$. The following result requires a deeper knowledge of the structure of $Cliff(\mathrm{V})_n$ and will be proven in §8.

Theorem 7.1. *If M admits the structure of a $\mathbb{Z}_2$ graded $Cliff(V)_k$ module then the dimension of M is divisible by $2a_k$ where a_k is given by the table*

(7.2)

k	1	2	3	4	5	6	7	8
a_k	1	2	4	4	8	8	8	8

, $\quad a_{k+8} = 16a_k$

$(a_k \sim 2^{[\frac{k-1}{2}]})$.

The main result of this section is

Theorem 7.3. *If $\mathbf{X}$ is a compact, boundaryless manifold of dimension $n = 4k$ with r linearly independent vector fields then Signature($\mathbf{X}$) is divisible by $2a_r$.*

Proof : We may take the fields to be orthonormal. Let $v_1, \cdots, v_r$ be these fields. (Note: we are identifying $T(\mathbf{X})$ with $T^{*]}(\mathbf{X})$ using the metric as in the proof of (5.3).) There are the operators $R_{v_i} : \Omega^* \to \Omega^*$.
Claim 1 R_{v_i} commutes with τ.
Proof : (of claim 1)We show that $L_\omega(\alpha) = (-1)^k\tau(\alpha)$ (L_ω = left Clifford multiplication with the volume form, ω).

This need only be checked for $\alpha = e_1 \wedge \cdots \wedge e_p$ for which it is clear that $L_\omega(\alpha) = \pm\tau(\alpha)$. The sigh is left to the reader. Claim 1 now follows from the associative law for Clifford algebras. So $R_{v_i} : \Omega^*_\pm \to \Omega^*_\pm$. Because $\{v_i\}$ are orthonormal, the relations defining $Cliff(\mathrm{V})_r$ (see (5.4)) imply

$$R_{v_i} \circ R_{v_j} = -R_{v_j} \circ R_{v_i} (i \neq j), R^2_{v_i} = -\mathrm{id}.$$

It was shown (c.f. (5.8)) that σ_{d+d^*} commutes with R_{v_i}. Let G be the group generated by $\{R_{v_i}\}$. The relations among the R_{v_i}'s imply G is finite. Define the operator

$$T = \frac{1}{|G|} \sum_{g \in G} g(d + d^*)g^{-1}$$

Since $R_{v_i} \in G$, T commutes with G. Also since σ_{d+d^*} and R_{v_i} commute $\sigma_T = \sigma_{d+d^*}$. Therefore by the stability of elliptic operators (3.4) Index(T)=Index(D_S). But for T, $ker\{T : \Omega_+ \to \Omega_-\}$ is a module over $G = \mathcal{C}_r$. Furthermore $\Omega_+ = \Omega_+^{\text{even}} \bigoplus \Omega_-^{\text{odd}}$ gives a $\mathbb{Z}_2$ grading to $ker(T)$. Similar comments apply to $coker(T) = ker\{\Omega_- \to \Omega_+\}$. Therefore by (7.2) $ker(T) - coker(T) =$ Index $T =$ Index $D_s =$ Signature($\mathbf{X}$) is divisible by $2a_r$. **q.e.d.**

8. Clifford Algebras

The reference for this section is [ABS]. Let V be a vector space with orthonormal basis $e_1, \cdots e_k$. Recall the definition of $\mathcal{C}_k$.

(8.1)

$\mathcal{C}_k$ is the free associative algebra generated by$\{e_i\}$ subject to therelations $e_ie_j + e_je_i = 0, i \neq j, e_i^2 = -1$

Definition 8.2. *$\mathcal{C}'_k$ is the free associative algebra generated by $\{e_i\}$ subject to the relations $e_ie_j + e_je_i = 0, i \neq j, e_i^2 = 1$*

$\mathcal{C}'_k$ is introduced in order to calculate $\mathcal{C}_k$. There are isomorphisms:

(8.3)

$$u : \mathcal{C}_{k+2} \to \mathcal{C}'_k \otimes \mathcal{C}_2 \quad v : \mathcal{C}'_{k+2} \to \mathcal{C}_k \otimes \mathcal{C}'_2$$

(8.4)

$$\mathcal{C}_k \otimes \mathbb{C} \approx \mathcal{C}'_k \otimes \mathbb{C}$$

(8.5)

$$\mathcal{C}_{k+4} \approx \mathcal{C}_k \otimes \mathcal{C}_4 \quad \mathcal{C}'_{k+4} \approx \mathcal{C}'_k \otimes \mathcal{C}'_4$$

(8.6)

$$\mathcal{C}_{k+8} \approx \mathcal{C}_k \otimes \mathbb{R}(16)$$

(We use the notation $F(n)$ for the algebra of $n \times n$ matrices over F, $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$[1] There are relations $F \otimes \mathbb{R}(n) = F(n)$, and $F \otimes (\mathbb{R} \bigoplus \mathbb{R}) = F \bigoplus F$.) All tensors in the following are over $\mathbb{R}$.

Proof : of (8.3). Let $e_1, \cdots, e_k$ be algebra generators for $\mathcal{C}_k$ and $e'_1, \cdots, e'_k$ algebra generators of $\mathcal{C}'_k$. Define $u : \mathbb{R}^{k+2} \to \mathcal{C}'_k \otimes \mathcal{C}_2$ by $u(e_1) = 1 \otimes e_1, \quad u(e_2) = 1 \otimes e_2, \quad u(e_i) = e'_{i-2} \otimes e_1e_2$ if $3 \leq i \leq k+2$. The u takes the defining relations to zero and passes a homomorphism

[1] $\mathbb{H}$ is generated by elements $1, i, j, k$ such that $i^2 = j^2 = k^2 = -1$, and

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

If $a = a_0 + a_1i + a_2j + a_3k, \overline{a} = a_0 - a_1i - a_2j - a_3k$

$\mathcal{C}_{k+2} \to \mathcal{C}'_k \otimes \mathcal{C}_2$. For dimension reasons u is an isomorphism. v is defined in a similar way.

Proof : of (8.4) $e_j \to ie'_j$ extends to an isomorphism.

Proof : of (8.5) $\mathcal{C}_4 \approx \mathcal{C}'_2 \otimes \mathcal{C}_2$ (by (8.3)) so $\mathcal{C}_{k+4} \approx \mathcal{C}'_{k+2} \otimes \mathcal{C}_2 \approx \mathcal{C}_k \otimes \mathcal{C}'_2 \otimes \mathcal{C}_2 \approx \mathcal{C}_k \otimes \mathcal{C}_4$ A similar argument calculates $\mathcal{C}'_{k+4}$.

To prove (8.6) we have to understand $\mathcal{C}_k$ for small k.

k=1 $\mathcal{C}_1$ is generated by 1 and e_1 with the relations $e_1^2 = -1$, i.e. $\mathcal{C}_1 = \mathbb{C}$.

k=2 $\mathcal{C}_2$ is generated by $1, e_1, e_2, e_1e_2$. By sending $1 \mapsto 1, e_1 \mapsto i, e_2 \mapsto j, e_1e_2 \mapsto k$ we obtain an isomorphism $\mathcal{C}_2 \approx \mathbb{H}$.

$\mathcal{C}'_2$ is generated by $1, e'_1, e'_2, e'_1e'_2$. By sending

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e'_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e'_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we obtain an isomorphism $\mathcal{C}'_2 \approx \mathbb{R}(2)$.

As a corollary of (8.4) and the case $k = 1$ we have $\mathbb{C} \otimes \mathbb{C} \approx \mathbb{C} \bigoplus \mathbb{C}$, and $\mathbb{H} \otimes \mathbb{C} \approx \mathbb{C} \otimes \mathbb{R}(2) = \mathbb{C}(2)$.

k=3

$$\mathcal{C}_3 \approx \mathcal{C}'_1 \otimes \mathcal{C}_2 = (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} = \mathbb{H} \oplus \mathbb{H}$$

$$\mathcal{C}'_3 \approx \mathcal{C}_1 \otimes \mathcal{C}'_2 = \mathbb{C} \otimes \mathbb{R}(2) = \mathbb{C}(2)$$

k=4

$$\mathcal{C}_4 \approx \mathcal{C}'_2 \otimes \mathcal{C}_2 = \mathbb{R}(2) \otimes \mathbb{H} = \mathbb{H}(2)$$

$$\mathcal{C}'_4 \approx \mathcal{C}_2 \otimes \mathcal{C}'_2 = \mathbb{H}(2).$$

To prove (8.6) we need

$$\mathbb{H} \otimes \mathbb{H} \approx \mathbb{R}(4).$$

To see this we think of $\mathbb{R}(4)$ as $L_{\mathbb{R}}(\mathbb{H})$, real, linear maps of $\mathbb{H}$ to $\mathbb{H}$. Define $w : \mathbb{H} \otimes \mathbb{H} \to L_{\mathbb{R}}(\mathbb{H})$ by $w(x_1 \otimes x_2)(x) = x_1 x \overline{x}_2$. By inspection w is onto. For dimension reasons w is an isomorphism.

Continuing with the proof of (8.6), we have $\mathcal{C}_{k+8} = \mathcal{C}_{k+4} \otimes \mathbb{H}(2) = \mathcal{C}_k \otimes \mathbb{H}(2) \otimes \mathbb{H}(2) = \mathcal{C}_k \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) = \mathcal{C}_k \otimes \mathbb{R}(16)$.

Claim 4 $\mathcal{C}_{k-1} = \mathcal{C}_k^0$

Proof : Map $x_0 \oplus x_1 \in \mathcal{C}_{k-1}^0 \bigoplus \mathcal{C}_{k-1}^1$ to $x_0 + e_k x_1 \in \mathcal{C}_k^0$

Putting all this together we have the following table.

(8.7)

k	1	2	3	4	5	6	7	8	9
$\mathcal{C}_k$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\bigoplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\bigoplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$,
$\mathbb{C}_k^0$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\bigoplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\bigoplus\mathbb{R}(8)$	$\mathbb{R}(16)$

$$\mathbb{C}_{k+8} = \mathbb{C}_k \otimes \mathbb{R}(16)$$

Now if M is a $\mathbb{Z}_2$-graded $\mathcal{C}_r$ module then M^0 is a $\mathbb{C}_r^0$ module and must be a direct sum of irreducible C_r^0 modules. These have dimensions divisible by a_r. (Notice that there are two different representations in dimensions 4 and 8). Since $M^0 \approx M^1$ (7.1) follows.

9. A Diversion: Constructing Vector Fields on Spheres using Clifford Algebras

In this section we show how Clifford algebras may be used to construct vector fields on spheres. We will make no use of previous sections, nor will it be used in subsequent sections. We simply include it for the readers amusement.

Write n as odd $2^{4d+c}, 0 \leq c < 4$.

Definition 9.1. *The Hurwicz-Radon number, $\rho(n)$ is defined by $\rho(n) = 2^c + 8d$*

It may be seen from table (7.2) that $\mathbb{R}^n$ is a $\mathcal{C}_{\rho(n)-1}$ module. i.e. there is an action $\mathbb{R}^n \times \mathcal{C}_{\rho(n)-1} \xrightarrow{\circ} \mathbb{R}^n$ and $\rho(n)$ is the largest such number.

Theorem 9.2. *There are $\rho(n) - 1$ linearly independent vector fields on S^{n-1}*

It is wonderful theorem of Frank Adams that there are not $\rho(n)$ independent vector fields on S^{n-1}.

Proof : We define a map

$$\mu : \mathbb{R}^{\rho(n)} \times \mathbb{R}^n \to \mathbb{R}^n$$

by

$$(e_i, r) \mapsto e_i \cdot r, \quad 1 \leq i < \rho(n)$$
$$(e_{\rho(n)}, r) \mapsto r$$

For $1 \leq i \leq \rho(n)$ let $u_i : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $u_i(r) = \mu(e_i, r)$. In particular $u_{\rho(n)}$ = the identity.

Notice we have an inner product on $\mathbb{R}^{\rho(n)}$ which was used to define $\mathcal{C}_{\rho(n)-1}$. We have not yet specified the inner product on $\mathbb{R}^n$. Let $\mathcal{U}$ be the group generated by $u_i \in GL(n)$. By the Clifford identities $\mathcal{U}$ is finite (in fact it has 2^n elements). For $\langle -, - \rangle'$ and metric on $\mathbb{R}^n$ we define

$$\langle x, y \rangle_n = \frac{1}{|\mathcal{U}|} \sum_{u \in \mathcal{U}} \langle u(x), u(y) \rangle'$$

with respect to this metric $u_i \in \mathcal{O}(n)$. For future reference we list the identities among the u_i which follow from the Clifford relations for

$i < \rho(n)$:

$$u_i^2 = -\text{id}\ , u_iu_j + u_ju_i = 0, \text{if } i \neq j.$$

We shall show that the above relations will imply a relation among the two norms. Specifically for $x \in S^{n-1}$

(9.3)

$$||\mu(y,x)||_n = ||y||_{\rho(n)}.$$

From this it will follow that for $x \in S^{n-1}, \quad y_1, y_2 \in \mathbb{R}^{\rho(n)}$ we have

$$\langle \mu(y_1,x), \mu(y_2,x)\rangle_{\mathbb{R}^n} = \langle y_1, y_2\rangle_{\mathbb{R}^{\rho(n)}}$$

Proof : We write $|| - ||_{\mathbb{R}^k}$ as $|| - ||_k$

(9.4) $\langle \mu(y_1,x), \mu(y_2,x)\rangle_n =$

$$\frac{1}{2}(||\mu(y_1,x) + \mu(y_2,x)||_n - ||\mu(y_1,x)||_n - ||\mu(y_2,x)||_n) =$$

$$\frac{1}{2}(||y_1 + y_2||_{\rho(n)} - ||y)1||_{\rho(n)} - ||y_2||_{\rho(n)}) = \langle y_1, y_2\rangle_{\rho(n)}$$

From (9.4) it follows that $\{u_i(x)\}_{i=1,\cdots\rho(n)}$ are mutually orthonormal. for $x \in S^{n-1}$. Since $u_{\rho(n)} =$ the identity we have proven (9.2) once we have proven (9.3).

Proof of (9.3). It suffices to show that $||\mu(u,x)||_n = 1$ for $x \in S^{n-1}, ||y||_{\rho(n)} = 1$. This is equivalent to proving that $\mu(y,-) \in O(n)$ for any $||y||_{\rho(n)} = 1$. i.e. we must show that

$$\mu(\sum_{i=1}^{\rho(n)} a_ie_i, x) = \sum_{i=1}^{\rho(n)} a_iu_i(x) = 1$$

for $\sum_{i=1}^{\rho(n)} a_i^2 = 1, \quad x \in S^{n-1}$. So we have to prove that $v = \Sigma a_iu_i \in O(n)$ for $\Sigma a_i^2 = 1$. We recall that a linear transformation, ω, belongs to $O(n)$ $\Leftrightarrow \omega\omega^* = 1$ where ω^* is the adjoint. In particular $\omega^* = \omega^{-1}$. For $\omega = u_i$ we have $u^* = u_i^{-1} = -u_i$. Now

(9.5) $\omega\omega^* = (\Sigma a_iu_i)(\Sigma(a_iu_i^*) = \Sigma a_i^2u_iu_i^* + \sum_{i<j} a_ia_j(u_iu_j^* + u_ju_i^*)$

Because $\Sigma a_i^2 = 1$ the first summation is 1. For $j = \rho(n)$ in the second summand $u_j = u_j^* =$ the identity and $u_iu_{\rho(n)}^* + u_{\rho(n)}u_i^* = u_i - u_i = 0$ for $i < \rho(n)$.

For $j < \rho(n)$

$$u_i u_j^* + u_j u_i^* = -(u_i u_j + u_j u_i) = 0.$$

So $\omega\omega^* = 1$ proving (9.2). **q.e.d.**

10. Topological Invariants of the Index. The Atiyah Singer Index Theorem

To discuss the topological part of the index theorem it is best to work in K-theory. For the purpose of these notes we only need the cohomology calculation of the index. Because of this we shall only outline the required K-theory. We also outline the needed facts about ordinary cohomology and Chern classes. Further references are: [A], [H].

Notation 10.1. *A complex vector bundle will be denoted by Greek letters, ξ, η etc. $Vect_n(X) = \{n-$ dimensional vector bundles over $X\}$ up to equivalence. Vect(X) = $\bigoplus_n Vect_n(X)$.*

Note: If X is not connected we allow different dimensions for the bundles over the various path components.

$Vect(X)$ is a semi group under Whitney sum.

For a semi group, G, $K(G)$ denotes the Grothendieck group of G. i.e. $G \times G/\sim$ where $(\alpha, \beta) \sim (\alpha', \beta')$ if $\exists c \in G$ such that $\alpha + \beta' + c = \alpha' + \beta + c$.

Definition 10.2. *For X compact $K(X) = K(Vect(X))$*

if $f : X \to Y$ is a continuous map then $K(f) : K(Y) \to K(X)$ is induced by the pullback of bundles.

Each element of $K(X)$ is of the form $[\xi] - [\eta]$ where $[-]$ denotes the class of a bundle in the Grothendieck group. $K(X)$ has a ring structure induced by the tensor product of bundles.

For a point, p, $K(p) = \mathbb{Z}$. If X has a base point, $*$, the map $* \hookrightarrow X$ induces a map $K(X) \xrightarrow{rank} \mathbb{Z}$.

Definition 10.3. $\widetilde{K}(X) =$ *kernel(rank)*

Let $\widetilde{Vect}(X)$ be $Vect(X)/\sim$ where $\xi \sim \eta$ if there are trivial bundles ϵ^a, ϵ^b such that $\xi \oplus \epsilon^a \approx \eta \oplus \epsilon^b$.

(10.4)

$$\widetilde{K}(X) = \widetilde{Vect}(X)$$

where $\xi \in \widetilde{Vect}(X)$ corresponds to $[\xi] - [\text{rank } \xi] \in \widetilde{K}(X)$.

Note: This identification does not behave well with respect to $\otimes$.

(10.5)

$$K(X, A) = \widetilde{K}(X/A)$$

Note: $X/\varnothing \underset{definition}{=} X \bigcup \{$ disjoint point$\})$.

For X locally compact X^+ denotes the one point compactification of X For X compact $X^+ = X/\varnothing$. Then $K(X) = \widetilde{K}(X^+)$.

(10.6) **The clutching construction**

Let $X = X_1 \bigcup X_2$, $A = X_1 \bigcap X_2$. We have to assume the triple (X_1, X_2, A) is reasonable. Assuming all spaces are C.W. complexes and subcomplexes will suffice.

Let η_i be a vector bundle over X_i and $\varphi : \eta_1|_A \to \eta_2|_A$ an isomorphism.

Definition 10.7. *The clutching of* η_1 *with* η_2 *via* φ , $\eta_1 \bigcup_\varphi \eta_2$, *is defined as* $\eta_1 \bigcup \eta_2 / \sim$ *where* $e_1 \in \eta_1|_A \sim \varphi(e_1) \in \eta_2|_A$.

$\eta_1 \bigcup_\varphi \eta_2$ is a vector bundle over X and $\eta_1 \bigcup_\varphi \eta_2|_{X_i} = \eta_i, i = 1, 2$. See ([H] or [A]) for details.

As an application of the clutching construction we describe the important difference construction.

Let η_1 and η_2 be vector bundles over X with $\varphi : \eta_1|_A \to \eta_2|_A$ an isomorphism. Let $Y = X_1 \bigcup_A X_2$ $(X_i = X)$. Then we have an exact sequence

$$K(X_1) \xleftarrow{i} K(Y) \xleftarrow{j} K(Y, X_1) \approx K(X_2, A) = K(X, A)$$

Since there is a folding map making the composite $X_1 \xrightarrow{i} Y \xrightarrow{\text{fold}} X_1$ the identity if follows that j is an injection (c.f. [A]). There is the element $\eta_1 \bigcup_\varphi \eta_2 - \text{fold}(\eta_1) \in K(Y)$. $i(\eta_1 \bigcup_\varphi \eta_2 - \text{fold}(\eta_1)) = 0$ so there is a unique element, $(\eta_1, \varphi, \eta_2) \in K(Y, X_1)$ mapping to $\eta_1 \bigcup_\varphi \eta_2 - \text{fold}(\eta_1)$ via j.

Definition 10.8. *The difference bundle is defined to be* $(\eta_1, \varphi, \eta_2) \in K(Y, X_1)$

Notice that the map $K(X, A) \to K(X)$ sends $(\eta_2, \varphi, \eta_1)$ to $[\eta_2] - [\eta_1]$.

Recall the definition of an elliptic operator, D (2.6). $\sigma_D : \pi^*(E) \to \pi^*(F)$ is an isomorphism away from the zero section. For any bundle, η let η_0 denote the non-zero vectors. Then the difference element, $(\pi^*(E), \sigma_D, \pi^*(F)$ is an element of $K(T^*(\mathbf{X}), T^*(\mathbf{X})_0)$.

Definition 10.9. *We shall denote this class by* $[\sigma_D]$.

The map $K(T^*(\mathbf{X}), T^*(\mathbf{X})_0) \to K(T^*(\mathbf{X}))$ sends $[\sigma_D]$ to $[\pi^*(E)] - [\pi^*(F)]$.

For a complex bundle, η, there are the *Chern classes*, $c_i(\eta)$ (see [MS]). The total Chern class is defined to be $c(\eta) = 1 + c_1(\eta) + \cdots + c_k(\eta)$ (η is a k plane bundle). We formally factor $c(\eta)$ as

$$c(\eta) = \prod_{i=1}^{k} (1 + w_1) \quad w_i \text{ is given degree } 2$$

The $w_i's$ do not have any meaning, however the elementary symmetric functions in the $w_i's$ do (as the Chern classes). Hence any polynomial in the elementary symmetric functions in the $w_i's$ has meaning. Conversely any symmetric polynomial in the $w_i's$ is a unique polynomial in the elementary symmetric functions (see [H]) therefore makes sense. The important polynomials for us are

$$Ch(\eta) = \Sigma e^{w_i} \tag{10.10}$$

$$\mathcal{T}(\eta) = \prod \frac{w_i}{1 - e^{-w_i}} \tag{10.11}$$

Ch is the Chern character, $\mathcal{T}$ is the Todd genus. Notice that Ch and $\mathcal{T}$ lie in $H^{**}(\mathbf{X}, \mathbb{Q})$ where $H^{**} = \prod H^*$.

Let $\eta^k \xrightarrow{\pi} B$ be a real, oriented, k dimensional real bundle over a compact base B. (We write η^k for the total space as well as for the bundle.) $\pi^* : H^*(B) \to H^*(\eta)$ is an isomorphism. There is a class $U \in H^k(\eta, \eta_0)$ such that there is a Thom isomorphism map, i.e.

$$\varphi(a) = \pi^*(a) \cup U \tag{10.12}$$

is an isomorphism ($\varphi : H^\ell(B;\mathbb{Q}) \to H^{\ell+k}(\eta,\eta_0;\mathbb{Q})$)

The Thom isomorphism is true with integral coefficients. Since we are only interested in rational cohomology it has been stated for cohomology with $\mathbb{Q}$ coefficients.

Let $\chi(\eta) \in H^k(B)$ be the Euler class of η (see [MS]) and $\iota : H^*(\eta,\eta_0) \to H^*(\eta)$ Then the fundamental relationship between $\chi(\eta)$ and the Thom class is

$$\pi^*(\chi(\eta)) = \iota(U). \tag{10.13}$$

Let D be an elliptic operator over $\mathbf{X}^{2\ell}$.

Definitions

•$Ch(D) = (-1)^\ell \varphi^{-1} ch([\sigma_D])$ (see 10.9).

•$\mathcal{T}(\mathbf{X}) = \mathcal{T}(T(\mathbf{X}) \otimes \mathbb{C})$

Definition 10.14. $Index_t(D) = \langle Ch(D)\mathcal{T}(X), [\mathbf{X}]\rangle$

where $\langle -,-\rangle$ is the Kronecker pairing and $[\mathbf{X}]$ is the fundamental class.

Theorem 10.15. *(Atiyah Singer Index Theorem)*

$$Index_t(D) = Index(D).$$

Note: $Index_t(D)$ can be described entirely in terms of $K-$theory. While we will not discuss this formulation here, it is quite important.

11. Borel Hirzebruch Theory and Characteristic Classes

A references for much of this material are [MS], [BH].

Let $G = SO(2\ell)$ or$U(\ell)$. There is the maximal torus $T^\ell \subset G$. For $G = U(\ell)$, T^ℓ corresponds to the diagonal matrices. For $G = SO(2\ell)$ T^ℓ corresponds to matrices with 2×2 blocks

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, x + iy \in S^1$$

on the diagonal. B_G and B_T denote the classifying spaces.

$\bullet B_{T^\ell} = \underbrace{CP^\infty \times \cdots CP^\infty}_{\ell}$

$\bullet H^*(B_{T^\ell}; \mathbb{Q}) \cong \mathbb{Q}[x_1, \cdots x_\ell], x_i \in H^2(B_{T^\ell}; \mathbb{Q})$

$T^\ell \hookrightarrow G$ induces $\rho : H^*(B_G; \mathbb{Q}) \to H^*(B_{T^\ell}\mathbb{Q})$

Theorem 11.1. *(Borel-Hirzebruch) ρ is an injection. The image is as follows:*

- *(a) $G = U(\ell)$ The image of ρ is the ring of symmetric polynomials over $\mathbb{Q}$. The element $\in H^{2i}(B_G; \mathbb{Q})$ corresponding to the $i-th$ elementary symmetric function is the universal Chern class, c_i. $1 + c_1 + \cdots + c_\ell = \prod(1 + x_i)$ via ρ*
- *(b) $G = SO(2\ell)$ The image of ρ is the ring of polynomials symmetric in x_i^2 and the element $x_1 \cdots x_\ell$. The element $\in H^{4i}(B_G\mathbb{Q})$ corresponding to the $i - th$ elementary symmetric function in the x_i^2's is the universal Pontrijagin class, $\wp_i$. $1 + \wp_1 + \cdots \wp_\ell = \prod(1 + x_i^2)$. The universal Euler class corresponds to $\prod x_i$*

In the sequel we shall identify $H^*(B_G; \mathbb{Q})$ with its image via ρ. A (complex) $G-$module is a (complex) vector space, M with an action of G on M.

Definition 11.2. *$\widetilde{M} = E_G \times_G M$ ($E_G \times_G M = E_G \times M/\sim$, where $(eg, g^{-1}m) \sim (e, m)$).*

We wish to understand the characteristic classes of $\widetilde{M}$. For this we introduce an isomorphism

$$\nu : Hom(T, S^1) \to H^2(B_T; \mathbb{Z})$$

where T is the torus and $S^1 = \mathbb{R}/\mathbb{Z}$. We identify S^1 with $U(1)$ via the map $s \mapsto e^{2\pi i s}$, $(s \in \mathbb{R}/\mathbb{Z})$. Then for $f : T \to S^1 = U(1)$we obtain a map $B_f : B_T \to B_{U(1)}$ which defines a line bundle, ξ_f over B_T. $\nu(f) = c_1(\xi_f)$, the first Chern class.

Case 1 Suppose M is a complex vector space of dimension n. By restriction M is a T module $(T \subset G)$. Then M is a direct sum of irreducible T modules of dimension 1

$$M = M_1 \bigoplus \cdots \bigoplus M_n$$

and T acts, for $m \in M_j$ by

$$t \cdot m = e^{2\pi i w_j(t)} m$$

for $w_j \in Hom(T, S^1)$. The w_i's are called the weights of M. Using ν we may think of the w_i's as elements of $H^2(B_T; \mathbb{Q})$. We then have the formula

$$1 + c_1(\widetilde{M}) + \cdots + c_n(\widetilde{M}) = \prod (1 + w_i)$$

[Here we re identifying $H^*(B_G)$ with its image via ρ). So $Ch(\widetilde{M}) = \Sigma e^{w_i}$.

The Pontrijagin classes of the underlying real bundle, $\widetilde{M}'$ are

$$1 + \wp_1(\widetilde{M}') + \cdots + \wp_n(\widetilde{M}') = \prod (1 + w_i^2)$$

Case 2 Suppose M is a real vector space of dimension $2n$. By restriction M is a T module. Hence M is a direct sum of $2-$dimensional T modules.

$$M = M_1 \bigoplus \cdots \bigoplus M_n \tag{11.3}$$

Give M a G invariant metric. Then with respect to an orthonormal basis of M_j

$$t \cdot m = \begin{pmatrix} cos 2\pi w_j(t) & -sin 2\pi w_j(t) \\ sin 2\pi w_j(t) & cos 2\pi w_j(t) \end{pmatrix} m$$

$m \in M_j, w_j \in Hom(T, S^1)$. Again we call w_j the weights of M which now depend on the orientation of M_j. The total Pontrijagin class is

$$1 + \wp_1(\widetilde{M}) + \cdots + \wp_n(\widetilde{M}) = \prod(1 + w_j^2)$$

If M is oriented and (11.3) agrees with the orientations then we have for the Euler class

$$\chi(\widetilde{M}) = \prod(w_i).$$

The weights of $M \otimes \mathbb{C}$ as a complex bundle are $\pm w_1, \cdots, \pm w_n$ and

$$1 + c_1(\widetilde{M \otimes \mathbb{C}}) + \cdots + c_n(\widetilde{M \otimes \mathbb{C}}) = \prod(1 + w_i)(1 - w_i) = \prod(1 - w_i^2).$$

So we have the formula

$$\wp_i(\widetilde{M}) = (-1)^i c_{2i}(\widetilde{M} \otimes \mathbb{C})$$

which in [MS] is taken as the definition of the $\wp_i$'s.

As a consequence if $\wp(T(\mathbf{X})) = \prod(1 + y_i^2)$ then

$$\mathcal{T}(\mathbf{X}) = \mathcal{T}(T(\mathbf{X}) \otimes \mathbb{C}) = \prod(\frac{y_i}{1 - e^{-y_i}})(\frac{-y_i}{1 - e^{y_i}}). \tag{11.4}$$

With these preliminaries behind us we are now ready to calculate $index_t(D)$. The secret is to pass to the universal case. For this we assume the Riemannian structure on $\mathbf{X}^{2\ell}$ is specified as follows:

- There is an fixed oriented, real $SO(2\ell)$ module ,V.
- We have a principle $SO(2\ell)$ bundle. P over $\mathbf{X}$.
- There is an orientation preserving isomorphisms $\chi : P \times_G V \cong T(\mathbf{X})$.

12. Verification of the Index Theorem for D_χ

Convention 12.1. *The methods of this and the following sections apply to bundles, $\eta \to \mathbf{X}$ which are of the form $P \times_G M$ for some G-module $M \quad (G = SO(2\ell)), P$ a principle G bundle over $\mathbf{X}$.*

For example:

- $\Lambda^*(\mathbf{X}) = P \times_G \Lambda^*(V)$
- $\Lambda^*_\pm(\mathbf{X}) = P \times_G \Lambda^*_\pm(V)$

Let $f : \mathbf{X} \to B_G$ classify $P \to \mathbf{X}$. Then $T(\mathbf{X}) = f^*(\widetilde{V})$ (see (11.2) for definition of $\widetilde{V}$). Recall the diagram involving the symbol σ_D

(12.2)

$$\begin{array}{ccccccc} \pi^*(E) & \xrightarrow{\sigma_D} & \pi^*(F) & & E & & F \\ & \searrow & \swarrow & & & \searrow \; \swarrow & \\ & T^*(\mathbf{X}) & & \xrightarrow{\pi} & & \mathbf{X} & \end{array}$$

This gave us an element $[\sigma_D] \in K(T^*(\mathbf{X}), T^*(\mathbf{X})_0)$,(10.9).

Let $\overline{E}$ and $\overline{F}$ be $SO(2\ell)$ modules defining E and F as described in (12.1). $\widetilde{E}, \widetilde{F}$ the vector bundles over B_G (as in (11.2)).

We have the diagram

(12.3)

$$\begin{array}{ccccccc} \widetilde{\pi}^*(\widetilde{E}) & \xrightarrow{\sigma_D} & \widetilde{\pi}^*(\widetilde{F}) & & \widetilde{E} & & \widetilde{F} \\ & \searrow & \swarrow & & & \searrow \; \swarrow & \\ & \widetilde{V}^* & & \xrightarrow{\widetilde{\pi}} & & B_G & \end{array}$$

We assume there is an isomorphism

$$\widetilde{\sigma}_D : \widetilde{\pi}^*(\widetilde{E})|_{\widetilde{V}_0^*} \to \widetilde{\pi}^*(\widetilde{F})|_{\widetilde{V}_0^*}$$

so that the resulting diagram pulls back via f to diagram (12.2). This can be done if $\sigma : V_0^* \to Iso(\overline{E}, \overline{F})$ is $G-$ equivariant as in our case where $\sigma(v) = iL_v$. The difference construction gives an element $\gamma \in K(\widetilde{V}^*, \widetilde{V}_0^*)$ with $f^*(\gamma) = [\sigma_D]$. (Here $f : (T^*(\mathbf{X}), T^*(\mathbf{X})_0) \to (\widetilde{V}^*, \widetilde{V}_0^*)$). The effect of all this is to pull all the data back to the universal situation over $BSO(2\ell)$. We fist calculate $Ch(D)$ (see (10.14)).

Theorem 12.4. *Suppose $\chi(\widetilde{V}^*) \in H^*(B_G, \mathbb{Q})$ is not zero, then*

$$Ch(D) = f^{**}((-1)^\ell \frac{Ch(\widetilde{M}) - Ch(\widetilde{N})}{\chi(\widetilde{V}^*)}) = f^{**}(\frac{Ch(\widetilde{M}) - Ch(\widetilde{N})}{\prod(-w_j)})$$

Here $\mathbf{X}^{2\ell}$ is a manifold with classifying map f and $w_1, \cdots, w_\ell$ are the weights of V. Note the difference between $Ch(D)$, the character of an operator and $Ch(\widetilde{M})$ or $Ch(\widetilde{N})$, the character of a complex vector bundle.

First, by the Borel Hirzebruch theorem (11.1) $H^*(B_G;\mathbb{Q})$ has no zero divisors so the quotient is well defined. Furthermore the last equality follows from section §11.

By the universality of Ch and the Thom isomorphism it suffices to prove the theorem in $H^{**}(B_G;\mathbb{Z})$. That is to show that

$$\chi(\widetilde{V}^*) \cup \varphi^{-1} Ch(\gamma) = Ch(\widetilde{M}) - Ch(\widetilde{N}).$$

Let $i : (\widetilde{V}^*, \phi) \hookrightarrow (\widetilde{V}^*, \widetilde{V}_0^*)$ be the inclusion. We look at the diagram:

$$\begin{array}{ccc} K(\widetilde{V}^*, \widetilde{V}_0^*) & \xrightarrow{i^*} & K(\widetilde{V}^*, \phi) \\ \downarrow{\scriptstyle Ch} & & \downarrow{\scriptstyle Ch} \\ H^{**}(\widetilde{V}^*, \widetilde{V}_0^*) & \xrightarrow{i^{**}} & H^{**}(\widetilde{V}^*, \phi) \\ \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \pi^{**}} \\ H^{**}(B_G;\mathbb{Q}) & \xrightarrow{\cup\chi(\widetilde{V}^*)} & H^{**}(B_G;\mathbb{Q}) \end{array}$$

φ^* and π^{**} are isomorphisms. The upper square commutes by the naturality of Ch. The bottom square commutes by (10.13).

The commutativity of the diagram yields

$$\varphi^{-1}(Ch(\gamma)) \cup \chi(\widetilde{V}^*) = (\pi^{**})^{-1} Ch(i\gamma)$$

But by the remark after (10.8) $i(\gamma) = \pi^{**}(\widetilde{E}) - \pi^{**}(\widetilde{F})$ and (12.4) follows. **q.e.d.**

We now have to calculate $Ch(\widetilde{E}) - Ch(\widetilde{F})$ for $\widetilde{E} = \Lambda^{\text{even}}(\widetilde{V}), \widetilde{F} = \Lambda^{\text{odd}}(\widetilde{V})$.

Proposition 12.5. *Let M be a complex G bundle with weights $w_1, \cdots, w_n$. then*

$$\sum_{i=1}^{n} (-1)^i Ch(\Lambda^i(\widetilde{M})) = \prod (1 - e^{w_i})$$

In our example $M = V \otimes \mathbb{C}$. If the Pontrijagin classes of $\widehat{V}$ are $\prod(1 + y_i^2)$ then the complex weights of $\widetilde{V} \otimes \mathbb{C}$ are $\pm y_i$. So (12.5) implies

Theorem 12.6. *Let $\mathbf{X}^{2n}$ be an oriented, Riemannian manifold, D_χ the Euler class operator then $index_t(D_\chi) = \chi(\mathbf{X})[\mathbf{X}]$ ($\chi(\mathbf{X})$ is the Euler class of $T(\mathbf{X})$) and the index theorem is true for D_χ.*

Proof : By (12.5) $Ch(\Lambda^{\text{even}}(\widetilde{V} \otimes \mathbb{C})) - Ch(\Lambda^{\text{odd}}(\widetilde{V} \otimes \mathbb{C})) = \prod(1 - e^{y_i})(1 - e^{-y_i})$. Hence $Ch(D) = \prod \frac{(1-e^{y_i})(1-e^{-y_i})}{-y_i}$, see (12.4). Here the y_i are the elements in the formal factorization of $\wp(T(\mathbf{X})) = \prod(1+y_i^2)$. Since $\mathcal{T}(\mathbf{X}) = \prod \frac{y_i(-y_i)}{(1-e^{y_i})(1-e^{-y_i})}$ we have $\langle Ch(D)\mathcal{T}(\mathbf{x}), [\mathbf{X}]\rangle = \langle \prod y_i, [\mathbf{X}]\rangle = \chi(\mathbf{X})$ and (12.6) follows. **q.e.d.**

Proof : of (12.5) Let $e_1, \cdots, e_n$ be a basis for M such that $t \cdot e_j = e^{2\pi i w_j(t)} e_j$ (see section 11). Then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} | i_1 < \cdots < i_k \leq n\}$$

is a basis for $\Lambda^k(M)$ and

$$t \cdot e_{i_1} \wedge \cdots \wedge e_{i_k} = e^{2\pi i(w_{i_1}+\cdots+w_{i_k})} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

So the basis of elements $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ can be used to describe the weights of $\Lambda(M)$. That is to say the weights of $\Lambda^k(M)$ are $\{w_{i_1} + \cdots + w_{i_k} | i_1 < \cdots < i_k \leq n\}$. Therefore

$$\sum_{j=o}^{n} (-1)^j Ch(\Lambda^j(M)) = \sum_{j=o}^{n} \Big[\sum_{i_1 < \cdots < i_j \leq n} (-1)^j e^{w_{i_1}+\cdots+w_{i_j}} \Big] = \prod (1 - e^{w_i})$$

q.e.d.

13. The Index of D_S is the Hirzebruch L-genus

The reader is referred to §6 for the definitions of $D_S, *, \tau$ and $\Omega_\pm(\mathbf{X})$.

Proposition 13.1. *Let V be an oriented $2-$dimensional real vector bundle. Let y be the Euler class of V (so the Pontrijagin class $\wp(V) = y^2$) then*

$$Ch(\Lambda_+(V \otimes \mathbb{C})) - Ch(\Lambda_-(V \otimes \mathbb{C})) = e^{-y} - e^y.$$

Proof : : By naturality we may assume V is the universal oriented $\mathbb{R}^2$ bundle over $BSO(2)$. Then the Euler class of V may be described as follows:

Let $y \in Hom(T^1, S^1)$ $(T^1 = SO(2))$ be the weight which sends

$$A = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$$

to $\theta \in S^1$. Where A is an arbitrary matrix in $SO(2)$. Then the bundle ξ_y defined by y is V and the element identified with $y \in H^2(BSO(2); \mathbb{Z})$ is the Euler class (i.e. the first Chern Class of the canonical line bundle over $BSO(2)$) of V. Let e_1, e_2 be an orthonormal basis of $\mathbb{R}^2$. Then

$$*e_1 = e_2, *e_2 = -e_1, *1 = e_1 \wedge e_2, *(e_1 \wedge e_2) = 1.$$

Since the dimensions of V is 2 we have

$$\tau(\alpha) = (-1)^{\frac{k(k-1)}{2} + \frac{1}{2}} * (\alpha)$$

for $\alpha \in \Lambda^k, k = 0, 1, 2$. Thus a basis for Λ_+ is

$$\{1 + i(e_1 \wedge e_2), e_1 + ie_2\}$$

and a basis for Λ_- is

$$\{1 - i(e_1 \wedge e_2), e_1 - ie_2\}$$

Now the weights of $\widetilde{V} \otimes \mathbb{C}$ are defined by the formula $A \cdot v = e^{2\pi i w_j(\tau)} \cdot v$. So for A as above we have

$$A \cdot (1 + ie_1e_2) = 1 + i(\cos 2\pi\theta e_1 + \sin 2\pi\theta e_2) \wedge (-\sin 2\pi\theta e_1 + \cos 2\pi\theta e_2) = 1 + i(e_1 \wedge e_2).$$

So the weight $= 0$ for this generator.

A similar calculation shows that

$$A \cdot (e_1 \pm ie_2) = e^{\mp 2i\pi\theta}(e_1 \pm ie_2).$$

Now $\Lambda_+(V \otimes \mathbb{C})$ is the direct sum, $[1+ie_1\wedge e_2]\bigoplus[e_1+ie_2]$. (Here $[-]$ denotes the 1-dimensional vector space spanned by.) So the weights of Λ_+ are 0 on the first factor and $-y$ on the second. Hence $Ch(\Lambda_+) = 1+e^{-y}$. Similarly $Ch(\Lambda_-) = 1+e^{y}$ and (13.1) follows. **q.e.d.**

For V and W even dimensional real vector spaces, let $\alpha \in \Lambda^k(V), \beta \in \Lambda^\ell(W)$. The the definition of $*$ implies

$$*_{V\bigoplus W}(\alpha\wedge\beta) = (-1)^{k\ell} *_V(\alpha)\wedge *_W(\beta).$$

it follows that

$$\tau_{V\bigoplus W}(\alpha\wedge\beta) = \tau_V(\alpha)\wedge\tau_W(\beta).$$

From this is clear that

(13.2)

$$\Lambda_+(V\bigoplus W) = (\Lambda_+(V)\otimes\Lambda_+(W))\bigoplus(\Lambda_-(V)\otimes\Lambda_-(W))$$
$$\Lambda_-(V\bigoplus W) = (\Lambda_+(V)\otimes\Lambda_-(W))\bigoplus(\Lambda_-(V)\otimes\Lambda_+(W))$$

By restricting to $T\subset SO(2\ell)$ we may assume V is a direct sum of 2 dimensional bundles V_i. The multiplicative properties of the Chern character show that

$$Ch(\Lambda_+(\widetilde{V}) - \Lambda_-(\widetilde{V})) = \prod Ch(\Lambda_+(\widetilde{V_i}) - \Lambda_-(\widetilde{V_i}))$$

(Here we are using (13.2.) Now we use (13.1) to prove

Theorem 13.3. *Let V be a real oriented $SO(2n)$ module of dimension $2n$ Let $y_1,\cdots y_n$ be the weights of V, then*

$$Ch(\Lambda_+(\widetilde{V}) - Ch(\Lambda_-(\widetilde{V})) = \prod \frac{y_i}{tanh(y_i)}[\mathbf{X}]$$

Corollary 13.4. $Index_t(D_S) = \prod \frac{y_i}{tanh(y_i)}[\mathbf{X}]$ *In particular the Atiyah-Singer Index theorem implies the Hirzebruch L-genus theorem.*

$$Sign(\mathbf{X}) = \prod \frac{y_i}{tanh(y_i)}[\mathbf{X}]$$

Proof : :

$$Ch(D_S)\mathcal{T}(\mathbf{X}) = \prod \frac{e^{-y_i}-e^{y_i}}{-y_i}\cdot\frac{y_i}{1-e^{-y_i}}\cdot\frac{-y_i}{1-e^{y_i}}$$

$$= \prod y_i \left(\frac{e^{-y_i} - e^{y_i}}{(1 - e^{y_i})(1 - e^{-y_i})} \right)$$

$$= \prod y_i \cdot \frac{cosh(y_i/2)}{sinh(y_i/2)} = \prod \frac{y_i}{tanh(y_i/2)}$$

If we replace y_i with $2y_i$, then the term in dimension $2k$ is multiplied by 2^k. Therefore if the dimension of $\mathbf{X}$ is $2n$ we have

$$2^n index_t(D_S) = \prod \frac{2y_i}{tanh(y_i)}[\mathbf{X}]$$

Cancelling the factor of 2^n gives the result. **q.e.d.**

14. The Hirzebruch Riemann Roch Theorem

Notation 14.1. Notation and Preliminaries

Let $\mathbf{X}$ be a complex analytic manifold of real dimension $2n$. Let $z_1, \cdots, z_n$ be holomorphic coordinates, $z_i = x_i + iy_i$. Then the collection $\{z_i, \overline{z_i}\}$ form a local coordinate system for $\mathbf{X}$ as a real manifold.

The bundle $\xi = \Lambda(T^*(\mathbf{X})) \otimes \mathbb{C}$ has a direct sum decomposition $\xi = \bigoplus_{p,q} \xi^{p,q}$ where locally $\xi^{p,q}$ looks like

$$\Sigma a_{i_1,\cdots,i_p,j_1,\cdots j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$$

($a_{I,J}$ are C^∞ functions). $\overline{\partial} : \Gamma(\xi)^{p,q} \to \Gamma(\xi)^{p,q+1}$ is defined as $\overline{\partial}(f) = \Sigma \frac{\partial h}{\partial \overline{z}_i} d\overline{z}_i \wedge w$ where $f = hw, h \in C^\infty(M, \mathbb{C}), w = dz_{i_1} \wedge \cdots \wedge d\overline{z}_{j_1} \wedge \cdots$.

let η be a holomorphic vector bundle over $\mathbf{X}$. $\xi^{p,q} \otimes \eta$ is called the *bundle of differential forms of type* (p,q) *with coefficients in* η.

Proposition 14.2. *There is a unique* $\overline{\partial}; \Gamma(\xi \otimes \eta) \to \Gamma(\xi \otimes \eta)$ *such that if* $\mathcal{O}$ *is an open neighborhood in* $\mathbf{X}$, $f \in \Gamma(\xi|_{\mathcal{O}})$, *and* $g \in \Gamma(\eta|_{\mathcal{O}})$ *is holomorphic, then* $\overline{\partial}(f \otimes g) = \overline{\partial}(f) \otimes g$

Sketch of proof: (For details see [P] page 79.) The Cauchy Riemann equations imply $\frac{\partial h}{\partial \overline{z}} = 0 \Longleftrightarrow h$ is holomorphic. One extends $\overline{\partial}$ by the product law. Since the transition functions for η are holomorphic, two extensions agree on intersections of coordinate patches. This allows us to define a global extension. **q.e.d.**

Remark 14.3. *If* $\mathbf{X}$ *has a Riemannian structure,* η *a Hermitian structure then* $\xi \otimes \eta$ *has a canonical Hermitian structure. There is a formal adjoint of the operator* $\overline{\partial}$ *which we denote by* $\mathcal{V}$

The definition of the Hermitian structure on $\xi^{p,q} \otimes \eta$, the construction of the adjoint and much of what follows are analogous to the constructions in §2 - §4. We leave the proof of the following to the reader

Proposition 14.4. $\overline{\partial} + \mathcal{V}$ *is elliptic.*

The proof is similar to the proof that $d + d^*$ is elliptic.

Let $\Box = \mathcal{V}\overline{\partial} + \overline{\partial}\mathcal{V}$ (this is analogous to the definition of $\triangle$.) Then

$$\Box(\varphi) = 0 \Leftrightarrow (\overline{\partial} + \mathcal{V})(\varphi) = 0 \Leftrightarrow \overline{\partial}(\varphi) = \mathcal{V}(\varphi) = 0. \tag{14.5}$$

Analogous to the DeRham complex we have

Definition 14.6. *The Dolbeault complex is the chain complex*

$$\xrightarrow{\overline{\partial}} \eta \otimes \xi^{0,q} \xrightarrow{\overline{\partial}} \eta \otimes \xi^{0,q+1} \xrightarrow{\overline{\partial}}$$

The analogue of the DeRham theorem is

Theorem 14.7. *(Dolbeault's theorem) The cohomology of the Dolbeault complex is isomorphic to the sheaf theoretic cohomology group* $H^*(\mathbf{X}; \Omega(\eta))$.

Definition 14.8. $\phi \in \Gamma(\xi^{0,q} \otimes \eta)$ *is harmonic if* $\Box\phi = 0$.

By (14.5) a harmonic form $\in ker\overline{\partial} \Rightarrow \phi$ represents an element of $H^*(\mathbf{X}; \Omega(\eta))$. The complex analogue of Höge's theorem is the converse.

Theorem 14.9. *Every element of* $H^*(\mathbf{X}; \Omega(\eta))$ *has a unique harmonic representative.*

Definition 14.10.

$$\xi^e = \bigoplus_q \xi^{0,2q} \otimes \eta$$

$$\xi^o = \bigoplus_q \xi^{0,2q+1} \otimes \eta$$

We have an operator $D_\eta = \overline{\partial} + \mathcal{V} : \Gamma(\xi^e) \to \Gamma(\xi^o)$. Analogous to theorem (5.2) we have

Theorem 14.11. $Index(D_\eta) = \Sigma(-1)^i dim H^*(\mathbf{X}; \Omega(\eta)) = \chi(\mathbf{X}; \Omega(\eta))$, *the Euler characteristic of* $\mathbf{X}$ *with coefficients in* $\Omega(\eta)$.

We now calculate $Index_t(D_\eta)$. Let $G = U(n)$, V a G-module of (complex) dimension n. Let $P \to \mathbf{X}$ be a principle $G-$bundle associated to $T(\mathbf{X})$. Then $T(X) = P \times_G V$ as a complex bundle.

Let $M' = \underset{k \equiv 0 \text{ mod } 2}{\Sigma} \Lambda^k(\overline{V}^*)$, $N' = \underset{k \equiv 1 \text{ mod } 2}{\Sigma} \Lambda^k(\overline{V}^*)$

Remark 14.12. $\bigoplus_q \xi^{0.2q} = P \times_G M'$ *since* $\xi^{0.2q}$ *involves the conjugate basis* $\{d\overline{z}_i\}$. *Similarly for* $\bigoplus_q \xi^{0,2q+1}$

Suppose η is a Hermitian complex vector bundle over $\mathbf{X}$ of complex dimension m. Let P' be a principal $U(m)$ bundle over $\mathbf{X}$ and W a $U(m)$ module such that $\eta = P' \times_{U(m)} W$. This amounts to choosing the Hermitian structure on η. P' and W are constructed analogously to P

and V. Let $M = M' \otimes W$, $N = N' \otimes W$. Write $\mathcal{T}(\mathbf{X})$ for the Todd class of the complex vector bundle $T(X)$ and $\mathcal{T}(X_r)$ for the Todd class of $T(\mathbf{X})_r \otimes \mathbb{C}$. ($T(X)_r$ is the underlying real bundle of $T(\mathbf{X})$.)

Theorem 14.13. ***(Hirzebruch Riemann Roch Theorem)***

$$Index_t(D_\eta) = \langle Ch(\eta)\mathcal{T}(\mathbf{X}), [\mathbf{X}]\rangle$$

Remark 14.14. *The pre-Atiyah Singer index version of the Hirzebruch Riemann Roch theorem was know only for* $\mathbf{X}$ *a projective algebraic variety.*

Proof : (of (14.13)) Let $f : \mathbf{X} \to BU(n) \times BU(m)$ classifying $P \times P' \to \mathbf{X}$. ($P \times P' \to \mathbf{X}$. is the pullback of $P \times P'$ via $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$.)

Let V_r be the underlying real bundle associated to the complex bundle V. Let the weights of V be $x_1, \cdots x_n$. Then the weights of the real module, V_r are also $x_1, \cdots x_n$. So $C(V) = \prod(1+x_i)$, $\wp(V_r) = \prod(1+x_i^2)$ and $\chi(V_r) = \prod x_i$.

V has a Hermitian inner product, $\langle -, - \rangle$ compatible with the action of $U(n)$. As a consequence $x \to \langle -, x\rangle$ sets up a $U(n)$ isomorphism $\overline{V} \simeq V^*$. Thus $\overline{V}^* \simeq \overline{\overline{V}} \simeq V$ as $U(n)$ modules. Now apply (12.5) and we get:

$$Ch(\widetilde{M}) - Ch(\widetilde{N}) = Ch(\widetilde{W})\prod_{i=1}^{n}(1 - e^{x_i})$$

From (12.4) we have

$$\begin{aligned} Ch(D_\eta) &= f^{**}\left(Ch(\widetilde{W})\prod_{i=1}^{n}\frac{(1-e^{x_i})}{-x_i}\right) \\ &= f^{**}Ch(\widetilde{W}) \cdot f^{**}\left(\prod \frac{(1-e^{x_i})}{-x_i}\right) \\ &= Ch(\eta) f^{**}\left(\prod \frac{(1-e^{x_i})}{-x_i}\right). \end{aligned}$$

Let $\mathbf{X}_r$ be the underlying real C^∞ manifold of $\mathbf{X}$. $T(\mathbf{X}_r) = f^*(\widetilde{V}_r)$ (Note $\widetilde{V}_r = E_{U(n)} \times_G V_r$ and f is applied first factor.)

Since the weights of V_r are $x_1, \cdots, x_n$ we have by (11.4)

$$\mathcal{T}(\mathbf{X}_r) = f^{**}\prod\left(\frac{x_i}{1-e^{-x_i}}\frac{-x_i}{1-e^{x_i}}\right)$$

and

$$\langle Ch(D_\eta)\mathcal{T}(\mathbf{x}_r),[\mathbf{X}]\rangle = Ch(\eta)f^{**}\left[\prod\left(\frac{1-e^{x_i}}{-x_i}\right)\left(\frac{x_i}{1-e^{-x_i}}\frac{-x_i}{1-e^{x_i}}\right)\right][\mathbf{X}]$$

$$= Ch(\eta)f^{**}\prod\left(\frac{x_i}{1-e^{-x_i}}\right)[\mathbf{X}] = Ch(\eta)\mathcal{T}(\mathbf{X})[\mathbf{X}].$$

q.e.d.

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