a rigorous proof of this last assertion.

7 Proof of Theorem 5.1

Let S be a compact surface. We shall demonstrate Theorem 5.1 by proving that S is homeomorphic to a polygon with the edges identified in pairs as indicated by one of the symbols listed at the end of Section 5.

First step. From the discussion in the preceding section, we may assume that S is triangulated. Denote the number of triangles by n. We assert that we can number the triangles $T_1, T_2, \ldots, T_n$, so that the triangle T_i has an edge e_i in common with at least one of the triangles $T_1, \ldots, T_{i-1}$, $2 \leqq i \leqq n$. To prove this assertion, label any of the triangles T_1; for T_2 choose any triangle that has an edge in common with T_1, for T_3 choose any triangle that has an edge in common with T_1 or T_2, etc. If at any stage we could not continue this process, then we would have two sets of triangles $\{T_1, \ldots, T_k\}$, and $\{T_{k+1}, \ldots, T_n\}$ such that no triangle in the first set would have an edge or vertex in common with any triangle of the second set. But this would give a partition of S into two disjoint nonempty closed sets, contrary to the assumption that S was connected.

We now use this ordering of the triangles, $T_1, T_2, \ldots, T_n$, together with the choice of edges $e_2, e_3, \ldots, e_n$, to construct a "model" of the surface S in the Euclidean plane; this model will be a polygon whose sides are to be identified in pairs. Recall that for each triangle T_i there exists an ordinary Euclidean triangle T'_i in $\mathbf{R}^2$ and a homeomorphism φ_i of T'_i onto T_i. We can assume that the triangles $T'_1, T'_2, \ldots, T'_n$ are pairwise disjoint; if they are not, we can translate some of them to various other parts of the plane $\mathbf{R}^2$. Let

$$T' = \bigcup_{i=1}^{n} T'_i;$$

then T' is a compact subset of $\mathbf{R}^2$. Define a map $\varphi : T' \to S$ by $\varphi \mid T'_i = \varphi_i$; the map φ is obviously continuous and onto. Because T' is compact and S is a Hausdorff space, φ is a closed map, and hence S has the quotient topology determined by φ (see Section 1 of Appendix A). This is a rigorous mathematical statement of our intuitive idea that S is obtained by gluing the triangles $T_1, T_2, \ldots$ together along the appropriate edges.

The polygon we desire will be constructed as a quotient space of T'. Consider any of the edges e_i, $2 \leqq i \leqq n$. By assumption, e_i is an edge of the triangle T_i and one other triangle T_j, for which $1 \leqq j < i$. Therefore, $\varphi^{-1}(e_i)$ consists of an edge of the triangle T'_i and an edge of the triangle T'_j. We identify these two edges of the triangles T'_i and T'_j by identifying points which map onto the same point of e_i (speaking intuitively, we glue together the triangles T'_i and T'_j). We make these identifications for each of the edges $e_2, e_3, \ldots, e_n$. Let D denote the resulting quotient space of T'. It is clear that the map $\varphi : T' \to S$ induces a map ψ of D onto S, and that S has the quotient topology induced by ψ (because D is compact and S is Hausdorff, ψ is a closed map).

We now assert that topologically D is a closed disc. The proof depends on two facts:

(a) Let E_1 and E_2 be disjoint spaces, which topologically are closed discs (i.e., they are homeomorphic to E^2). Let A_1 and A_2 be subsets of the boundary of E_1 and E_2, respectively, which are homeomorphic to the closed interval [0, 1], and let $h : A_1 \to A_2$ be a definite homeomorphism. Form a quotient space of $E_1 \cup E_2$

by identifying points that correspond under h. Then, topologically, the quotient space is also a closed disc. The reader may either take this very plausible fact for granted, or construct a proof using the type of argument given in II.4. Intuitively, it means that if we glue two discs together along a common segment of their boundaries, the result is again a disc.

(b) In forming the quotient space D of T', we may either make all the identifications at once, or make the identifications corresponding to e_2, then those corresponding to e_3, etc., in succession. This is a consequence of Lemma 2.4 of Appendix A [see application (a) of this lemma].

We now use these facts to prove that D is a disc as follows. T_1' and T_2' are topologically equivalent to discs. Therefore, the quotient space of $T_1' \cup T_2'$ obtained by identifying points of $\varphi^{-1}(e_2)$ is again a disc by (a). Form a quotient space of this disc and T_3' by making the identifications corresponding to the edge e_3, etc.

It is clear that S is obtained from D by identifying certain paired edges on the boundary of D.

Example

7.1 Figure 1.15 shows an easily visualized example. The surface of a cube has been triangulated by dividing each face by a diagonal into two triangles.

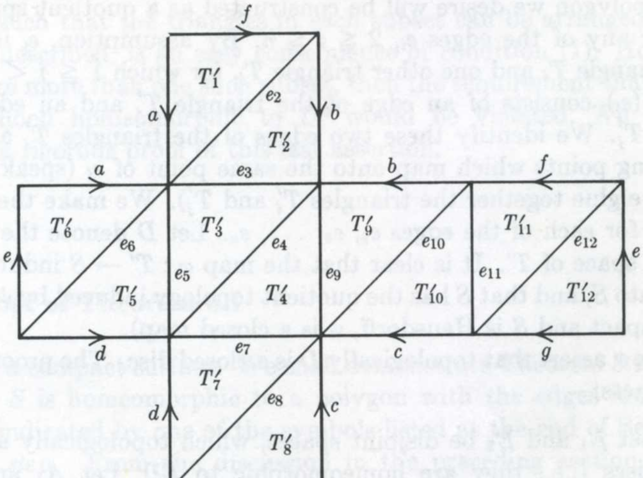


FIGURE 1.15 Example illustrating the first step of the proof of Theorem 5.1.

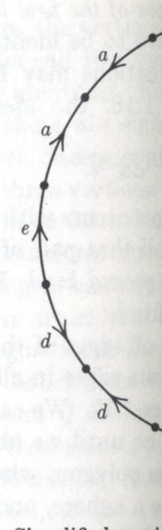


FIGURE 1.16 Simplified versi

The resulting disc D might look
the triangles were enumerated, a
edges of D that are to be identifie
can forget about the edges e_2, e_3,
1.15, we could work equally well

Exercises

Carry out the above process f
given below. (NOTE: these exam

7.1	124	236	134	2
	367	347	469	4
	698	678	457	2
	289	578	358	1
	238	135		
7.2	123	234	341	4
7.3	123	234	345	4
	136	246	356	4
7.4	124	235	346	4
	713	134	245	3
	126	237		
7.5	123	256	341	4
	156	268	357	4
	167	275	374	4
	172	283	385	4

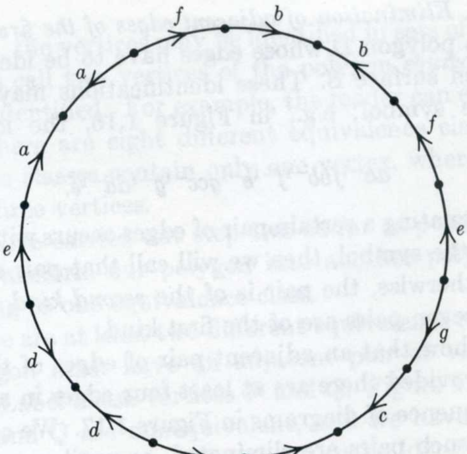


FIGURE 1.16 Simplified version of polygon shown in Figure 1.15.

The resulting disc D might look like the diagram, depending, of course, on how the triangles were enumerated, and how the edges $e_2, \ldots, e_{12}$ were chosen. The edges of D that are to be identified are labeled in the usual way. At this stage, we can forget about the edges $e_2, e_3, \ldots, e_{12}$. Thus, instead of the polygon in Figure 1.15, we could work equally well with the one in Figure 1.16.

Exercises

Carry out the above process for each of the surfaces whose triangulations are given below. (NOTE: these examples will be used later.)

7.1	124	236	134	246		
	367	347	469	459		
	698	678	457	259		
	289	578	358	125		
	238	135				
7.2	123	234	341	412		
7.3	123	234	345	451	512	
	136	246	356	416	526	
7.4	124	235	346	457	561	672
	713	134	245	356	467	571
	126	237				
7.5	123	256	341	451		
	156	268	357	468		
	167	275	374	476		
	172	283	385	485		

Second step. Elimination of adjacent edges of the first kind. We hav now obtained a polygon D whose edges have to be identified in pairs t obtain the given surface S. These identifications may be indicated b the appropriate symbol; e.g., in Figure 1.16, the identifications a described by

$$aa^{-1}fbb^{-1}f^{-1}e^{-1}gcc^{-1}g^{-1}dd^{-1}e.$$

If the letter designating a certain pair of edges occurs with *both* exponent $+1$ and -1, in the symbol, then we will call that pair of edges a pair the *first kind;* otherwise, the pair is of the *second kind*. For example, i Figure 1.16, all seven pairs are of the first kind.

We wish to show that an adjacent pair of edges of the first kind ca be eliminated, provided there are at least four edges in all. This is easil seen from the sequence of diagrams in Figure 1.17. We can continue th process until all such pairs are eliminated, or until we obtain a polygo with only two sides. In the latter case, this polygon, whose symbol wi be aa or aa^{-1}, must be a projective plane or a sphere, and we have com pleted the proof. Otherwise, we proceed as follows.

Third step. Transformation to a polygon such that all vertices must b identified to a single vertex. Although the edges of our polygon must b

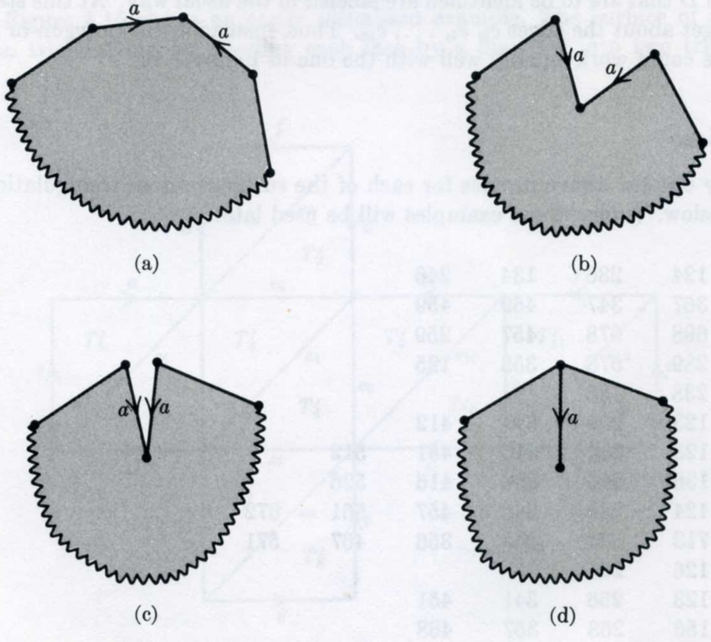


FIGURE 1.17 Elimination of an adjacent pair of edges of the first kind.

identified in pairs, the vertices may be identified in sets of one, two, three, four, ... Let us call two vertices of the polygon *equivalent* if and only if they are to be identified. For example, the reader can easily verify that in Figure 1.16 there are eight different equivalence classes of vertices. Some equivalence classes contain only one vertex, whereas other classes contain two or three vertices.

Assume we have carried out step two as far as possible. We wish to prove we can transform our polygon into another polygon with all its vertices belonging to one equivalence class.

Suppose there are at least two different equivalence classes of vertices. Then, the polygon must have an adjacent pair of vertices which are nonequivalent. Label these vertices P and Q. Figure 1.18 shows how to proceed. As P and Q are nonequivalent, and we have carried out step two, it follows that sides a and b are *not* to be identified. Make a cut along the line labeled c, from the vertex labeled Q to the other vertex of the edge a (i.e., to the vertex of edge a, which is distinct from P). Then, glue the two edges labeled a together. A new polygon with one less vertex in the equivalence class of P and one more vertex in the equivalence class of Q results. If possible, perform step two again. Then carry out step three to reduce the number of vertices in the equivalence class of P still further, then do step two again. Continue alternately doing step three and step two until the equivalence class of P is eliminated entirely. If more than one equivalence class of vertices remains, we can repeat this procedure to reduce the number by one. If we continue in this manner, we ultimately obtain a polygon such that all the vertices are to be identified to a single vertex.

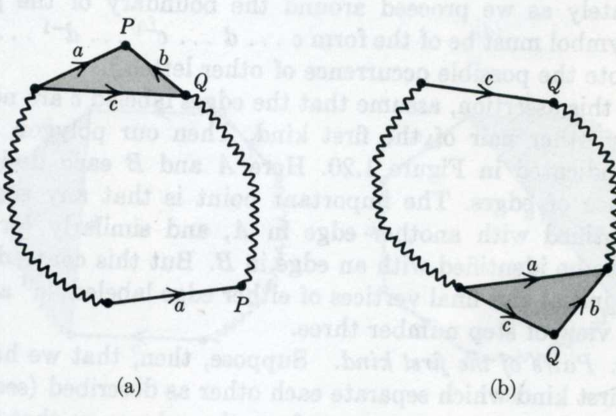


FIGURE 1.18 Third step in the proof of Theorem 5.1.

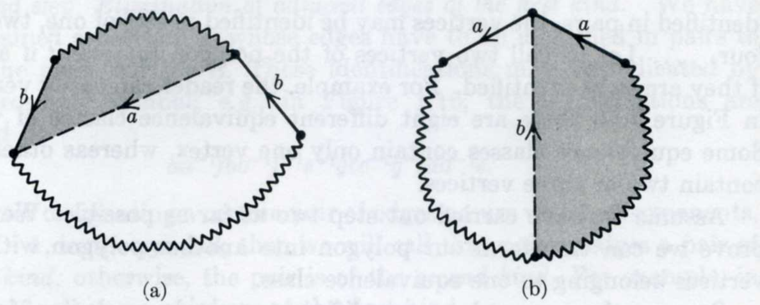


FIGURE 1.19 Fourth step in the proof of Theorem 5.1.

Fourth step. How to make any pair of edges of the second kind adjacent. We wish to show that our surface can be transformed so that any pair of edges of the second kind are adjacent to each other. Suppose we have a pair of edges of the second kind which are nonadjacent, as in Figure 1.19(a). Cut along the dotted line labeled a and paste together along b. As shown in Figure 1.19(b), the two edges are now adjacent.

Continue this process until all pairs of edges of the second kind are adjacent. If there are no pairs of the first kind, we are finished, because the symbol of the polygon must then be of the form $a_1a_1a_2a_2 \ldots a_na_n$, and hence S is the connected sum of n projective planes.

Assume to the contrary that at this stage there is at least one pair of edges of the first kind, each of which is labeled with the letter c. Then we assert that there is at least one other pair of edges of the first kind such that these two pairs separate one another; i.e., edges from the two pairs occur alternately as we proceed around the boundary of the polygon (hence, the symbol must be of the form $c \ldots d \ldots c^{-1} \ldots d^{-1} \ldots$, where the dots denote the possible occurrence of other letters).

To prove this assertion, assume that the edges labeled c are not separated by any other pair of the first kind. Then our polygon has the appearance indicated in Figure 1.20. Here A and B each designate a whole sequence of edges. The important point is that any edge in A must be identified with another edge in A, and similarly for B. No edge in A is to be identified with an edge in B. But this contradicts the fact that the initial and final vertices of either edge labeled "c" are to be identified, in view of step number three.

Fifth step. Pairs of the first kind. Suppose, then, that we have two pairs of the first kind which separate each other as described (see Figure 1.21). We shall show that we can transform the polygon so that the four sides in question are consecutive around the perimeter of the polygon.

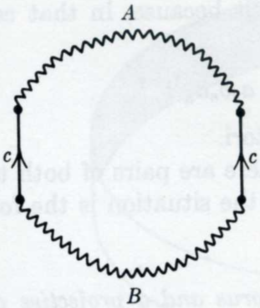


FIGURE 1.20 A pair of edges of the first kind.

First, cut along c and paste together along b to obtain Figure 1.21(b). Then, cut along d and paste together along a to obtain (c), as desired.

Continue this process until all pairs of the first kind are in adjacent groups of four, as $cdc^{-1}d^{-1}$ in Figure 1.21(c). If there are no pairs of the

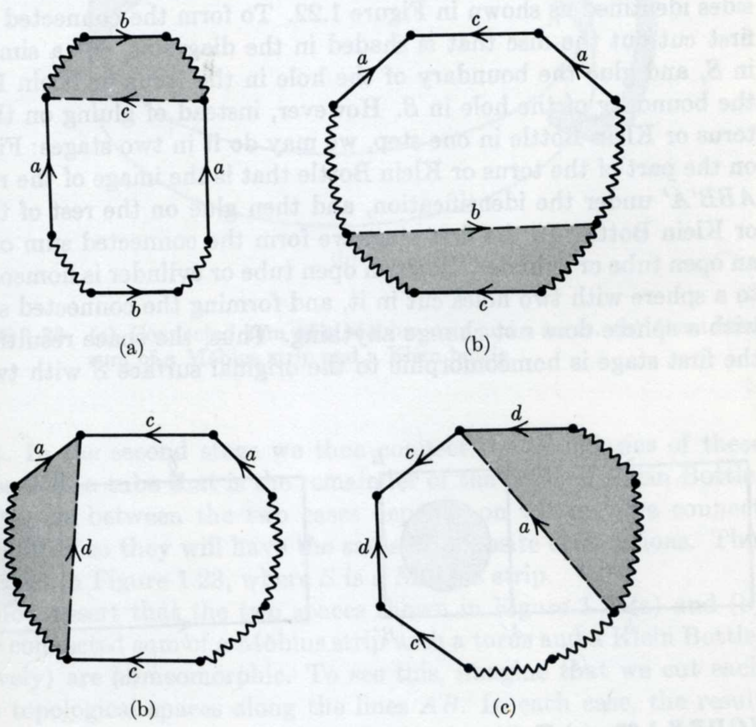


FIGURE 1.21 Fifth step in the proof of Theorem 5.1.

second kind, this leads to the desired result because, in that case, the symbol must be of the form

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \ldots a_nb_na_n^{-1}b_n^{-1}$$

and the surface is the connected sum of n tori.

It remains to treat the case in which there are pairs of both the first and second kind at this stage. The key to the situation is the following rather surprising lemma:

Lemma 7.1 *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*

PROOF: We have remarked that the connected sum of two projective planes is homeomorphic to a Klein Bottle (see Example 4.3). Thus, we must prove that the connected sum of a projective plane and a torus is homeomorphic to the connected sum of a projective plane and a Klein Bottle. To do this it will be convenient to give an alternative construction for a connected sum of any surface S with a torus or a Klein Bottle. We can represent the torus and Klein Bottle as rectangles with opposite sides identified as shown in Figure 1.22. To form the connected sum, we first cut out the disc that is shaded in the diagrams, cut a similar hole in S, and glue the boundary of the hole in the torus or Klein Bottle to the boundary of the hole in S. However, instead of gluing on the entire torus or Klein Bottle in one step, we may do it in two stages: First, glue on the part of the torus or Klein Bottle that is the image of the rectangle $ABB'A'$ under the identification, and then glue on the rest of the torus or Klein Bottle. In the first stage we form the connected sum of S with an open tube or cylinder. Such an open tube or cylinder is homeomorphic to a sphere with two holes cut in it, and forming the connected sum of S with a sphere does not change anything. Thus, the space resulting from the first stage is homeomorphic to the original surface S with two holes

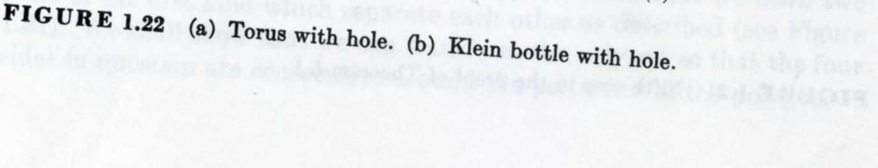
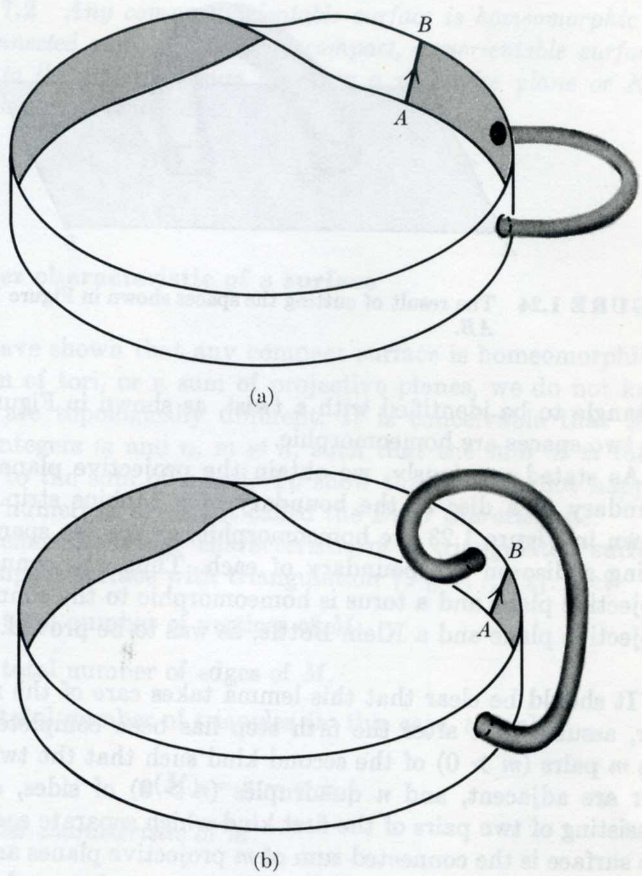


FIGURE 1.22 (a) Torus with hole. (b) Klein bottle with hole.



IGURE 1.23 (a) Connected sum of a Möbius strip and a torus. (b) Connected sum of a Möbius strip and a Klein bottle.

t in it. In the second stage we then connect the boundaries of these vo holes with a tube that is the remainder of the torus or Klein Bottle. he difference between the two cases depends on whether we connect e boundaries so they will have the same or opposite orientations. This illustrated in Figure 1.23, where S is a Möbius strip.

We now assert that the two spaces shown in Figure 1.23(a) and (b) e., the connected sum of a Möbius strip with a torus and a Klein Bottle, spectively) are homeomorphic. To see this, imagine that we cut each these topological spaces along the lines AB. In each case, the result the connected sum of a rectangle and a torus, with the two ends of the