A *word* is a sequence of elements of $(G - \{e\}) \cup (H - \{e\})$; it is *reduced* if no two adjacent elements are in G or in H.

We will denote a word with commas:

$$w = a_1, a_2, \ldots, a_n.$$

If a and b are both in G or both in H, let $\overline{a, b}$ denote ab if $ab \neq e$ and the empty word otherwise. By an *immediate descendent* of w we mean a word obtained by replacing a subword a, b by $\overline{a, b}$. By a descendent of w we mean either w itself or a word which can be reached from w by a chain of immediate descendents.

Proposition If x and y are reduced descendents of w then $x = y$.

Assuming the proposition, we define ww' (for reduced words w and w') to be the reduced descendent of the word w, w'. This is obviously associative, since $(ww')w''$ and $w(w'w'')$ are reduced descendents of the word w, w', w''.

The proof of the proposition is by induction on the length of w. Let the chain from w to x (resp., y) begin with x_1 (resp., y_1), and let x_1 (resp., y_1) be obtained from w by replacing a, b by $\overline{a, b}$ (resp., c, d by $\overline{c, d}$).

First we observe that if x_1 and y_1 have a common descendent z then we are done, because if u is the reduced descendent of z then u and x are reduced descendents of x_1, so they are equal by the inductive hypothesis, and similarly $u = y$, so $x = y$.

If a, b and c, d are the same subword then $x_1 = y_1$.

If a, b and c, d don't overlap, we obtain a common descendent z by replacing c, d in x_1 by $\overline{c, d}$.

If a, b and c, d overlap but are not the same subword, we may assume that a is to the left of c in w, and then $b = c$ and the triple a, b, d is in G or in H. There are four cases. If $ab \neq e$ and $bd \neq e$ then the word z obtained from x_1 by replacing ab, d by $\overline{ab, d}$ is a common descendent of x_1 and y_1. If $ab = e$ and $bd \neq e$ then x_1 is an immediate descendent of y_1. If $ab \neq e$ and $bd = e$ then y_1 is an immediate descendent of x_1. Finally, if $ab = e$ and $bd = e$ then $x_1 = y_1$. This completes the proof.