We state the Hodge theorem in the following form:

Theorem 1 *Let M be a Riemannian manifold, let $H^p(M)$ be the space of harmonic p-forms on M, and let $E^p(M)$ be the space of smooth p-forms on M. Then, for each integer p, $H^p(M)$ is finite-dimensional, and we have the following direct sum decomposition:*
$E^p(M) = \triangle(E^p) \oplus H^p = d(E^{p-1}) \oplus d^*(E^{p+1}) \oplus H^p$
Furthermore, this decomposition is orthogonal wrt the inner product $\int \alpha \wedge * \beta$.

This implies the following:
Corollary: Each de Rham cohomology class on a compact oriented Riemannian manifold M has a unique harmonic representative.
Proof: Suppose α and β are harmonic forms in the same cohomology class. Then $\alpha - \beta = d\gamma$. But $< \alpha - \beta, d\gamma >=< d * \alpha - d * \beta, \gamma >=< 0, \gamma >= 0$. So dB is orthogonal to itself, and thus zero. Now, for an arbitrary cohomology class α, we can take $H(\alpha)$ the projection of α on the space of harmonic forms. This is in the same cohomology class as α, because, if $\alpha = H(\alpha) + d\alpha_1 + d * \alpha_2$, then $0 =< d\alpha, \alpha_2 >=< dd^*\alpha_2, \alpha_2 >=< d^*\alpha_2, d^*\alpha_2 >$, so $\alpha = H(\alpha) = d\alpha_1$.
We now embark on a brief discussion of the machinary necessary to prove the Hodge Theorem in a compact subset of Euclidean space. We will be content with this special case because: a) the case of Euclidean space already contains all the essential analytic ideas and b) the extension to arbritrary Riemannian manifolds is a messy argument involving partitions of unity and Jacobians that I would almost certainly mangle irretreivably.
We will state several lemmas that we will not prove. More detailed proofs of the Hodge theorem can be found in Gilkey, "Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem", or Warner, "Foundations of differential Manifolds and Lie Groups", or Griffiths-Harris, "Fundamentals of Algebraic Geometry". Our point of view is closest to Gilkey. Furthermore, some of these results can be found in any good book on functional analysis.
The fundamental calculational tool in modern linear PDEs is the Fourier Transform, defined as follows:

Definition 1 (Fourier Transform) *for functions f for which it makes sense, is defined as $f(\zeta) = \int e^{-ix\zeta} f(x)dx$*

Lemma 1 *a)The fourier transform $f(x) \to Ff(x)$ extends to an isometric isomorphism of L^2 onto itself, with inverse given by $fx \to Ff(-x)$. b)The fourier transform converts differentiation to multiplication by ζ; ie. $D^\alpha_\zeta \hat{f(\zeta)} = \zeta^\alpha \widehat{f(x)}$ c)Using a) and b) we find that for a PDE P, $Pf = \int e^{i(x-y)\zeta} p(x,\zeta) f(y) dy d\zeta$ for some unique function $p(x,\zeta)$. If P is a linear differential operator, p is a polynomial, often called its "symbol". The highest order part of the polynomial is what we called the symbol in class.*

When we are working with the Fourier Transform, we can replace the C^k spaces with more explicit and general spaces.

Definition 2 (Sobolev spaces) *If f is a compactly supported smooth function and $s \in R$, then define: $||f||_s^2 = \int (1 + |\zeta|^2)^s |\hat{f}(\zeta)|^2 d\zeta$ The completion with respect to this norm is called the Sobolev space H_s.*

Lemma 2 *a) (Sobolev Lemma) If $f \in H_s$ and $s > k + n/2$, (where n is the real dimension) then f is C^k. Consequently, if f is in arbritrarily high Sobolev spaces, then f is smooth. b) (Rellich Lemma) The inclusion of H_s into H_{s+d} is a compact operator for all s and all positive d; ie, it takes bounded sequences into sequences with cauchy subsequences.*

Using the fourier transform, we can also generalize the concept of an operator, by not requiring the symbol to be a polynomial.

Definition 3 (Pseudo-Differential Operators) *For any $p(x, \zeta)$ smooth with compact x support such that $|D_x^\alpha D_\zeta^\beta p(x, \zeta)| \leq C_{\alpha,\beta}(1 + |\zeta|)^{d-|\beta|}$, we define the associated Psuedo-Differential operator as $Pf(x) = \int e^{i(x-y)\zeta} p(x, \zeta) f(y) dy d\zeta$. d is called the order of P.*

An operator is said to be smoothing if it is of order -d for abritrarily large d. An operator with symbol T is said to be elliptic if there exists a symbol S such that $ST - I$ is the symbol of a smoothing Psuedo-Differential operator.

Lemma 3 *a)if the symbol of an operator is a polynomial restricted to a compact set, then the operator is elliptic iff the highest order part of the polynomial is invertible. b)An operator P of order d extends to a map from H_s to H_{s-d}. Using the Sobolev Lemma, this implies that if f is smooth then Pf is also smooth, and if P is smoothing, it maps every sobelev space to the smooth functions. Using the Rellich Lemma, this implies that smoothing operators are compact.*

Lemma 4 (Fundamental fact on elliptic psudes) *If P is an elliptic psuedo-differential operator of order d, there exists a "parametrix" Q, such that Q is a psuedo-diff of order -d, and $PQ - I$ is smoothing.*

The whole point to using psuedo diff operators is that even if P is a linear operator, Q will, in general, not be.

Lemma 5 (Fundamental fact about Fredholm operators) *An operator S is Fredholm iff there exists an operator T such that $(ST - I)$ is a compact operator, ie. it takes bounded sequences to sequences with cauchy subsequences.*

We can now prove the Hodge Theorem.

Proof: The finite dimensionality of the Harmonic forms follows from the fact that elliptic operators are Fredholm, which follows from the above fundamental facts. Since the Laplacian is Fredholm, $N(\triangle *)^\perp = R(\triangle)$, but since the Laplacian is self-adjoint, we have $N(\triangle *)^\perp = R(\triangle)$. Every harmonic form is certainly smooth, so if f is a smooth form, we have the decomposition $f = f_0 + (dd * + d * d)\omega = f_0 + d\omega_1 + d * \omega_2$ where f_0 is smooth. The proof will be complete if we can show ω is smooth. But $\triangle\omega = f - f_0$ is smooth, and $\triangle$ has a parametrix Q s.t. $\omega = (I - Q\triangle)\omega + Q\triangle\omega$, and $(I - Q\triangle)\omega$ is smooth since $(I - Q\triangle)$ is smoothing, and $Q\triangle\omega$ is smooth since $\triangle\omega$ is smooth and Q is a psuedo-pde.