Tychonoff's Theorem

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1 Introduction

The main goal of these notes is to provide an elementary proof of Tychonoff's theorem.

Tychonoff's Theorem. *The product of arbitrarily many compact spaces is compact.*

It is not hard to prove that the product of finitely many compact spaces is compact, it follows immediately from the compactness of of two compact spaces. It is harder to prove that the Tychonoff theorem. In fact, one must use the axiom of choice (or its equivalent) to prove the general case. It's not an overstatement to say "must use the axiom of choice" since in 1950, Kelley proved that Tychonoff's theorem implies the axiom of choice [3]. I include Kelley's proof at the end of these notes.

2 Preliminaries from set theory

2.1 Axiom of choice and Zorn's lemma

In addition to the standard axioms of set theory (axiom of unions, axiom of subsets, etc...), there is the axiom of choice:

The Axiom of Choice. *Given any collection of nonempty sets $\{X_\alpha\}_{\alpha\in\Lambda}$, there exists a function $f : \Lambda \to \cup_{\alpha\in\Lambda} X_\alpha$ with $f(\alpha) \in X_\alpha$ for each $\alpha \in \Lambda$.*

This axiom says that $\prod_{\alpha\in\Lambda} X_\alpha$ is nonempty. This is something you probably feel is natural to assume—that there's at least one element in the product of spaces!—but the axiom of choice has some admittedly strange consequences, such as the Banach-Tarski paradox. At any rate, we assume it as an axiom. One equivalent formulation of the axiom of choice is *Zorn's Lemma*. Before stating it and proving its equivalence of the axiom of choice, I'll remind you about a little about partially ordered sets.

A partially ordered set is a set P together with a relation $\leq$ on P that is reflexive, transitive, and antisymmetric. Reflexive means that for all $a \in P$, $a \leq a$; transitive means that for all $a, b, c \in P$, if $a \leq b$ and $b \leq c$ then $a \leq c$; antisymmetric means that for all $a, b \in P$, if $a \leq b$ and $b \leq a$ then $a = b$. A subset C of a partially ordered set P is called a *chain* iff for every $a, b \in C$ either $a \leq b$ or $b \leq a$. An element b of a partially ordered set P is called an upper bound for a subset $A \subset P$ provided $a \leq b$ for all $a \in A$. We say that m is a maximal element of a partially ordered set P iff there exists no element $a \in P$ with $m \leq a$ and $m \neq a$. One can write $a < b$ if $a \leq b$ and $a \neq b$. One can use the notation $\geq$ and $>$ with the obvious meaning. Then one can say m is a maximal element if there is no element a with $a > m$.

Zorn's Lemma. *If every chain in a nonempty partially ordered set P has an upper bound, then P has a maximal element.*

Theorem 1. *The axiom of choice $\Leftrightarrow$ Zorn's lemma.*

Proof. Assume the axiom of choice and let P be a partially ordered set in which every chain has an upper bound. For each $\alpha \in P$, define a set $E_\alpha := \{\beta \in P : \alpha < \beta\}$. If $E_\alpha = \emptyset$ for any α, then α is a maximal element of P and we're done. If, however, there $E_\alpha \neq \emptyset$ for any α, then the axiom of choice says there is a function $f : P \to \cup_{\alpha\in\Lambda} E_\alpha$ with $f(\alpha) \in E_\alpha$. This means that $f(\alpha) > \alpha$ for every α. So, we create a chain

$$\alpha < f(\alpha) < f(f(\alpha)) < \cdots .$$

We know this chain has an upper bound since every chain in P has an upper bound. Call it β. But then $f(\beta) > \beta$ and we can extend the chain

$$\alpha < f(\alpha) < f(f(\alpha)) < \cdots \beta < f(\beta) < f(f(\beta)) < \cdots$$

Now this chain has an upper bound, call it γ and we can add $\gamma, f(\gamma), \ldots$ to the chain. We can continue transfinitely, which shows that that cardinality of P is greater than that of any other set, which is impossible.

Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a collection of nonempty sets. Define a partial choice function to be a function $f : I \to \cup_{\alpha\in I} X_\alpha$ where $I \subset \Lambda$. The set P of partial choice functions is partially ordered by extension: for two partial choice function f and g one has $f \leq g$ iff the domain of f is a subset of the domain of g and they agree on their common domain. If C is a chain of partial choice functions, then the union of the functions in C is an upper bound for C (the union of a chain of partial choice functions is a function whose domain is the union of the domains of the functions in C and whose value at a point is the common value at that point of any one of the functions in C defined at the point.) Then Zorn's Lemma implies that there is a maximal element of P, which must be a choice function with domain Λ. This function satisfies the conclusion of the axiom of choice. $\square$

There are a few other statements that are equivalent to the axiom of choice. One favored by many topologists is the *Hausdorff maximum principle* which states that every partially ordered set has a maximal chain. Here, a maximal chain means a chain that is not properly contained in any other chain. Another statement equivalent to the axiom of choice is the *well ordering principle* which states that every set can be well ordered. A well ordering is a partial order in which every two elements are comparable and every nonempty subset has a least element. This puts in mind the joke:

> *The axiom of choice is obviously true, the well ordering principle is obviously false, and who can tell about Zorn's lemma?*

2.2 Finite intersection property

Let $\mathcal{S}$ be a collection of sets. We say that the collection $\mathcal{S}$ has the *finite intersection property* if and only if for every finite subcollection $A_1, \ldots, A_n \subset \mathcal{S}$, the intersection $A_1 \cap \cdots \cap A_n \neq \emptyset$. We abbreviate the finite intersection property by FIP.

3 Preliminaries about compact sets

3.1 Inventory of basics

Definition 1. A space is compact if and only if every open cover has a finite subcover.

Here's a quick review of the basic theorems about compactness. For the proofs, which are all short and direct, see your class notes or any topology book. Better yet, reprove them as you read them.

Theorem 2. *The continuous image of a compact space is compact.*

Corollary 1. *Compactness is a topological property*

Corollary 2. *The quotient of a compact space is compact.*

Corollary 3. *If the product of spaces is compact, then each factor must be compact.*

Theorem 3. *The unit interval is compact. The real line is not compact.*

Theorem 4. *A closed subset of a compact space is compact.*

Theorem 5. *A compact subset of a Hausdorff space is closed.*

Theorem 6. *A continuous, real valued function on a compact space attains both a maximum and a minimum.*

Theorem 7. *The product of two compact spaces is compact.*

The Heine-Borel Theorem. *Closed and bounded subsets of* $\mathbb{R}^n$ *are compact and vice versa.*

Example 1. The circle, torus, and all spheres are compact.

3.2 Nets and compactness

I think of compact sets as somehow being small—not in terms of cardinality, but in terms of roominess. For example, if you squeeze an infinite set of points into the unit interval, they get cramped—for any $\epsilon > 0$, there are two points that are less than ϵ apart. But, it's easy to fit an infinite number of points in the real line so that they're all spread out. This idea is summarized in the following simple theorem.

The Bolzano-Weierstrauss Theorem. *Every infinite set in a compact space has a limit point.*

Proof. Suppose that F is an infinite subset in a compact space and suppose that F has no limit points. Note that F is closed since it has no limit points and hence it (trivially) contains all its limit points. Therefore F itself is compact. Now, for each $x \in F$, there is an open set U_x with $U_x \cap F = \{x\}$. Then $\{U_x\}_{x\in F}$ is an open cover of F. Notice that there can be no finite subcover $U_{x_1}, \ldots, U_{x_n}$ since $(U_{x_1} \cup \cdots \cup U_{x_n}) \cap F = \{x_1, \ldots, x_n\}$, and cannot contain the infinite set F. ☐

Now, there are noncompact spaces for which every infinite subset has a limit point. For instance, take $\mathbb{R}$ with topology $\{(x, \infty) : x \in \mathbb{R}\}$, together with $\emptyset$ and $\mathbb{R}$. This space is not compact, but any set (infinite or not) has a limit point (infinitely many, in fact). In order to more sharply relate compactness to infinite sets, we introduce the notion of a cluster point of a nets.

Definition 2. Let X be a space and let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net. We say that a point $x \in X$ is a *cluster point* (or *accumulation point*) of the net $\{x_\alpha\}_{\alpha\in\Lambda}$ if and only if for every open set U containing x and every $\alpha \in \Lambda$, there exists $\beta \in \Lambda$ with $\alpha \leq \beta$ and $x_\beta \in U$.

Example 2. Consider the net $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ given by

$$x_n = \begin{cases} \frac{1}{n} & \text{if } n = 1, 5, 9, 13, \ldots \\ 1 - \frac{1}{n} & \text{if } n = 2, 6, 10, 14, \ldots \\ -n & \text{if } n = 3, 7, 11, 15, \ldots \\ 2 & \text{if } n = 4, 8, 12, 16, \ldots \end{cases}$$

The first few terms of $\{x_n\}$ are

$$1, \frac{1}{2}, -3, 2, \frac{1}{5}, \frac{5}{6}, -7, 2, \frac{1}{9}, \frac{9}{10}, -11, 2, \frac{1}{13}, \frac{13}{14}, -15, 2 \ldots$$

Then $0, 1, 2$ are all cluster points of this net. Note that if we view the range of this net as a subset of $\mathbb{R}$, then the limit points of this set are 0 and 1.

Theorem 8. *Let X be a topological space. The following are equivalent:*

(a) X is compact.

(b) Every collection of closed subsets of X with the FIP have nonempty intersection.

(c) Every net has a cluster point.

Proof. $(a) \Rightarrow (b)$. Let $\{E_\alpha\}$ be a collection of closed sets with $\cap E_\alpha = \emptyset$. Then $\{U_\alpha\}$ where $U_\alpha := X \setminus E_\alpha$ is an open cover of X. If X is compact, there is a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$. Then $E_{\alpha_1} \cap \cdots \cap E_{\alpha_n} = \emptyset$. This proves that any collection of closed sets with empty intersection does not have FIP.

$(b) \Rightarrow (c)$. Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net. For each $\alpha \in \Lambda$ define

$$F_\alpha = \{x_\beta : \alpha \leq \beta\} \text{ and } E_\alpha = \overline{F_\alpha}.$$

Since Λ is a directed set, if $\alpha, \alpha' \in \Lambda$ there exists $\beta \in \Lambda$ with $\alpha \leq \beta$ and $\alpha' \leq \beta$. Then $x_\beta \in F_\alpha \cap F_{\alpha'} \subset E_\alpha \cap E_{\alpha'}$. It follows that $\{E_\alpha\}$ is a collection of closed sets with the FIP. Therefore, there is an element $x \in \cap E_\alpha$. I claim that x is a cluster point of the net $\{x_\alpha\}$. To see this, let U be an open set containing x and let $\alpha \in \Lambda$. Since $x \in E_\alpha = \overline{F_\alpha}$, there is a point $y \in F_\alpha \cap U$. By definition of F_α, $y = x_\beta$ for some $\alpha \leq \beta$, as needed.

$(c) \Rightarrow (a)$. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets for which no finite subset covers X. We'll prove that $\{U_\alpha\}$ is not a cover of X. Let $\mathcal{D}$ be the set of finite subsets of Λ directed by inclusion: $F \leq G \Leftrightarrow F \subset G$. For each $F \in \mathcal{D}$, $\cup_{\alpha\in F} U_\alpha$ is not a cover of X, therefore there exists a point $x_F \in X \setminus (\cup_{\alpha\in F} U_\alpha)$. The assignment $F \mapsto x_F$ defines a net in X, which by hypothesis has a cluster point, call it x. I claim that $x \notin \cup_{\alpha\in\Lambda} U_\alpha$. To see this, fix $\alpha \in \Lambda$. Then the singleton set $\{\alpha\} \in \mathcal{D}$, therefore, for any open set U containing x, there is a finite set F with $\{\alpha\} \subset F$ and a point $x_F \in U$. This says that $x_F \in X \setminus (\cup_{\beta\in F} U_\beta) \subset X \setminus U_\alpha$. This proves that $x \in \overline{X \setminus U_\alpha} = X \setminus U_\alpha$. Since this is true for every α, $x \notin \cup_{\alpha\in\Lambda} U_\alpha$. $\square$

Problem 1. Give a direct proof that $(b) \Rightarrow (a)$.

Definition 3. Let $f : \Lambda \to X$ be a net. A subnet of the net f is a composition $f \circ \phi$ where $\phi : \mathcal{D} \to \Lambda$ is an increasing cofinal function from a directed set $\mathcal{D}$ to the directed set Λ. *Increasing* means that $\phi(a) \leq \phi(b)$ whenever $a \leq b$ and *cofinal* means that for every $\alpha \in \Lambda$, there exists an $a \in \mathcal{D}$ with $\alpha \leq \phi(a)$.

Problem 2. A sequence is an example of a net. Show that a subsequence of a sequence is a subnet, but not all subnets of a sequence are subsequences.

Problem 3. Prove that x is a cluster point of a net if and only if there exists a subnet converging to x.

Corollary 4. *A space is compact if and only if every net has a convergent subnet.*

This corollary should be compared with the familiar statement that a metric space X is compact if and only if every sequence has a convergent subsequence.

4 A proof of Tychonoff's Theorem

Here, I give the proof due to Chernoff in 1992 [1]. There's a "standard" proof that uses the well ordering principle, instead of Zorn's lemma, but I think it's more difficult—it's outlined in Exercise 5 on page 236 of Munkres *Topology* [4].

Proof of Tychonoff's theorem. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of compact spaces, let $X = \prod_{\alpha\in\Lambda}$ and let $\{f_d\}_{d\in\mathcal{D}}$ be a net in X. As in the proof of Theorem 2.1, we define an element of $\prod_{\alpha\in I} X_\alpha$ where $I \subset \Lambda$ to be a partial function. We say that a partial function f is a partial cluster point if it is a cluster point of the net $\{f_d|_I\}$ for some $I \subset \Lambda$. If there is a partial cluster point with domain $I = \Lambda$, then we have found a cluster point of the net $\{f_d\}$ and proved that X is compact.

Let P be the set of partial cluster points ordered by inclusion. Note that $P \neq \emptyset$ for if we let $\alpha \in \Lambda$ be one index and set $I = \{\alpha\}$, then $\{f_d(\alpha)\}$ is a net in X_α. Since X_α is compact, there is a cluster point $p \in X_\alpha$ of the net $\{f_d(\alpha)\}$. Then for $I = \{\alpha\}$, the partial function $f : I \to X$ defined by $f(\alpha) = p$ is a partial cluster point. Also, every chain in P has an upper bound since the union of the partial cluster points in a chain will also be a partial cluster point. Thus P satisfies the hypotheses of Zorn's Lemma.

Let g be a maximal element of P. If the domain I of $g = \Lambda$ then we are done. If the domain of $g \neq \Lambda$, choose an index $\alpha \in \Lambda \setminus I$. Then $\{f_d(\alpha)\}$ is a net in X_α. Since X_α is compact, there is a cluster point $p \in X_\alpha$ of the net $\{f_d(\alpha)\}$. Then h defined by

$$h(\beta) = \begin{cases} p & \text{if } \beta = \alpha \\ g(\beta) & \text{if } \beta \in I \end{cases}$$

is a partial cluster point with domain $I \cup \{\alpha\}$ extending g. This contradicts the maximality of g. Therefore, the domain of g is all of Λ and the proof is complete. □

Problem 4. What's wrong with the following easy "proof" of Tychonoff's theorem?

Wrong proof. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of compact spaces, let $X = \prod_{\alpha\in\Lambda}$ and let $\{f_d\}_{d\in\mathcal{D}}$ be a net in X. For each $\alpha \in A$, we have a net $\{f_d(\alpha)\}$ in X_α. Since X_α is compact, the net $\{f_d(\alpha)\}$ has a cluster point $p_\alpha \in X_\alpha$. Then, the function $f \in X$ defined by $f(\alpha) = p_\alpha$ is a cluster point of the net $\{f_d\}$, proving that X is compact. □

Hint: You can give a clear illustration of why this proof is wrong using the product of just two compact spaces.

5 Tychonoff's theorem implies the axiom of choice

We used Zorn's lemma to prove the Tychonoff theorem. In 1950, Kelley proved that the Tychonoff theorem is equivalent to the axiom of choice [3]. In order to prove that the Tychonoff theorem implies the axiom of choice, one begins with an arbitrary collection of sets and from them creates a collection of compact topological spaces. Then, the compactness of the product leads to the existence of a choice function. Originally, Kelley used an augmented cofinite topology. Here, is an even easier proof.

Theorem 9. *The Tychonoff theorem* $\Leftrightarrow$ *the axiom of choice.*

Proof. We used Zorn's lemma to prove Tychonoff's theorem, which shows that Tychonoff's theorem $\Leftarrow$ the axiom of choice.

To prove that Tychonoff's theorem $\Rightarrow$ the axiom of choice, let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a collection of nonempty sets. We need to make a bunch of compact spaces so we can apply the Tychonoff theorem. First, let $Y_\alpha = X_\alpha \cup \{\infty_\alpha\}$; we add a new element to X_α called "∞_α" We make each Y_α into a topological space by defining the topology to be $\{\emptyset, \{\infty_\alpha\}, X_\alpha, Y_\alpha\}$. Note that Y_α is compact—there are only finitely many open sets so every open cover is finite. So, by Tychonoff's theorem, $Y := \prod_{\alpha\in\Lambda} Y_\alpha$ is compact.

Now consider a collection of open sets $\{U_\beta\}_{\beta\in\Lambda}$ of Y where U_β is the basic open set in Y obtained by taking the product of all Y_α's for $\alpha \neq \beta$ and in the β factor put the open set $\{\infty_\beta\}$. Notice that any finite subcollection $\{U_{\beta_1}, \dots, U_{\beta_n}\}$ cannot cover Y for the function f defined as follows is not in $\cup_{i=1}^n U_{\beta_i}$: choose a partial function $\overline{f} \in \prod_{i=1}^n X_{\beta_i}$ which is possible without the axiom of choice since the product is finite. Then extend $\overline{f}$ to a function $f \in Y$ be setting $f(\alpha) = \infty_\alpha$ for all $\alpha \neq \beta_1, \dots, \beta_n$, which is possible since we're not making any choices.

Therefore, the collection $\{U_\beta\}$ cannot cover Y. So, there is a function $f \in Y$ not in the $\cup_{\alpha\in\Lambda} U_\alpha$. This says that for no $\alpha \in \Lambda$ does $f(\alpha) = \infty_\alpha$. Therefore, $f(\alpha) \in X_\alpha$ for each α, which is a desired choice function. □

5.1 Two weaker versions of the Tychonoff theorem that imply AC

Notice that this proof actually proves a stronger theorem. Let us call the statement "The arbitrary product of spaces with finite topologies is compact" the *finite Tychonoff theorem*. Then, the proof above actually shows that

Theorem 10. *The finite Tychonoff theorem* $\Leftrightarrow$ *the axiom of choice.*

Problem 5. Let us call the statement "If X is compact and Λ is any index set, then X^Λ is compact" the *weak Tychonoff theorem*. This problem outlines a proof due to E. L. Ward, Jr. [2] that the uniform Tychonoff theorem is equivalent to the axiom of choice.

(a) Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be any collection of disjoint, nonempty sets and let $X = \cup_{\alpha\in\Lambda} X_\alpha$. Call a set $U \subset X$ if $U = \emptyset, X$, or the union of finitely many X_α. Prove that this defines a topology on X and that X is compact.

(b) Let $Y = X^\Lambda$, which assuming the weak Tychonoff theorem, will be compact. For each $\alpha \in \Lambda$, define $F_\alpha := \{f \in Y : f(\alpha) \in X_\alpha\}$. Show that $\{F_\alpha\}_{\alpha\in\Lambda}$ is a collection of closed sets with FIP.

(c) Conclude that $f \in \cap_{\alpha\in\Lambda} F\alpha$ is a desired choice function.

References

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