# Group theory

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*Dedicated to Conacyt.*

ABSTRACT. This course was given at GC.

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# Preface

This course was given at Graduate Center by Prof. Olga Kharlampovich.

# Part 1

# The First Part

CHAPTER 1

# The First Chapter

## 1. Universal property

Definition 1. *Let $S$ be a set. A free group on $S$ is a group $F_S$ with a set map*

$$i : S \to F_S$$

*such that, whenever $G$ is a group and*

$$\phi : S \to G$$

*is a set map, there exists a unique group homomorphism*

$$\tilde{\phi} : F_S \to G$$

*such that $\tilde{\phi} \circ i$ extends $\phi$.*

Consider the following diagram,

$$\begin{array}{ccc} S & \hookrightarrow & F_S \\ & \phi \searrow & \downarrow \tilde{\phi} \\ & & G \end{array}$$

This says that there is a correspondence between group homomorphisms

$$F_S \to G,$$

and set maps

$$S \to G.$$

I.e.

$$\begin{array}{c} \text{Groups} \\ \uparrow \;\; \downarrow \\ \text{Sets} \end{array}$$

Which is a left adjoint to the forgetful functor:

$$Hom\left(F\left(S\right) \to G\right) \longleftrightarrow Hom\left(S \to \underbrace{G}_{\text{free group}}\right).$$

As usual with objects defined via a universal property, it's immediate that if $F_S$ exists then it's unique up to isomorphism. If we were to take the categorical point of view we could guarantee the existence of free groups by appealing to an adjoint functor theorem.

Example 1. *If $S = \varnothing$, then $F_S \cong 1$.*

Example 2. *If $S = \{s\}$, then $F_S \cong \mathbb{Z}$*

## 2. Algebraic definition, and construction of a free group with basis $S$

The above was via the universal property, but now construct free groups on large sets in a different way.

Let $S$ be an arbitrary set. We define the free groups $F(S)$ generated by $S$, as follows. A word $w$ in $S$ is a finite sequence of elements which we write as

$$w = y_1 \cdots y_n,$$

where $y_i \in S$.

The number $n$ is called the length of the word $w$, we denote it by $|w|$. The empty sequence of elements is also allowed. We denote the empty word by $e$ and set its length to be $|e| = 0$. Consider the set $S^{-1} = \left\{ s^{-1} | s \in S \right\}$ where $s^{-1}$ is just a formal expression. We call $s^{-1}$ the formal inverse of $s$. The set

$$S^{\pm 1} = S \cup S^{-1},$$

is called the alphabet of $F$, and an element $y \in S^{\pm 1}$ of this set is called a letter. By $s^1$ we mean $s$, for each $s \in S$. An expression of the type

$$w = s_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n}, \;\; s_{i_j} \in S; \epsilon_j \in \{1, -1\},$$

is called a group word in $S$. A word $w$ is called reduced if it contains no subword of the type $ss^{-1}$ or $s^{-1}s$, for all $s \in S$.

Definition 2. *A group $G$ is called a free group if there exists a generating set $S$ in $G$ such that every nonempty reduced group word in $S$ defines a nontrivial element of $G$. If this is the case, then one says that $G$ is freely generated by $S$ (or that $G$ is free on $S$), and $S$ is called a free basis of $G$.*

**Reduction Process**

Let $S$ be an arbitrary set. To construct a free group with basis $S$, we need to describe a reduction process which allows one to obtain a reduced word from an arbitrary word. An elementary reduction of a group word $w$ consists of deleting a subword of the type $yy^{-1}$ where $y \in S^{\pm 1}$ from $w$. For instance, let $w = uyy^{-1}v$ for some words $u$ and $v$ in $S$. Then the elementary reduction of $w$ with respect to the given subword $yy^{-1}$ results in the word $uv$. In this event we write

$$uyy^{-1}v \to uv.$$

A reduction of $w$ (or a reduction process starting at $w$) consists of consequent applications of elementary reductions starting at $w$ and ending at a reduced word:

$$w \to w_1 \to \cdots \to w_n,$$

$w_n$ is reduced. The word $w_n$ is termed a reduced from of $w$.

Lemma 1. *For any two elementary reductions $w \to w_1$ and $w \to w_2$ of a group word $w$ in $S$ there exist elementary reductions $w_1 \to w_\circ$ and $w_2 \to w_\circ$, so that the following diagram commutes.*

$$\begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ w_1 \arrow[dr] & & w_2 \arrow[dl] \\ & w_\circ & \end{tikzcd}$$

PROOF. Let $\lambda_1 : w \to w_1$ and $\lambda_2 : w \to w_2$ be elementary reductions of a word $w$. We distinguish the following two cases.

(1) Disjoint reductions. In this case $w = u_1 y_1 y_1^{-1} u_2 y_2 y_2^{-1} u_3$ where $y_i \in S^{\pm 1}$, and $\lambda_i$ deletes the subword $y_i y_i^{-1}$, $i = 1, 2$. Then

$$\lambda_2 \circ \lambda_1 \quad : \quad w \to u_1 u_2 y_2 y_2^{-1} u_3 \to u_1 u_2 u_3 \tag{2.1}$$

$$\lambda_1 \circ \lambda_2 \quad : \quad w \to u_1 y_1 y_1^{-1} u_2 u_3 \to u_1 u_2 u_3; \tag{2.2}$$

(2) Overlapping reductions. In this case $y_1 = y_2$ and $w$ takes on the following form $w = u_1 y y^{-1} y u_2$. Then

$$\lambda_2 \quad : \quad w = u_1 y y^{-1} y u_2 \to u_1 y u_2, \text{ and} \tag{2.3}$$

$$\lambda_1 \quad : \quad w = u_1 \left( y y^{-1} \right) y u_2 \to u_1 y u_2. \tag{2.4}$$

□

PROPOSITION 1. *Let $w$ be a group word in $S$. Then any two reductions of $w$ :*

$$w \to w_\circ^{'} \to \cdots \to w_n^{'}, \text{ and} \tag{2.5}$$

$$w \to w_\circ^{''} \to \cdots \to w_n^{''}, \tag{2.6}$$

*result in the same reduced form, in other words, $w_n^{'} = w_n^{''}$.*

PROOF. The proof is by induction on $|w|$. If $|w| = 0$ then $w$ is reduced and there is nothing to prove. Let now $|w| > 1$. Then by the Lemma (1), there are elementary reductions $w_\circ^{'} \to w_\circ$, and $w_\circ^{''} \to w_\circ$. Consider a reduction process for

$$w_\circ \to w_1 \to \cdots \to w_k. \tag{2.7}$$

This corresponds to the following diagram:

$$\begin{tikzcd}
& & w \arrow[dl] \arrow[dr] & & \\
& w_\circ^{'} \arrow[dl] \arrow[dr] & & w_\circ^{''} \arrow[dl] \arrow[dr] & \\
w_1^{'} \arrow[d] & & w_\circ \arrow[d] & & w_1^{''} \arrow[d] \\
\cdots \arrow[d] & & w_1 \arrow[d] & & \cdots \arrow[d] \\
w_n^{'} & & \cdots \arrow[d] & & w_m^{'} \\
& & w_k & &
\end{tikzcd}$$

□

For a group word $w$, by $\bar{w}$ we denote the unique reduced form of $w$. Let $F(S)$ be the set of all reduced words in $S$. For $u, v \in F(S)$ we define multiplication $u \bullet v$ as follows:

$$u \bullet v = \overline{uv}.$$

THEOREM 1. *The set $F(S)$ forms a group with respect to the multiplication "$\bullet$". This group is free on $S$.*

THEOREM 2. *Let $F$ be a group with a generating set $S \subset S$. Then $F$ is freely generated by $S$ iff $F$ has the following universal property. Every map $\phi : S \to G$ from $S$ into a group $G$ can be extended to a unique homomorphism $\tilde{\phi} : F \to G$ so that the diagram below commutes*

$$\begin{array}{ccc} S & \overset{i}{\hookrightarrow} & F_S \\ & \phi \searrow & \downarrow \tilde{\phi} \\ & & G \end{array}$$

*Where $i$ is the inclusion of $S$ into $F$. Where $\tilde{\phi} \circ i = \phi$.*

PROOF. Let $F$ be a group freely generated by $S$ and let $\phi : S \to G$ be a map from $S$ into a group $G$. Since $F = F(S)$ is freely generated by $S$, every element $g \in F$ is defined by a unique reduced word in $S^{\pm 1}$. Let

$$g = s_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n},\ s_{i_j} \in S; \epsilon_j \in \{1, -1\}.$$

We set $\tilde{\phi}(g)$ to be

$$\begin{aligned} \tilde{\phi}(g) &= \tilde{\phi}\left(s_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n}\right) \\ &= \tilde{\phi}\left(s_{i_1}^{\epsilon_1}\right) \cdots \tilde{\phi}\left(s_{i_n}^{\epsilon_n}\right) \\ &= \tilde{\phi}\left(s_{i_1}\right)^{\epsilon_1} \cdots \tilde{\phi}\left(s_{i_n}\right)^{\epsilon_n}. \end{aligned} \tag{2.8}$$

We claim that $\tilde{\phi}$ is a homomorphism. Indeed, let $g, h \in F$ be so that

$$g = y_1 \cdots y_n z_1 \cdots z_m,$$

and

$$h = z_m^{-1} \cdots z_1^{-1} y_{n+1} \cdots y_k$$

are the corresponding reduced words in $S$, where $y_i, z_j \in S^{\pm 1}$ and $y_n \neq y_{n+1}^{-1}$ (we allow the subwords $y_1 \cdots y_n$, $z_1 \cdots z_m$ and $y_{n+1} \cdots y_k$ to be empty). Then

$$\begin{aligned} gh &= y_1 \cdots y_n z_1 \cdots z_m z_m^{-1} \cdots z_1^{-1} y_{n+1} \cdots y_k \\ &= y_1 \cdots y_n y_{n+1} \cdots y_k. \end{aligned}$$

$$\begin{aligned} \tilde{\phi}(gh) &= \tilde{\phi}(y_1 \cdots y_n y_{n+1} \cdots y_k) \\ &= \tilde{\phi}(y_1) \cdots \tilde{\phi}(y_n) \tilde{\phi}(y_{n+1}) \cdots \tilde{\phi}(y_k) \\ &= \tilde{\phi}(y_1) \cdots \tilde{\phi}(y_n) \tilde{\phi}(z_1) \cdots \tilde{\phi}(z_m) \tilde{\phi}\left(z_m^{-1}\right) \cdots \tilde{\phi}\left(z_1^{-1}\right) \tilde{\phi}(y_{n+1}) \cdots \tilde{\phi}(y_k) \\ &= \tilde{\phi}(g) \tilde{\phi}(h). \end{aligned}$$

Hence $\tilde{\phi}$ is a homomorphism. Clearly, $\tilde{\phi}$ extends $\phi$ and the corresponding diagram commutes. Observe that any homomorphism

$$\tilde{\phi} : F \to G,$$

that makes the diagram commutative, must satisfy the equality $(1.8)$, so $\tilde{\phi}$ is unique. This shows that $F$ satisfies the required universal property. Suppose now that a group $F = F_S$ with a generating set $S$ satisfies the universal property. Take $G = F(S)$ and define a map

$$\begin{aligned} \phi &: S \to G, \text{ by} \\ \phi(s) &= s, \text{ for each } s \in S. \end{aligned}$$

Then by the universal property $\phi$ extends to a unique homomorphism

$$\tilde{\phi}: F \to F(S).$$

Let $w$ be a nonempty reduced group word on $S$. Then $w$ defines an element $g$ in $F$ for which $\tilde{\phi}(g) = w \in F(S)$. Hence $\tilde{\phi}(g) \neq 1$ and $i(w) = g$, then $\tilde{\phi} \circ i(w) = \tilde{\phi}(g) = w = \phi(w)$, so that the diagram commutes

$$\begin{array}{ccc} S & \overset{i}{\hookrightarrow} & F_S \\ & \phi \searrow & \downarrow \tilde{\phi} \\ & & G \end{array}$$

□

## 3. Word problem and conjugacy problem

DEFINITION 3. *(Cyclically reduced word) Let $w = y_1 y_2 \cdots y_n$ be a word in the alphabet $S^{\pm 1}$. The word $w$ is cyclically reduced, if $w$ is reduced and $y_n \neq y_1^{-1}$.*

EXAMPLE 3. *The word $w = s_1 s_3 s_2^{-1}$ is cyclically reduced, whereas neither $u = s_1 s_2^{-1} s_1 s_3 s_2 s_1^{-1}$, nor $v = s_1 s_3^{-1} s_3 s_2^{-1}$ is a cyclically reduced.*

LEMMA 2. *The word and the conjugacy problem in a free group are solvable.*

Observed that there is an (obvious) algorithm to compute both reduced and cyclically reduced forms of a given word $w$. Our algorithm to solve the word problem is based on

PROPOSITION 2. *A word $w$ represents the trivial element in $F(S)$ iff the reduced form of $w$ is the empty word.*

Two cyclically reduced words are conjugate iff one is cyclic shift of the other.

EXAMPLE 4. *$u = s_1 s_2 s_3 s_4 s_2^{-1} s_1^{-1}$, then $u$ is conjugate to $s_3 s_4$, i.e.*

$$\begin{aligned} u &= (s_1 s_2) s_3 s_4 \left(s_2^{-1} s_1^{-1}\right) \\ &= (s_1 s_2) s_3 s_4 (s_1 s_2)^{-1}. \end{aligned}$$

EXAMPLE 5. *$s_3 s_4 s_3^{-1} s_5$ is conjugate to $s_3^{-1} s_5 s_3 s_4$, i.e.*

$$(s_3 s_4)^{-1} \left(s_3 s_4 s_3^{-1} s_5\right) s_3 s_4.$$

*which is of the form $uv = u (vu)^{-1} u$.*

## 4. The isomorphism problem

THEOREM 3. *Let $G$ be freely generated by a set $S$, and let $H$ be freely generated by a set $U$. Then $G \cong H$ iff $|S| = |U|$.*

PROOF. Let $G \cong H$. Let $K = \left\langle g^2, g \in G \right\rangle \subseteq G \lhd G$, where $K$ is the subgroup generated by the squares.

$$\left(g_1 g g_1^{-1}\right)^2 = \left(g_1 g g_1^{-1}\right)\left(g_1 g g_1^{-1}\right) = g_1 g^2 g_1^{-1}.$$

Consider the quotient group $\frac{G}{K}$, we show that $\frac{G}{K}$ is Abelian. Take a commutator $[\bar{g}_1, \bar{g}_2] \in \frac{G}{K}$, then define a map

$$- : G \to \frac{G}{K},$$

and we show that the commutator has order 2.

$$\begin{aligned}[\bar{g}_1, \bar{g}_2] &= \bar{g}_1^{-1}\bar{g}_2^{-1}\bar{g}_1\bar{g}_2 = \bar{g}_1^{-1}\left(\bar{g}_1^2\bar{g}_2^2\right)\bar{g}_2^{-1}\bar{g}_1\bar{g}_2 \\ &= \left(\bar{g}_1\bar{g}_2\right)\bar{g}_1\bar{g}_2 = \left(\bar{g}_1\bar{g}_2\right)^2 = 1.\end{aligned}$$

Therefore $\frac{G}{K}$ is Abelian. If $s \in S$ then $\bar{s} \neq 1$ in $\frac{G}{K}$. This is different letters are send to different letters. Now $s_1 s_1^{-1}$ is not in $K$. So is reduced of length 2. This is

$$\frac{G}{K} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_2 = |S|.$$

Similarly for $\frac{H}{K_1}$, we have

$$\frac{H}{K_1} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_2 = |U|.$$

□

**Seifert-van Kampen Theorem**

Let $X$ be a path connected topological space. Suppose that $X = U \cup V$ where $U$ and $V$ are path-connected open subsets and $U \cap V$ is also path-connected. For any $x \in U \cap V$, the commutative diagram

$$\begin{array}{ccc} \pi_1(U \cup V, x) & \to & \pi_1(U, x) \\ \downarrow & & \downarrow \\ \pi_1(V, x) & \to & \pi_1(X, x) \end{array}$$

is a push out.

Definition 4. *The rose with $n$ petals, denoted $R_n$, is the graph with one vertex $*$ and $n$ oriented edges $e_1, \cdots, e_n$.*

Theorem 4. *Let $X$ be a rose with $|S|$ petals, that is, the wedge of $|S|$ copies of $S^1$ index by $S$. Then $\pi_1(X) = F_S$.*

Proof. Let $S$ be finite. The proof is by induction on $|S|$. If $S$ is empty, we take the wedge of 0 circles to be a point. Let $X$ be a wedge of $|S|$ circles, let $U$ be (a small open neighborhood of ) the circle corresponding to some fixed element $s_\circ$ and let $V$ be the union of the circles corresponding to $T = S \backslash s_\circ$. $\pi_1(U) \cong \mathbb{Z}$, $\pi_1(V) \cong F_T$ by induction. Let $i : S \to \pi_1(X)$ be the map sending $s$ to a path that goes around the circle corresponding to $s$.

Consider a set map $f$ from $S$ to a some group $G$. There is a unique homomorphism

$$\begin{aligned} f_1 &: \pi_1(U) \to G \ni \\ f_1 \circ i(s_\circ) &= f(s_\circ), \end{aligned}$$

and unique homomorphism

$$\begin{aligned} f_2 &: \pi_1(V) \to G \ni \\ f_1 \circ i(t) &= f(t), \end{aligned}$$

$\forall t \in T$. It follows from the Seifert-van Kampen theorem that there is a unique homomorphism

$$\widehat{f} : \pi_1(U) \to G,$$

extending $f_1$ and $f_2$. □

**Nielsen- Schreier Theorem**

This theorem implies that every free group is the fundamental group of a graph (i.e. a one dimensional CW complex). This has a strong converse.

THEOREM 5. *A group is free iff it is the fundamental group of a graph.*

It is enough to show that every graph is homotopy equivalent to a rose. Let $\Gamma$ be a graph, and let $T$ be a maximal subtree in $\Gamma$. Any tree is contractible to a point. Therefore $\Gamma$ is homotopy equivalent to a rose.

THEOREM 6. *Every subgroup of a free group is free.*

PROOF. Think of a free group $F$ as a fundamental group of a graph $X$. Let $H$ be a subgroup of $F$, and let $X^{'}$ be the covering space of $X$ corresponding to $H$. Then $X^{'}$ is a graph and $H = \pi_1 \left(X^{'}\right)$ so $H$ is free. □

DEFINITION 5. *The rank of an open surface is the least number of cuts required to make the surface homeomorphic to a disk.*

DEFINITION 6. *Alternative definition of a rank: Rank is the greatest number of non-intersecting cuts which can be made without making the surface disconnected.*

THEOREM 7. *(Schreier Index Formula) If $H$ is a subgroup of $F_r$ of finite index $k$ then the rank of $H$ is $1 + k(r-1)$.*

PROOF. Again, let $F_r = \pi_1(X)$ and let $H = \pi_1\left(X^{'}\right)$, where $X$ is the rose with $r$ petals and $X^{'}$ is a covering space of $X$. It is standard that

$$\chi\left(X^{'}\right) = k\chi(X), \tag{4.1}$$

Euler characteristic class function. If $X$ is the rose with $r$ petals it is clear that $\chi(X) = 1 - r$. Similarly, the rank of $H$ is $1 - \chi\left(X^{'}\right)$. Then we have $1 - \chi\left(X^{'}\right) = k(1-r) \Longrightarrow$ by (1.9) that

$$1 + k(r-1) = \chi\left(X^{'}\right),$$

□

Hanna Neumann conjecture (˜1954).

$\boxed{\text{Conjecture:}}$ Let $H, K \leq F(X)$ be two nontrivial finitely generating subgroups of a free group $F(X)$ and let $L = H \cap K$ be the intersection of $H$ and $K$. The conjecture says that in this case

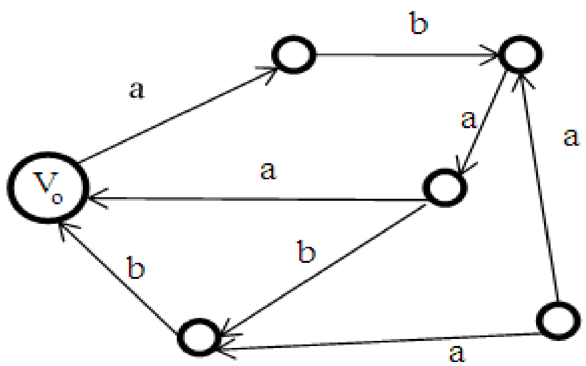
$$rank(L) - 1 \leq (rank(H) - 1)(rank(K) - 1).$$

Here the $rank(G)$ = the smallest size of a generating set for $G$, (where $G$ is a group). Now the $rank$( free group) =the size of any free basis of the free group.

THEOREM 8. *Every subgroup $H \leq F$ of a free group $F$ is free, and given generators of $H$ we can compute its basis.*

Before proving this theorem we need to develop some machinery that will allow us to associate to any subgroup of a free group an automaton, i.e. a finite oriented labeled graph, that accepts only elements of $H$. Such an automaton is called $\Gamma(H)$- the Stallings graph of $H$.

Example 6. *$\Gamma(H)$ for $H = \langle aba^2, a^{-1}b^2, aba^{-2}b \rangle$ :*




Remark 1. *Traversing an edge $\xrightarrow{a}$ forward, i.e. along its direction, we read $a$, and traversing it backward, we read $a^{-1}$.*

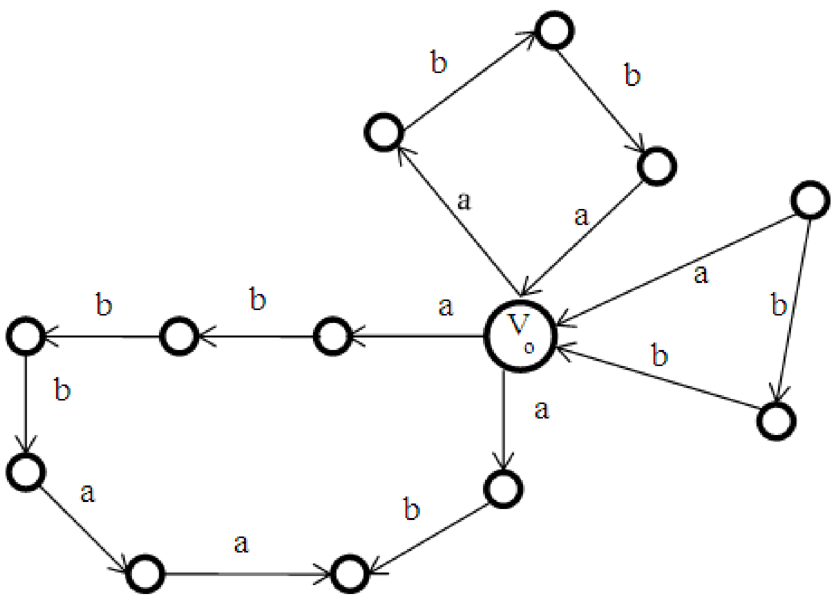
Definition 7. *One way reading property $(OR)$- no two edges outgoing from a vertex are labeled by the same symbol.*

Definition 8. *A path in $\Gamma(H)$ is a sequence $e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n}$, where $e_i$ are edges and $\epsilon_i = \pm 1$. We say that a path is reduced if it contains no subpaths $e_i e_i^{-1}$ or $e_i^{-1} e_i$. The subgroup $H$ corresponds to labels of loops beginning at $V_\circ$ in $\Gamma(H)$.*

$\boxed{\text{Fact:}}$ Every element of $H$ is a loop from $V_\circ$ in $\Gamma(H)$.

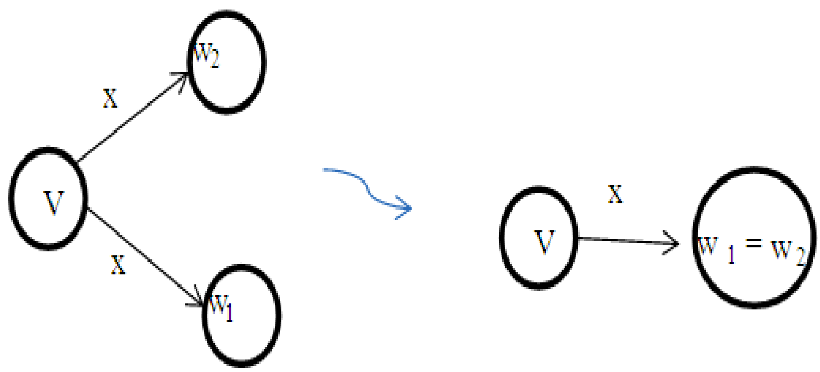
This gives an easy procedure to decide whether $g \in H$ of not. However, how does one construct $\Gamma(H)$, given $H \leq F$?

Example 7. *Suppose $H = \langle ab^2a, a^{-1}b^2, aba^{-2}b^{-3}a^{-1} \rangle$.*

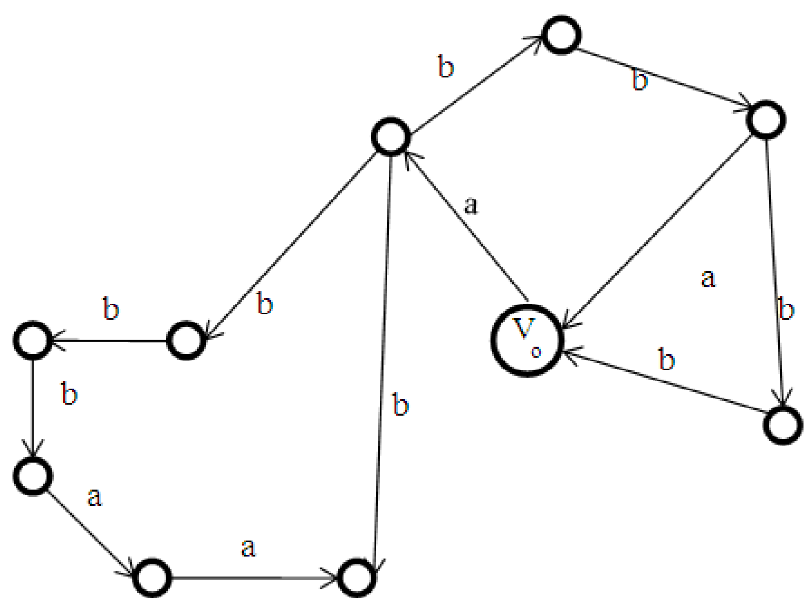



(♠)

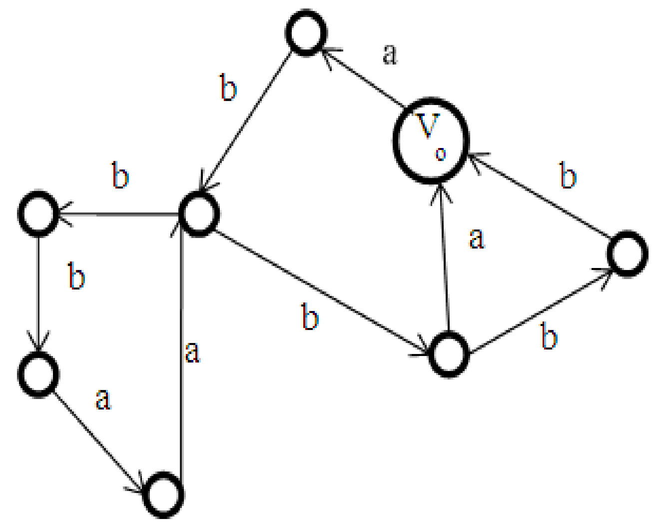
*we use the following reduction with $H$,*




*so that* (♠) *is of the form*



*or*

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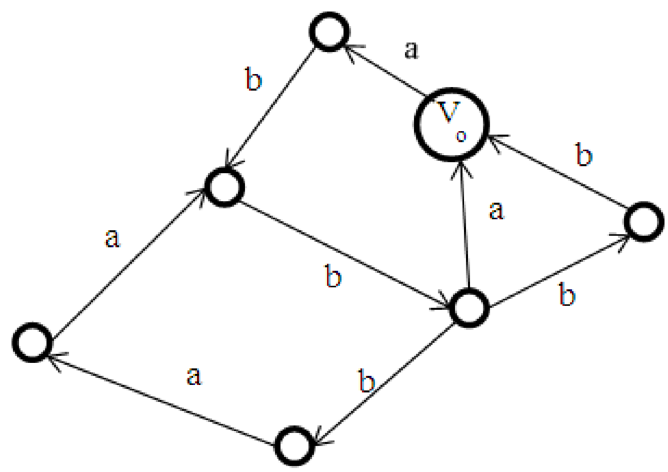


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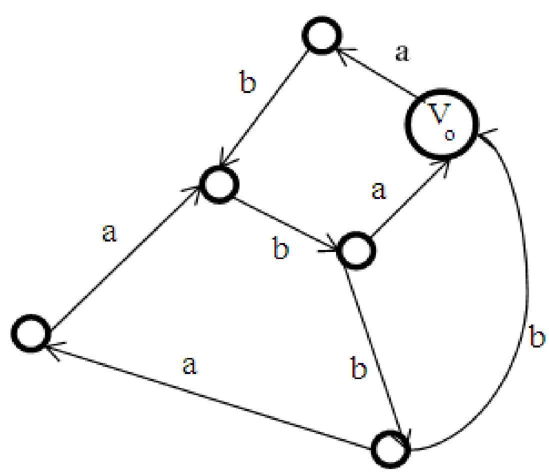


*which is the final result* $\Gamma(H)$, *the graph has the* $(OR)$ *property.*

Definition 9. *An oriented graph* $\Gamma$ *consists of a set of vertices* $V(\Gamma)$, *a set of edges* $E(\Gamma)$, *and two functions,*

$$\begin{array}{rcl} E & \to & V \times V \\ e & \mapsto & (\alpha(e), \tau(e)), \ \ (\textit{endpoints of } e), \end{array}$$

*and*

$$\begin{array}{rcl} E & \to & E \\ e & \mapsto & \bar{e} := (\tau(e), \alpha(e)), \ \ (\textit{inverse of } e). \end{array}$$

Definition 10. *For the graph,*

1. $u, v \in V$ *are adjacent if* $\exists\ (u,v) \in E$.
2. *A path in* $\Gamma$ *is a sequence of edges* $e_1 e_2 \cdots e_n \ \ni \tau(e_i) = \alpha(e_{i+1})$ *for* $i = 1, \cdots, n-1$.
3. *A path is simple if all* $\alpha(e_i)$ *are distinct.*
4. *A path is a loop if* $\alpha(e_1) = \tau(e_n)$.
5. *A path is reduced if* $\bar{e}_i \neq e_{i+1}\ \forall i$.
6. *A graph is connected if there is a path between any two vertices.*
7. *A graph is a forest if it does not have simple loops.*
8. *A graph is a tree if it is a connected forest.*

Remark 2. *In a tree* $T, \forall\ u, v \in V(T), \exists!$ *reduced path joining* $u$ *and* $v$.

Proof. Sketch. If $p_1, p_2$ are two different paths, $p_1\overline{p_2}$ is a loop. Removing pairs $e\bar{e}$ from it makes it reduced. Contradiction. □

Definition 11. *This unique path is called a geodesic between* $u$ *and* $v$.

By Zorn's Lemma, every connected graph $\Gamma$ has a maximal subtree $T^*$. If $\Gamma$ is finite, there is an algorithm to construct $T^*$ :

1. Take an edge.
2. Add edges without forming a simple loop.
3. Stop when no more edges can be added.

**Labeled graphs with orientation**

Definition 12. *Let* $G = (V, E)$ *be a graph.*

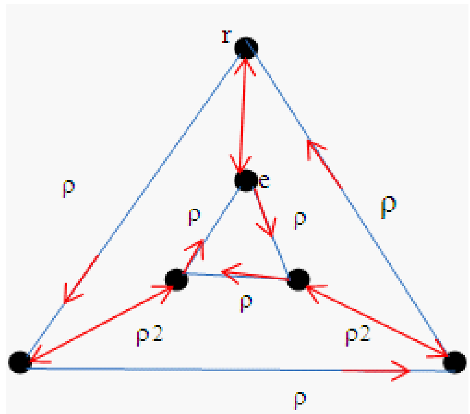
1. *An orientation is* $E_+ \subset E$ *such that*
$$\begin{array}{rcl} E_+ \cap \overline{E_+} & = & \varnothing, \ E_+ \cup \overline{E_+} = E, \ \textit{if} \\ e & = & (u,v), \ \textit{then } \bar{e} = (v,u). \end{array}$$
2. *Given an orientation* $E_+$, *and an alphabet* $S = \{a, b, \cdots\}$. *We set* $S^{-1} := \{a^{-1}, b^{-1}, \cdots\}$. *A labeling is a function*
$$\lambda : E_+ \to S,$$
*such that* $\forall e \in E_+$, $(\lambda(e) = x) \Longrightarrow (\lambda(\bar{e}) = x^{-1})$.

Definition 13. *A Cayley graph of a group* $G = \langle S | R \rangle$ *is the labeled graph* *Cayley*$(G) = (V, E, \lambda)$, *where* $V = G$ *and*

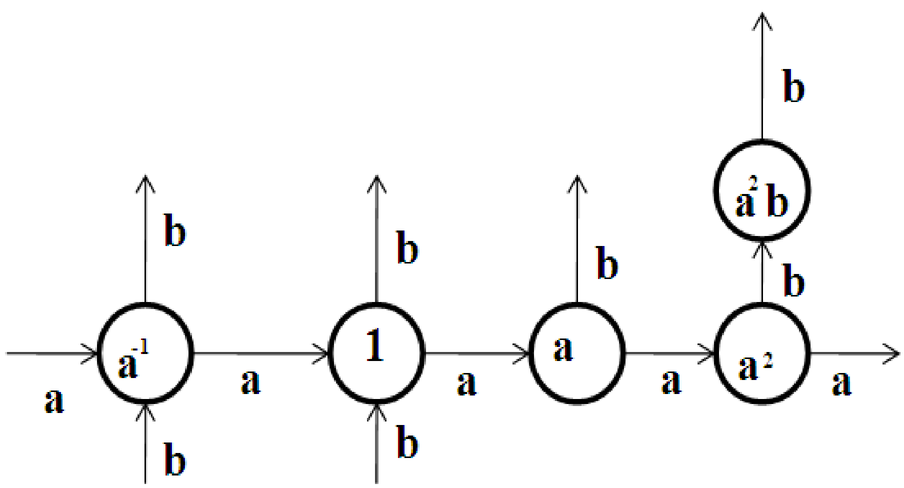
$$(\forall s \in S)(\forall g \in G)(\exists e \in E_+)(e = (g, gs), \lambda(e) = s).$$

EXAMPLE 8. *Cayley*$(D_3)$ *where* $D_3 = \left\langle r, \rho | \rho^3 = 1, r^2 = 1, \rho^3 \rho r = 1, \rho r = r\rho^{-1} \right\rangle$, *where* $|D_3| = 6$. *Then we see that*

$$D_3 = \left\{e, \rho, \rho^2, r, \rho r, \rho^2 r\right\}.$$




EXAMPLE 9. *Let* $F = F(a, b)$ *be a free group on two elements. Then Cayley* $(F)$ *is*




REMARK 3. *Cayley*$(G)$, *where* $G = \langle S|R \rangle$ *is a tree iff* $G$ *is a free group with basis* $S$.

PROOF. If Cayley$(G)$ is a tree, then every word in $S^{\pm}$ corresponds to a unique vertex in the graph, and this vertex is different from 1. Hence, this word is different than 1.

Conversely, if Cayley$(G)$ is not a tree, $\exists$ a simple loop from some $g \in G$ :

$$\begin{aligned} & e_1 e_2 \cdots e_n, \\ \tau(e_n) &= \alpha(e_1) = g. \end{aligned}$$

Thus, $g = g\lambda(e_1 e_2 \cdots e_n) = g\lambda(e_1)\lambda(e_2)\cdots\lambda(e_n)$, implies that

$$\lambda(e_1)\cdots\lambda(e_n) = 1.$$

That is there is a reduced word in $G$ that is equal to the identity, hence, $G$ is not free. $\square$

Let $H \leq F = F(S)$ be defined as $H = \langle g_1, \cdots, g_k \rangle$. Construct $\gamma(H)$ by drawing loops corresponding to the generatrix starting from $V_\circ$. As the generating set $H$ is finite, this process must stop, so we will end up with $\Gamma(H)$-the Stalling's graph of $H$ with the (OR) property.

In $\gamma(H)$, $g \in H$ iff $g$ is a label of a loop at $V_\circ$ in $\gamma(H)$. This property is preserved during foldings, thus it is also true for $\Gamma(H)$. This proves that fact above.

**Schreir's graph**

The graph of right cosets of $H$, denoted by $\Gamma_\circ(H)$, is called the Schreir's graph of $H$.

$$\begin{aligned} V(\Gamma_\circ) &= \frac{G}{H} = \{Hg | g \in \text{set of right cosets representatives}\}, \\ \forall Hg, \forall s &\in S, \exists e \in E_+,\ e = (Hg, Hgs),\ \lambda(e) = s. \end{aligned}$$

Theorem 9. *If $F = F(S)$, $H \leq F$, then $\Gamma(H)$ is the minimal subgraph of $\Gamma_\circ(H)$, containing all loops at $V_\circ$ in $H$.*

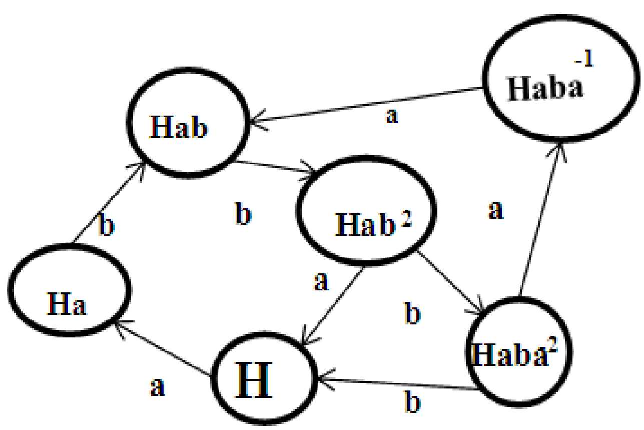
Proof. Sketch of the proof. $Core(\Gamma_\circ(H)) = \{\text{minimal subgraph containing all simple loops at vertex } V_\circ\}$. Then every coset corresponds to a vertex.

$$\Gamma(H) \subseteq Core(\Gamma_\circ(H)).$$

If $s_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n}$ is label of a loop of $\Gamma(H)$ then $Hs_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n} = H$, since $s_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n} \in H$.

Conversely, suppose we have a loop in $Core(\Gamma_\circ(H))$ then by definition $Core(\Gamma_\circ(H))$ contains all simple loops of $H$ at $V_\circ$. $\square$

Example 10. $\Gamma(H)$ *for* $H = \left\langle ab^2a, a^{-1}b^2, aba^{-2}b, a^{-1}b^{-1}a^{-2}b \right\rangle$



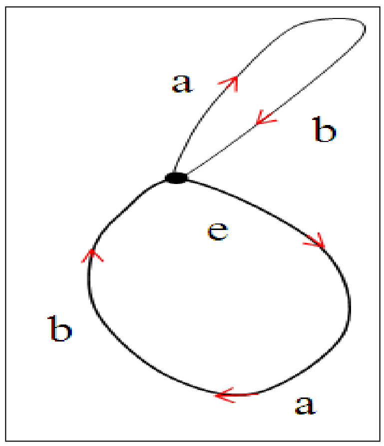

A label graph $\Gamma$ is complete if for any $v \in \Gamma, \forall s \in S_{\text{alphabet}}$ $\exists$ edge labeled by $s$ and $s^{-1}$ outgoing from $V_\circ$.

Exercise 1. *Prove* $[F : H] < \infty \Leftrightarrow \Gamma(H)$ *is complete.*

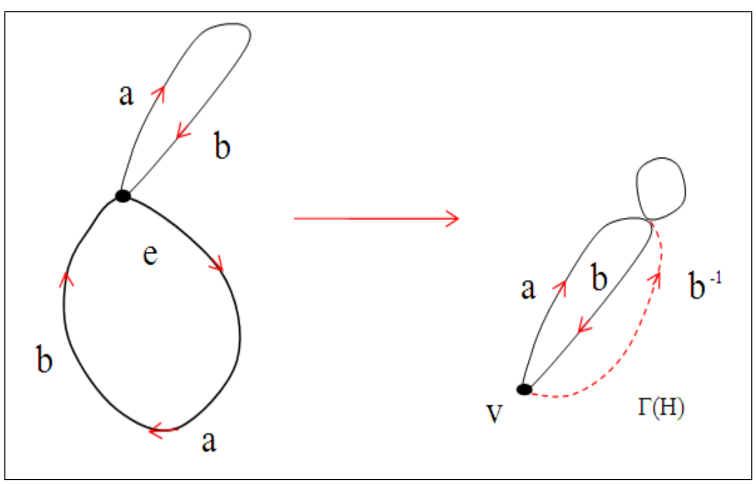
# Chapter 2

Theorem 10. *Every subgroup $H \leq F$ of a free group $F$ is free. Given a finite number of generators of $H$ we can compute its basis.*

Example 11. *Free group $F = F(a, b)$, $H = \langle ab, a^2 b \rangle$,*




*and then we form*




**Schreier's graph**

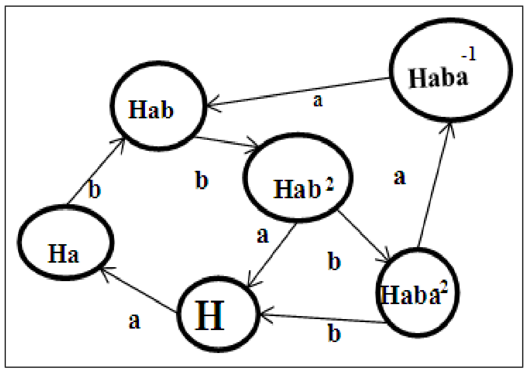
The graph of right cosets of $H$, denotes by $\Gamma_\circ(H)$, is called the Schreir's graph of $H$.

$$V(\Gamma_\circ) = \frac{G}{H} = \{Hg | g \in \text{set of right coset representatives}\},$$

$\forall Hg$, $\forall s \in S$, $\exists e \in E_+, e = (Hg, Hgs)$, $\lambda(e) = s$.

THEOREM 11. *If $F = F(S)$, $H \leq F$, then $\Gamma(H)$ is isomorphic to the minimal subgraph of $\Gamma_\circ(H)$, containing all loops at $v_\circ = H$, (we call that subgraph $Core(\Gamma_\circ(H))$. )*

EXAMPLE 12. *$\Gamma(H)$ for $H = \left\langle ab^2a, a^{-1}b^2, aba^{-2}b, a^{-1}b^{-1}a^{-2}b \right\rangle$, has three simple loops*

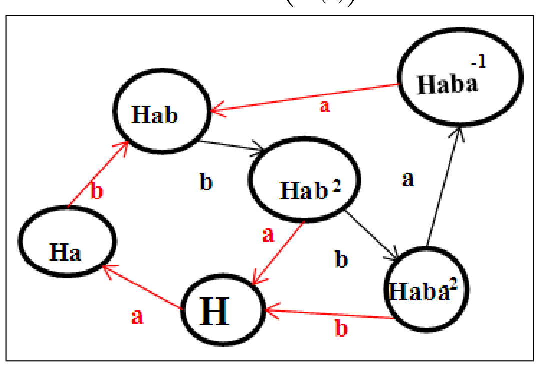



$\boxed{\textbf{Algorithm for finding a basis for H}}$

Let $H \leq F(a,b)$. Now for example $H = \left\langle ab^2a, a^{-1}b^2, aba^{-2}b, a^{-1}b^{-1}a^{-2}b \right\rangle$.

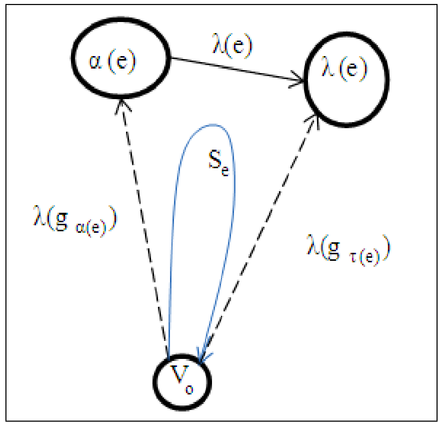
(1) Construct $\Gamma(H)$.
(2) Let $T$ be a maximal subtree in $\Gamma(H)$ ( not unique).
(3) If $v \in V(\Gamma)$, denote by $g_v$ the unique geodesic from $v_\circ$ to $v$ in $T$.
(4) Let $e \in E_+(\Gamma \backslash T)$ be a positively oriented edge that is not on the tree $T$. Set

$$S_e = \lambda\left(g_{\alpha(e)}\right)\lambda(e)\lambda\left(g_{\tau(e)}^{-1}\right) \Longrightarrow S_e \in H$$



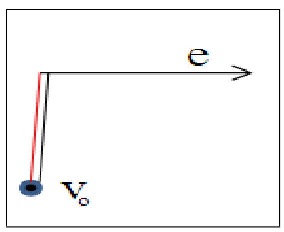

EXAMPLE 13. *For example in $H_\circ$,*

$$S_e = \underbrace{b^{-1}}_{\lambda\left(g_{\alpha(e)}\right)} \overbrace{a}^{\lambda(e)} \underbrace{ab^{-1}a^{-1}}_{\lambda\left(g_{\tau(e)}^{-1}\right)}$$




REMARK 4. *If $e \notin E_+ (\Gamma \backslash T)$, we define $S_e$ similarly, but then $S_e = 1$ (the empty word). Indeed, by uniqueness of geodesic,*

$$g_{\tau(e)}^{-1} = e^{-1} g_{\alpha(e)}^{-1}.$$




CLAIM 1. *The set $\{S_e | e \in E_+ (\Gamma \backslash T)\}$ form a free basis of $H$.*

PROOF. Let $w \in H$ be arbitrary. Since $\Gamma(H)$ accepts $w$. There exist a reduced loop at $v_\circ$, $e_1 e_2 \cdots e_n$ such that

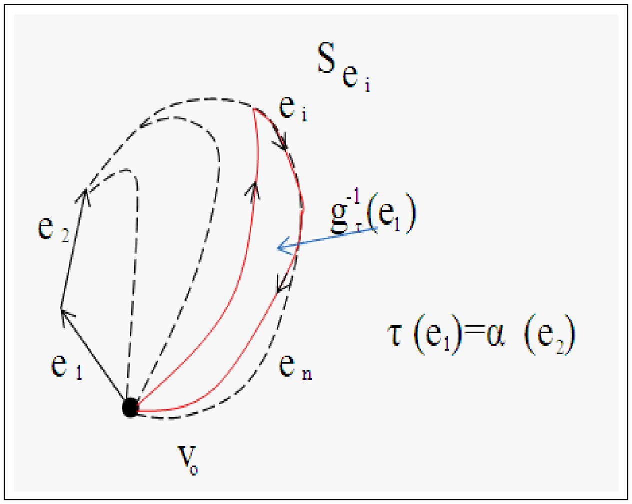
$$\lambda(e_1 e_2 \cdots e_n) = w.$$

But this loop can be written as

$$\begin{aligned}\prod_{i=1}^{n} \left(g_{\alpha(e_i)} e_i g_{\tau(e_i)}^{-1}\right) &= \left(g_{\alpha(e_1)} e_1 g_{\tau(e_1)}^{-1}\right) \cdots \left(g_{\alpha(e_n)} e_n g_{\tau(e_n)}^{-1}\right) \\ &= \left(g_{\alpha(e_1)} \left[\prod_{i=1}^{n} e_i\right] g_{\tau(e_n)}^{-1}\right) = \left[\prod_{i=1}^{n} e_i\right].\end{aligned}$$

Because $g_{\alpha(e_n)} = g_{\tau(e_{n-1})}$, $g_{\alpha(e_{n-1})} = g_{\tau(e_{n-2})}, \cdots$, $g_{\alpha(e_2)} = g_{\tau(e_1)}$ are trivial. Thus

$$\prod_{i=1}^{n} S_{e_i} = w.$$




However, as $S_{e_i} = 1$ if $e_i \in T$, the set $\{S_e | e \in E_+ (\Gamma \backslash T)\}$ is generating.

On the other hand, for the product $S_{e_1} S_{e_2}$, where $e_1 \neq e_2^{-1}$, $e_1, e_2 \in E_+ (\Gamma \backslash T)$, the corresponding path is

$$P = \underbrace{\left( g_{\alpha(e_1)} e_1 g_{\tau(e_1)}^{-1} \right)}_{S_{e_1}} \underbrace{\left( g_{\alpha(e_2)} e_2 g_{\tau(e_2)}^{-1} \right)}_{S_{e_2}}.$$

Every edge in $g_{\alpha(\bullet)}$, $g_{\tau(\bullet)}$ belongs to the tree, while $e_1 \neq e_2^{-1}$ do not belong to the tree. A geodesic in the tree but not in the graph. Thus

$e_1$, and $e_2$ cannot be cancelled in $P$. This implies that $S_{e_1} S_{e_2} \neq 1$. Hence

$$\{S_e | e \in E_+ (\Gamma \backslash T)\},$$

is a basis of $H$ and

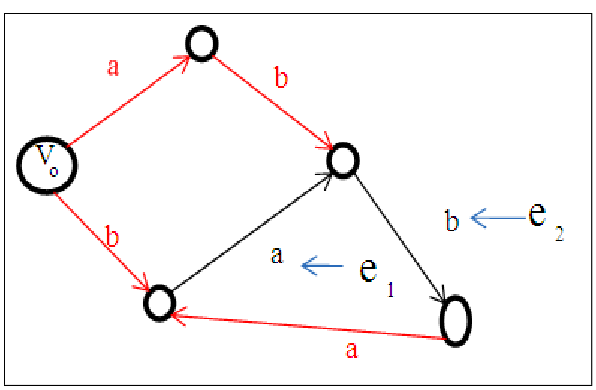
$$H = \langle S_e | e \in E_+ (\Gamma \backslash T) \rangle$$

$\square$

Corollary 1. *Every subgroup of a free group is free, rank* $(H) = E - V + 1$.

Example 14. *The rank of a tree is given by: rank* $(T) = E_T - V_T + 1 = 0$.

Example 15. *Consider* $F = F(a, b)$, *and* $H = \left\langle aba^{-1}b^{-1}, ab^2ab^{-1}, babab^{-1} \right\rangle$; *H has only two basis elements. Construct* $\Gamma(H)$ *and fold it.*



*The red edges indicate the maximal subtree chosen. Thus* $E_T = \{e_1, e_2\}$ *and*

$$S_{e_2} = b\,(a)\,b^{-1}a^{-1},\ S_{e_1} = ab\,(b)\,ab^{-1}.$$

*Then, the rank of* $H$ *is* 2.

Definition 14. *The graph* $\Gamma(H)$ *is called complete (or* $S-$ *regular) if* $\forall v \in \Gamma(H)$ *and for* $\forall s \in S$ *(the alphabet), there exists an edge labelled by* $s$ *and* $s^{-1}$ *from* $v$.

Theorem 12. *Suppose* $F, H$ *are finitely generated, then*

$$[F : H] < \infty \Leftrightarrow \Gamma(H) \text{ is complete,}$$

*in that case,* $[F : H] = |\Gamma(H)| = |V(\Gamma)|$.

Proof. $(\Longleftarrow)$

Suppose $\Gamma(H)$ is complete. Recall that $\Gamma(H) \subset \Gamma_\circ(H)$, the graph of right cosets of $H.\Gamma(H)$ implies that $\Gamma_\circ(H) = \Gamma(H)$. Indeed, for any coset $Hg$, $g$ is a reduced word in $F(S)$. By completeness, we can read $g$ as path (not necessarily a loop) in $\Gamma(H)$. The label of this path is a representative of $Hg$, which means that

$$\begin{aligned} \Gamma_\circ(H) &\subset \Gamma(H) \Longrightarrow \\ \Gamma_\circ(H) &= \Gamma(H). \end{aligned}$$

$H$ is finitely generated.

$$\begin{aligned} |\Gamma(H)| &< \infty \Longrightarrow \\ |\Gamma_\circ(H)| &< \infty \Longrightarrow \\ [F : H] &< \infty. \end{aligned}$$

$(\Longrightarrow)$

If $[F : H] < \infty$, $\Gamma_\circ(H) = Core(\Gamma_\circ(H))$ because every reduced word in $F$ is a beginning of the label of a loop at $H$. Therefore

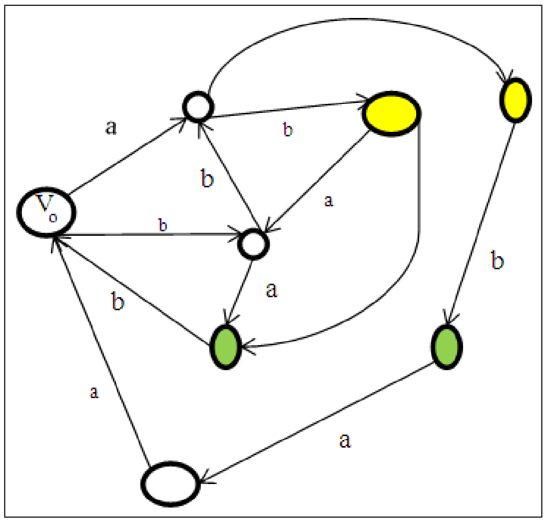
$$\Gamma_\circ(H) = \Gamma(H),$$

and $\Gamma(H)$ is complete. □

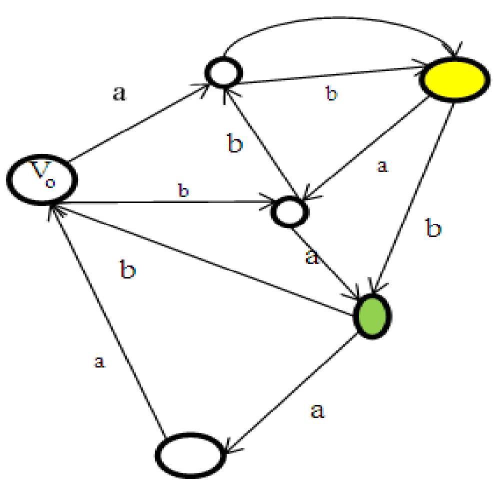
Example 16. $F = F(a, b)$, *and* $H = \left\langle ab^{-1}ab, a^2ba^2, ab^{-2}, ababa^{-1}, ba^{-1}b^2, a^2b^{-3}ba^{-1} \right\rangle$,

(1) Construct $\Gamma(H)$..
(2) Find $[F : H]$.
(3) The number of elements in the basis for $H$ is the rank of $H$, $rank(H) = E - V + 1$.
(4) Is $w = a^3ba^{-1}ba \in H$?

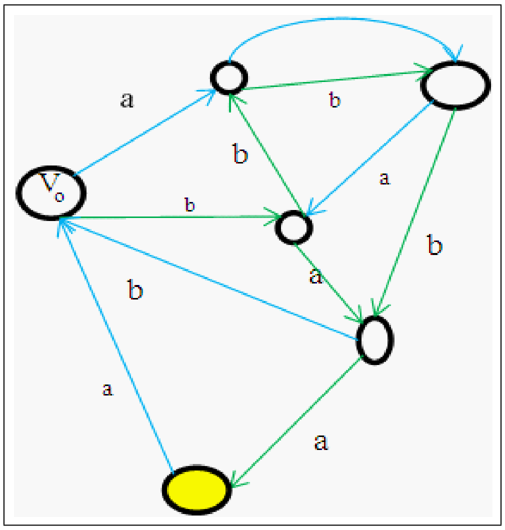
PROOF. $\Gamma(H)$ is given by the




As $a^2b^{-3} \in H$, we need to identify the yellow vertices, which implies that the green vertices must be identify too. This produces the final result:



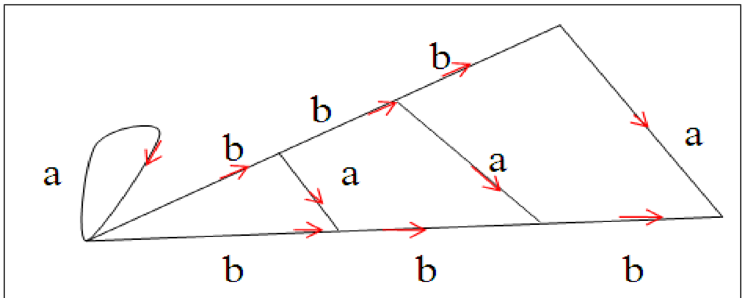
Thus, the rank of $H$ is the number of fundamental simple loops =6. To find the basis let us choose a maximal subtree (blue edges)




There are 6 non tree edges (green) and the basis corresponding to them is $\left\{aa\left[b^{-1}\right]a^{-1}, aa\left[a\right]ba^{-1}, aa\left[b\right]a^{-1}ba^{-1}, \left[b\right]ba^{-1}, ab^{-1}a\left[b\right], ab^{-1}a\left[a\right]a\right\}.$

There is a vertex of degree 2 (yellow), so $\Gamma(H)$ is not complete, i.e. $[F:H]=\infty$. Also, $w\notin H$, as it is not a label of a loop around $v_\circ$. □

Example 17.




*You can go forever.*

**Spanning trees and free bases**

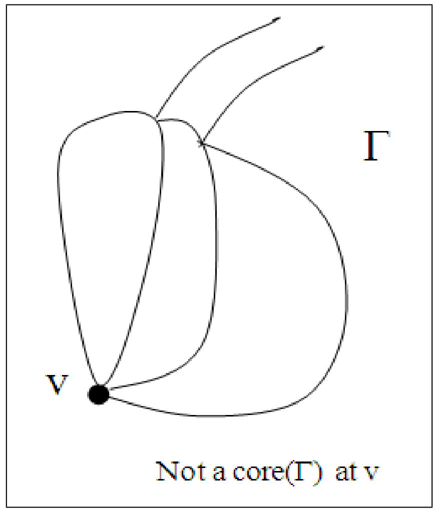
We will use notation $G=L(\Gamma,v)$ meaning that the group $G$ is a language accepted by the automaton $\Gamma(G)$. We say that $\Gamma$ is folded if it has OR property. When we say that $\Gamma$ is connected we mean that it is connected as a non directed graph.

Definition 15. *Let $\Gamma$ be an $X-$ digraph and let $v$ be a vertex of $\Gamma$. Then the core of $\Gamma$ at $v$ is defined as :*

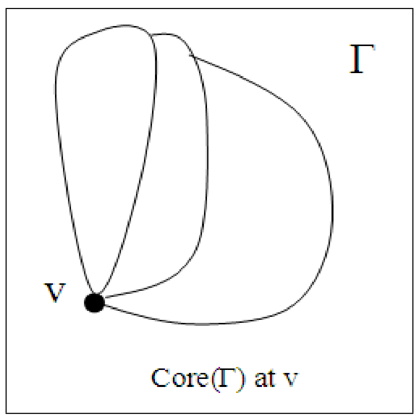
$$Core(\Gamma,v)=\cup\{p|\textit{where } p \textit{ is a path in } \Gamma \textit{ from } v \textit{ to } v\}.$$

*It is easy to see that $Core(\Gamma,v)$ is a connected subgraph of $\Gamma$ containing $v$. If $Core(\Gamma,v)=\Gamma$ we say that $\Gamma$ is a core graph with respect to $v$.*

EXAMPLE 18.




*but*




DEFINITION 16. *(Nielsen set) Let $S$ be a set of nontrivial elements of the free group $F(X)$ such that $S \cap S^{-1} = \varnothing$. We say that $S$ is Nielsen reduced with respect to the free basis $X$ if the following conditions hold:*

(1) *If $u, v \in S \cup S^{-1}$ and $u \neq v^{-1}$ then $|u \bullet v|_X \geq |u|_X$ and $|v \bullet u|_X \geq |v|_X$.*
(2) *If $u, v, w \in S \cup S^{-1}$ and $u \neq w^{-1}$, $v \neq w^{-1}$ then $|u \bullet w \bullet v|_X > |u|_X + |v|_X - |w|_X$.*

REMARK 5. *Condition (1) means that no more than a half of $u$ and no more than a half of $v$ freely cancels in the product $u \bullet v$. Condition (2) means that at least one letter of $w$ survives after all free cancellations in the product $u \bullet w \bullet v$.*

DEFINITION 17. *(Geodesic tree) Let $\Gamma$ be a connected graph with a base vertex $v$. A subtree $T$ in $\Gamma$ is said to be geodesic relative to $v$ if $v \in T$ and for any vertex $u$ of $T$ the path $[v, u]_T$ is geodesic in $\Gamma$, that is a path of the smallest possible length in $\Gamma$ from $v$ to $u$.*

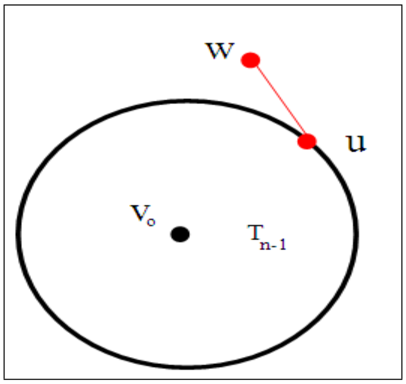
How to construct a Geodesic tree? Answer: by induction.
Let $B(v, n)$ be a ball of radius $n$ in $\Gamma$, this implies that

$$B(v, n) = \{u \in \Gamma | \, [u, v]_T \leq n\}.$$

Where $v \in T$; $VT_n = B(v, n)$. Suppose that $T_{n-1}$ is constructed and that the path $[w, v] = n$. Then there exists $e = (u, w)$, $[u, v] = n - 1$. Now add $e$ to $T$. Then we do it for all vertices so that

$$T = \bigcup_{i=1}^{\infty} T_n,$$

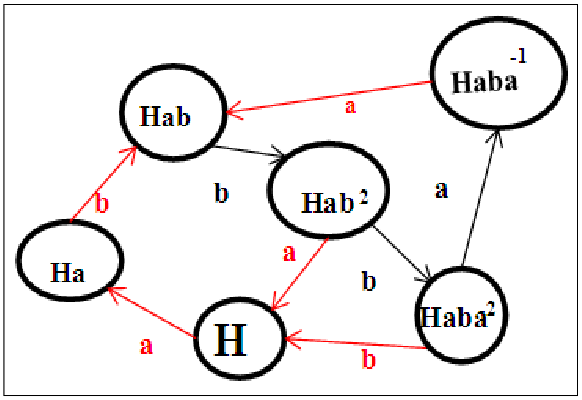
and such a tree can be constructed.



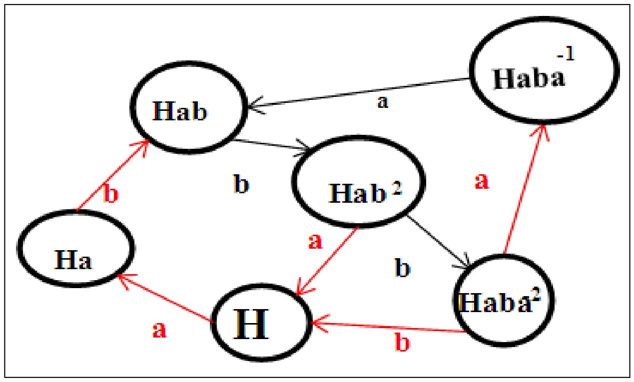

It is easy to see that geodesic spanning trees always exist:

Lemma 3. *Let $\Gamma$ be a graph (whether finite or infinite) with a base vertex $v$. Then there exist a geodesic relative to $v$ spanning tree $T$ for $\Gamma$.*

Example 19. *The first spanning tree is not geodesic*



*but if we choose that following,*




*it is.*

PROPOSITION 3. *Let $\Gamma$ be a folded $X-$ digraph which is a core graph with respect to a vertex $v$ of $\Gamma$. Let $H = L(\Gamma, v) \leq F(X)$ and let $T$ be a spanning tree in $\Gamma$ which is geodesic with respect to $v$. Then the set $Y_T$ is a Nielsen reduced free basis of the subgroup $H$.*

## 1. Embeddings of free groups

We say that a group $G$ embeds into a group $H$, if there is a monomorphism

$$\phi : G \to H.$$

If $\phi(G) \subsetneqq H$, then we say that $G$ properly embeds into $H$ and that $\phi$ is a proper embedding.

PROPOSITION 4. *Any countable free group $G$ can be embedded into a free group of rank* 2.

PROOF. To prove the result it suffices to find a free subgroup of countable rank in a free group of rank 2. Let $F_2$ be a free group with a basis $\{a, b\}$. Denote

$$x_n = b^n a b^{-n} \ \ (n = 0, 1, 2, \cdots)$$

and let $S = \{x_\circ, x_1, x_2, \cdots\}$. We claim that $S$ freely generates the subgroup $\langle S \rangle$ in $F_2$. Indeed, let

$$w = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_n}^{\epsilon_n},$$

be reduced nonempty word in $S^{\pm}$. Then $w$ can also be consider a word in $\{a, b\}$. Indeed,

$$w = b^{i_1} a^{\epsilon_1} b^{-i_1} b^{i_2} a^{\epsilon_2} b^{-i_2} \cdots b^{i_n} a^{\epsilon_n} b^{-i_n}.$$

Since $w$ is a reduced word in $S$, we have that either $i_j \neq i_{j+1}$, or $i_j = i_{j+1}$ and $\epsilon_j + \epsilon_{j+1} \neq 0$, for each $j = 1, 2, \cdots, n-1$. In either case, any reduction of $w$ (as a word in $\{a, b\}$) does not affect $a^{\epsilon_j}$ and $a^{\epsilon_{j+1}}$ in the subword

$$b^{i_j} a^{\epsilon_j} b^{-i_j} b^{i_{j+1}} a^{\epsilon_{j+1}} b^{-i_{j+1}},$$

i.e., the literals $a^{\epsilon_j}$ and $a^{\epsilon_{j+1}}$ are present in the reduced form of $w$ as a word in $\{a, b\}^{\pm 1}$. Hence the reduced form of $w$ is nonempty, so $w \neq 1$ in $F_2$. Clearly, $\langle S \rangle$ is a free group of countable rank. $\square$

## 2. Free products

Given a family of groups $\{G_i|i \in I\}$ we may assume that the $G_i$ are mutually disjoint sets. Let $X = \bigcup_{i\in I} G_i$ and let $\{1\}$ be a one element set disjoint from $X$. A word in $X$ is a finite string $(a_1, a_2, \cdots, a_n, 1, 1, \cdots) = a_1 a_2 \cdots a_n$, where $a_i \in X$. A word is reduced provided

(1) no $a_i$ is the identity in its group $G_j$,
(2) for all $i$, $a_i$ and $a_{i+1}$ are not in the same $G_j$,
(3) $a_k = 1$ implies $a_i = 1$ for $i \geq k$.

Let $\prod^{*}_{i\in I} G_i$ or $G_1 * G_2 * \cdots * G_n$ be the set of all reduced words on $X$. Then $\prod^{*}_{i\in I} G_i$ forms a group, called the free product of the family $\{G_i|i \in I\}$ under the operation "justaposition +cancellation+contraction",

Example 20. *If $a_i b_i \in G_i$ then $(a_1 a_2 a_3)\left(a_3^{-1} b_2 b_1 b_3\right) = a_1 c_2 b_1 b_3$, where $c_2 = a_2 b_2 \in G_2$.*

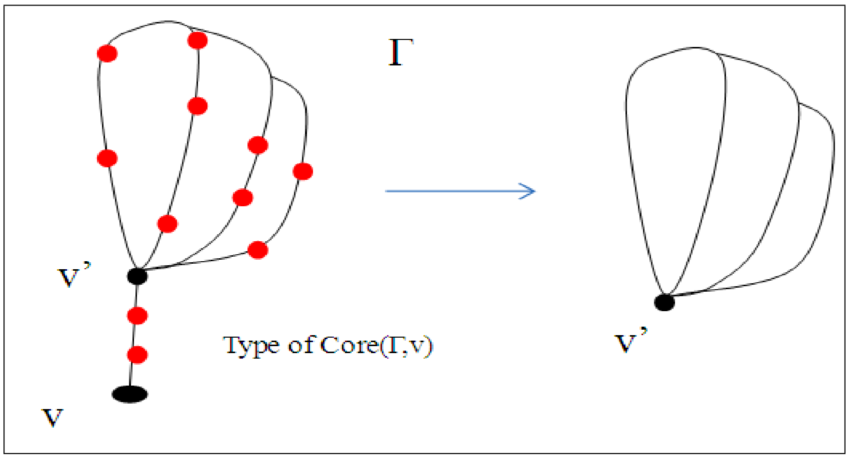
## 3. Basic properties of subgroup graph

Definition 18. *(Type of a core graph) Let $\Gamma$ be a folded $X-$ digraph which is a core graph with respect to some vertex. Suppose that $\Gamma$ has at least one edge. If every vertex of $\Gamma$ has degree at least two, we set the type of $\Gamma$, denoted $Type\left(\Gamma\right) = \Gamma$. Suppose now that $\Gamma$ has a vertex $v$ of degree one (such a vertex is unique since $\Gamma$ is a core graph). Then there exists a unique vertex $v^{'}$ of $\Gamma$ with the following properties:*

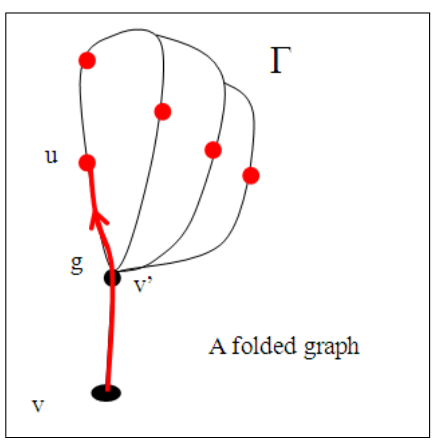
(1) *There is a unique $\Gamma-$geodesic path $\left[v, v^{'}\right]$ from $v$ to $v^{'}$ and every vertex of this geodesic, other than $v$ and $v^{'}$, has degree two.*
(2) *The vertex $v^{'}$ has degree at least three.*

*Let $\Gamma^{'}$ be the graph obtained by removing from $\Gamma$ all the edges of $\left[v, v^{'}\right]$ and all the vertices of $\left[v, v^{'}\right]$ except for $v^{'}$. Then $\Gamma^{'}$ is called the type of $\Gamma$ and denoted $Type\left(\Gamma\right)$. Finally, if $\Gamma$ consists of a single vertex, we set $Type\left(\Gamma\right) = \Gamma$.*

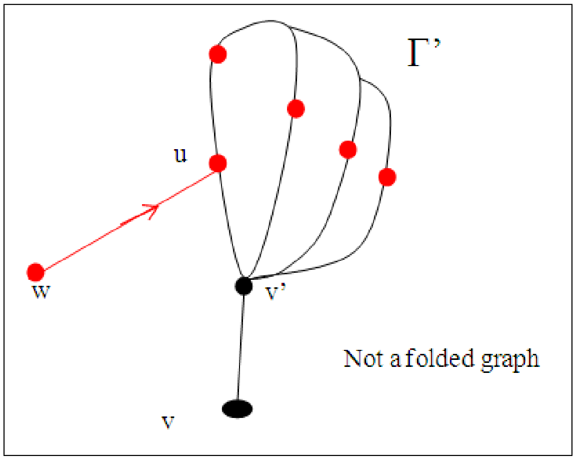
Type of core definition:




LEMMA 4. *Let* $\Gamma$ *be a folded core graph (with respect to one of its vertices). Let* $v$ *and* $u$ *be two vertices of* $\Gamma$ *and let* $q$ *be a reduced path in* $\Gamma$ *from* $v$ *to* $u$ *with label* $g \in F(X)$. *Let* $H = L(\Gamma, v)$ *and* $K = L(\Gamma, u)$. *Then* $H = gKg^{-1}$.



*We want* $H = gKg^{-1}$. *This is language (language consist of loops) accepted by the automaton by* $\Gamma$.

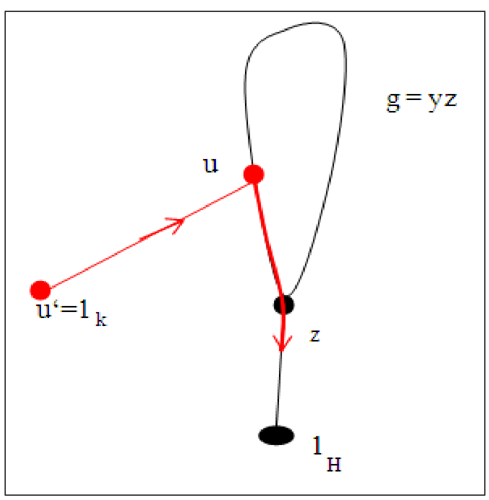



*This is* $K = L(\Gamma, v)$ *and* $H^{'} = L\left(\Gamma^{'}, w\right)$ *then it is clear that* $H^{'} = gKg^{-1}$. *We see that* $\Gamma$ *is the folded graph for* $\Gamma^{'}(w \to v)$. *Therefore we have* $H^{'} = H$.

If we consider several subgroups $(H, K)$ we often will denote distinguished vertices of their Stallings graphs by $1_H$, $1_K$.

Lemma 5. *Let* $H \leq F(X)$ *and let* $\Gamma = \Gamma(H)$. *Let* $g \in F(X)$ *be a nontrivial freely reduced word in* $X$. *Let* $g = yz$ *where* $z$ *is the maximal terminal segment of the word* $g$ *such that there is a path with label* $z^{-1}$ *in* $\Gamma$ *starting at* $1_H$ *(such a path is unique since* $\Gamma$ *is folded). Denote the end vertex of this path by* $u$. *Let* $\Delta^{'}$ *be the graph obtained from* $\Gamma$ *as follows. We attach to* $\Gamma$ *at* $u$ *the segment consisting of* $|y|$ *edges with label* $y^{-1}$, *as read from* $u$. *Let* $u^{'}$ *be the other end of this segment. Put* $\Delta^{''} = Core\left(\Delta^{'}, u^{'}\right)$. *Then* $\left(\Delta^{''}, u^{'}\right) = (\Gamma(K), 1_K)$ *where* $K = gHg^{-1}$.

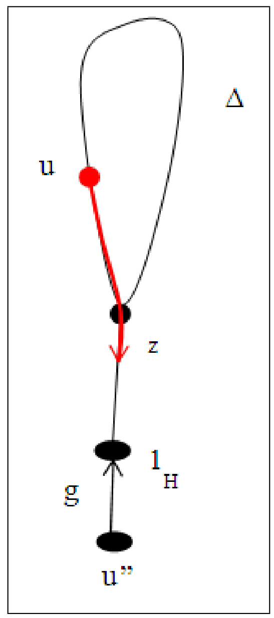
(3.1)




$L\left(\Delta^{''}, u\right) = (\Gamma(K), 1_K)$, *then* $K = gHg^{-1}$. *Every* $\overline{ghg^{-1}}$ *is a label of a loop originating in* $u^{'} = 1_K$ *then* $K \supseteq gHg^{-1}$ *this is* $(u^{'}u1_Hu^{-1}u^{'})$. *Now if do the fold*

(3.1) *we obtain*

(3.2)

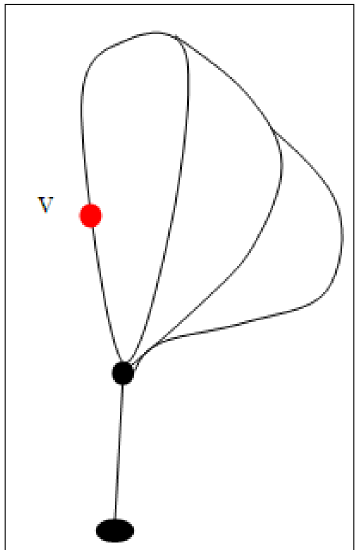



*loops at* $u''$ *in* $\Delta$ *are all* $s$ *of* $gHg^{-1}$.

Proposition 5. *(conjugate subgroups) Let* $H$ *and* $K$ *be subgroups of* $F(X)$. *Then* $H$ *is conjugate to* $K$ *in* $F(X)$ *if and only if the graphs* $Type(\Gamma(H))$ *and* $Type(\Gamma(K))$ *are isomorphic as* $X-$ *digraphs.*

Proof. Suppose that $Type(\Gamma(H)) = Type(\Gamma(K)) = \Gamma$. Let $v$ be a vertex of $\Gamma$. The subgroup $L(\Gamma, v)$ is conjugate to both $H$ and $K$, so that $H$ is conjugate to $K$. If we remove the hair from the figure we still have a subgroup conjugate to $H$and $K$. Because $Type(\Gamma(H)) = Type(\Gamma(K))$

Suppose now that $K$ is conjugate to $H, (H \cong K)$ that is $K = gHg^{-1}$ for some $g \in F(X)$. Then $Type(\Gamma(H)) = Type(\Gamma(K))$, as required.




□

## 4. Morphism of labelled graphs

Let $\Gamma$ and $\Delta$ be reduced $A-$labelled graphs as above. A mapping $\phi$ from the vertex set of $\Gamma$ to the vertex set of $\Delta$ (we write $\phi : \Gamma \to \Delta$) is a morphism of reduced $(A)-$ labelled graphs if it maps the designated vertex of $\Gamma$ to the designated vertex of $\Delta$ and if, for each $a \in A$,whenever $\Gamma$ has an $A-$labelled edge $e$ from vertex $u$ to vertex $v$, then $\Delta$ has an $A-$ labelled edge $f$ from vertex $\phi(u)$ to vertex $\phi(v)$. The edge $f$ is unique defined since $\Delta$ is reduced. We then extend the domain and range of $\phi$ to the edge sets of the two graphs, by letting $\phi(e) = f$. Note that such a morphism of reduced $A-$ labelled graphs is necessarily locally injective (an immersion), in the following sense: for each vertex $v$ of $\Gamma$, distinct edges starting (respectively ending) at $v$ have distinct images.

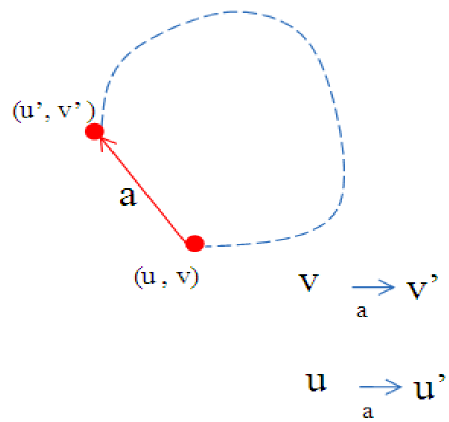
Further we say that the morphism

$$\phi : \Gamma \to \Delta$$

is a cover if it is locally bijective, that is, if the following holds: for each vertex $v$ of $\Gamma$, each edge of $\Delta$ starting (respectively ending) at $\phi(v)$ is the image under $\phi$ of an edge of $\Gamma$ starting (respectively ending) at $v$.

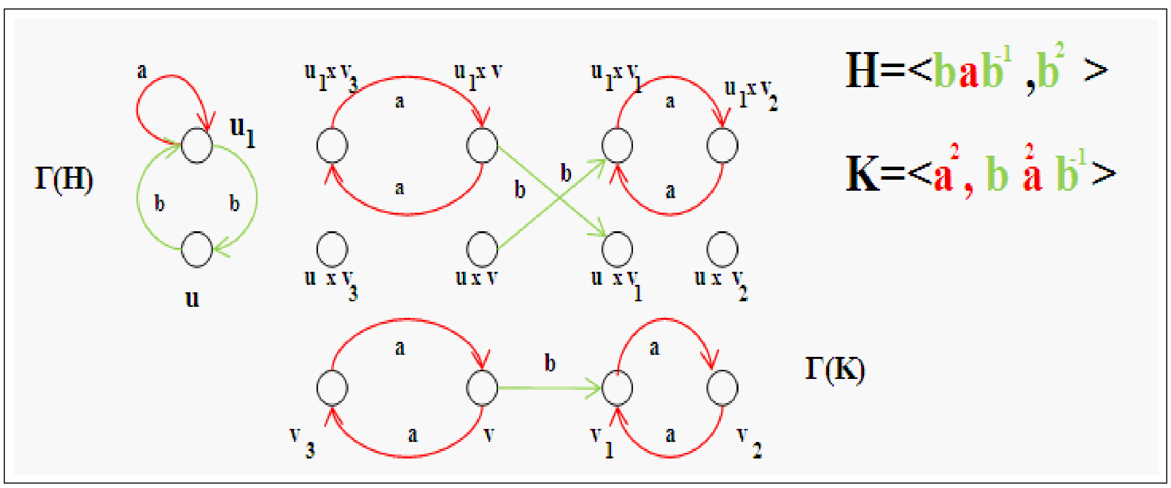
## 5. Intersection of subgroups

If $H$ and $K$ are finitely generated subgroups of $F(A)$, then $\Gamma_A(H \cap K)$ can be easily constructed from $\Gamma_A(H)$ and $\Gamma_A(K)$ : one first considers the $A-$ labeled graph whose vertices are pairs $(u, v)$ consisting of a vertex $u$ of $\Gamma_A(H)$ and a vertex $v$ of $\Gamma_A(K)$, with an $A-$ labelled edge from $(u, v)$ to $\left(u', v'\right)$ if and only if there are $A-$labelled edges from $u$ to $u'$ in $\Gamma_A(H)$ and from $v$ to $v'$ in $\Gamma_A(K)$. Finally, one considers the connected component of vertex $(1, 1)$ in this product, and we repeatedly remove the vertices of valence 1, other than the distinguished vertex $(1, 1)$ itself, to make it a reduced $A-$labelled graph.




Intersection of subgroups

EXAMPLE 21. $H = \left\langle bab^{-1}, b^2 \right\rangle$, $K = \left\langle a^2, ba^2b^{-1} \right\rangle$




DEFINITION 19. *(Product graph) Let* $\Gamma$ *and* $\Delta$ *be* $X-$ *digraphs. We define the product graph* $\Gamma \times \Delta$ *as follows. The vertex set of* $\Gamma \times \Delta$ *is the set* $V\Gamma \times V\Delta$*. For a pair of vertices* $(v, u), \left(v', u'\right) \in V(\Gamma \times \Delta)$, *so that* $v, v' \in V\Gamma$ *and* $u, u' \in V\Delta$ *and a letter* $x \in X$ *we introduce an edge labeled* $x$ *with origin* $(u, v)$ *and terminus*[1] $\left(v', u'\right)$ *provided there is an edge, labeled* $x$*, from* $v$ *to* $v'$ *in* $\Gamma$ *and there is an edge, labeled* $x$*, from* $u$ *to* $u'$ *in* $\Delta$*. Thus* $\Gamma \times \Delta$ *is an* $X-$ *digraph. In this situation we will sometimes denote a vertex* $(v, u)$ *of* $\Gamma \times \Delta$ *by* $v \times u$*.*

LEMMA 6. *Let* $\Gamma$ *and* $\Delta$ *be folded* $X-$ *digraphs. Let* $H = L(\Gamma, v)$ *and* $K = L(\Delta, u)$ *for some vertices* $v \in V\Gamma$ *and* $u \in V\Delta$*. Let* $y = (v, u) \in V(\Gamma \times \Delta)$*. Then* $\Gamma \times \Delta$ *is folded and* $L(\Gamma \times \Delta, y) = H \cap K$*.*

---

[1] end of the line in French

COROLLARY 2. *There exists an algorithm which, given finitely many freely reduced words*

$$h_1, \cdots, h_s, k_1, \cdots, k_m \in F(X),$$

*finds the rank and a Nielsen reduced free basis of the subgroup*

$$\langle h_1, \cdots, h_s \rangle \cap \langle k_1, \cdots, k_m \rangle,$$

*of $F(X)$. In particular, this algorithm determinates whether or not*

$$\langle h_1, \cdots, h_s \rangle \cap \langle k_1, \cdots, k_m \rangle = 1.$$

COROLLARY 3. *(Howson Property) The intersection of any two finitely generated subgroups of $F(X)$ is again finitely generated.*

## 6. Exercises

EXERCISE 2. *This exercise have three parts:*

(1) *Prove that the reduced form of a given element is unique.*
(2) *Prove that the map*

$$i_k : G_k \to \prod_{i \in I}^{*} G_i,$$

*given by $e \to 1$ and $a \to a = (a, 1, 1, \cdots)$ is a monomorphism. by $G_i$ with its image in $\prod_{i \in I}^{*} G_i$.*

(3) *Prove that $\prod_{i \in I}^{*} G_i$ is a coproduct in the category of groups.*

EXERCISE 3. *Show that a subgroup $H$ of $F(A)$ has finite index if and only if this natural morphism from $\Gamma_A(H)$ to the bouquet of $A$ circles is a cover, and in that case, thee index of $H$ in $F(A)$ is the number of vertices of $\Gamma_A(H)$.*

EXERCISE 4. *(Lemma) Let $H, K$ be subgroups of a free group $F$ with basis $A$. Then $H \leq K$ if and only if there exist a morphism of labeled graphs $\phi_{H,K}$ from $\Gamma_A(H)$ to $\Gamma_A(K)$. If it exist, this morphism is unique.*

EXERCISE 5. *(Proposition) Let $\Gamma$ be a folded $X-$ digraph which is a core graph with respect to a vertex $v$ of $\Gamma$. Let $H = L(\Gamma, v) \leq F(X)$ and let $T$ be a spanning tree in $\Gamma$ which is geodesic with respect to $v$. Then the set $Y_T$ is a Nielsen reduced free basis of the subgroup $H$.*

# Chapter 3

## 1. Intersection of Subgroups

Proposition 6. *Let $H, K \leq F(X)$ and $g \in F(X)$. Let $g = yz$ where $z$ is the maximal terminal segment of $g$ such that $z^{-1}$ is the label of a path in $\Gamma(H)$ with origin $1_H$. Let $y = y^{'} y^{''}$ where $y^{'}$ is the maximal initial segment of $y$ which is the label of a path in $\Gamma(K)$ with origin $1_K$.*

*Suppose that the word $y^{''}$ is nontrivial. Then*

$$\left\langle gHg^{-1}, K \right\rangle = gHg^{-1} * K.$$

Remark 6. *In particular the intersection is non-trivial, i.e., $gHg^{-1} \cap K = \{e\}$.*

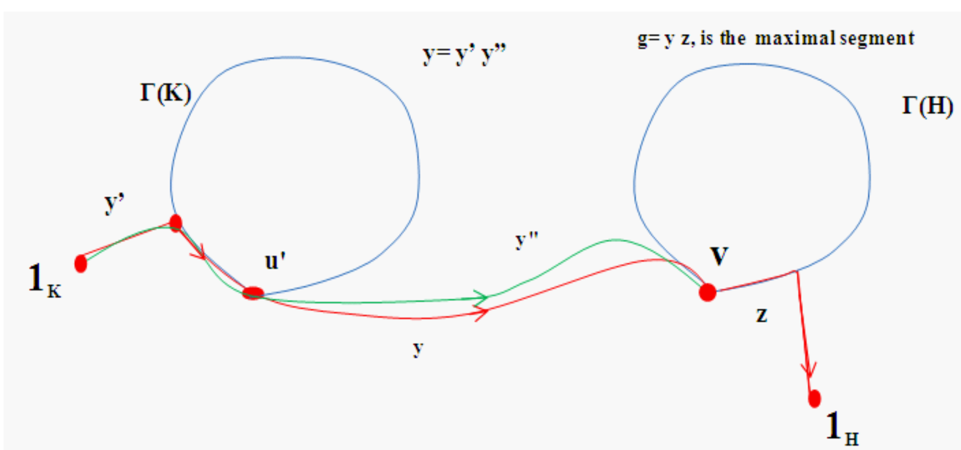
Remark 7. *Also $\Gamma$ accepts $\left\langle K \cup gHg^{-1} \right\rangle$.*

Remark 8. *$rank\left(\left\langle K \cup gHg^{-1} \right\rangle\right) = rank(K) + rank(H)$; all the generators of $H$ are independent since it is a folded graphs implies that*

$$gHg^{-1} \cap K = \{e\}$$

*which implies that it is a free product.*

Proof. (Proposition 6).




□

Proposition 7. *Let $H, K \leq F(X)$. Let $g \in F(X)$ be such that the double cosets $KgH$ and $KH$ are distinct. Suppose that $gHg^{-1} \cap K \neq 1$. Then there is a* *vertex $v \times u$ in $\Gamma(H) \times \Gamma(K)$ which does not belong to the connected component of $1_H \times 1_K$ such that the subgroup*

(✠) $$L(\Gamma(H) \times \Gamma(K), v \times u) \text{ is conjugate to } gHg^{-1} \cap K,$$

*in $F(X)$.*

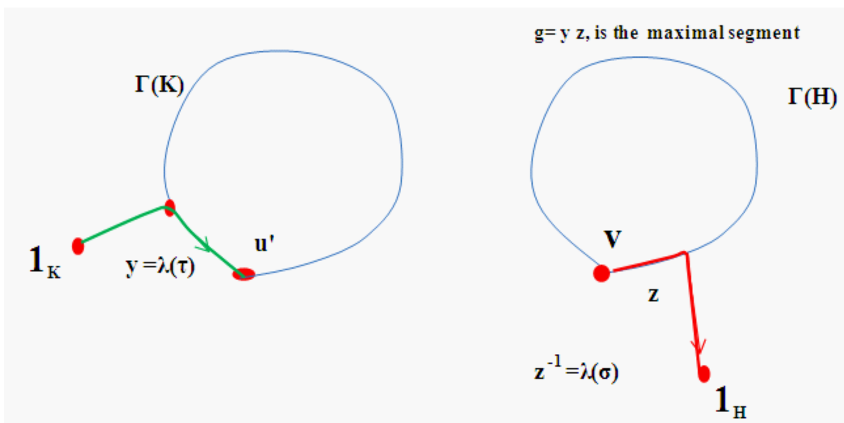
Proof. If $g \in KgH$, then

(♣) $$gHg^{-1} \cap K, \text{ and } g_1 H g_1^{-1} \cap K, \text{ are conjugate}$$

Let $g_1 = kgh$ for some $k \in K$ and $h \in H$, then

$$\begin{aligned} g_1 H g_1^{-1} \cap K &= kg\left(hHh^{-1}\right) g^{-1} k^{-1} \cap K \\ &= kg(H)(kg)^{-1} \cap K \\ &= k\left(gHg^{-1} \cap K\right) k^{-1}, \end{aligned}$$

this implies (♣)Conversely




Let $g = yz$ where $z$ is the largest terminal segment of $g$ such that $z^{-1}$ is the label of a path in $\Gamma(H)$ with origin $1_H$. Denote this path by $\sigma$ and the terminal vertex of $\sigma$ by $v$, see picture. If the word $y$ is not the label of a path in $\Gamma(K)$ with origin $1_K$, then by Proposition (6) we have that

$$\left\langle gHg^{-1}, K\right\rangle = gHg^{-1} * K.$$

This implies that

$$gHg^{-1} \cap K = \{e\},$$

contrary to our assumptions.

Thus $y$ is the label of a path $\tau$ in $\Gamma(K)$ from $1_K$ to some vertex $u$. Then set $L(\Gamma(H), v) = zHz^{-1}$ and $L(\Gamma(K), u) = y^{-1}Ky$. Note also that

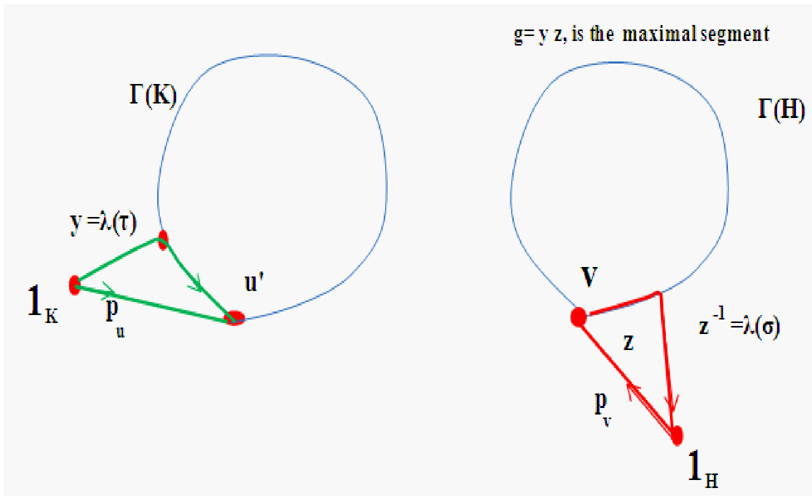
$$\begin{aligned} y\left(zHz^{-1} \cap y^{-1}Ky\right) y^{-1} &= yzHz^{-1}y^{-1} \cap K \\ &= gHg^{-1} \cap K, \end{aligned}$$

and therefore $gHg^{-1} \cap K$ and $zHz^{-1} \cap y^{-1}Ky$ are conjugate in $F(X)$.

$$\begin{aligned} L &= L(\Gamma(H) \times \Gamma(K), v \times u) = zHz^{-1} \cap y^{-1}Ky \\ &= gHg^{-1} \cap K, \end{aligned}$$

as required.

It remains to show that $v \times u$ does not belong to the connected component of $1_H \times 1_K$ in $\Gamma(H) \times \Gamma(K)$. Suppose this is not the case.




Then there exist a reduced path $p_v$ in $\Gamma(H)$ from $1_H$ to $v$ and a reduced path $p_u$ in $\Gamma(K)$ from $1_K$ to $u$ such that their labels are the same, that is

$$\lambda(p_v) = \lambda(p_u) = \alpha.$$

Note that $p_v\sigma^{-1}$ is a path in $\Gamma(H)$ from $1_H$ to $1_H$ and therefore

$$\overline{\lambda(p_v\sigma^{-1})} = \overline{\alpha z} = \alpha \bullet z = h \in H.$$

Similarly, $\tau p_u^{-1}$ is a path in $\Gamma(K)$ from $1_K$ to $1_K$ and hence $y \bullet \alpha^{-1} = k \in K$. Thus we have

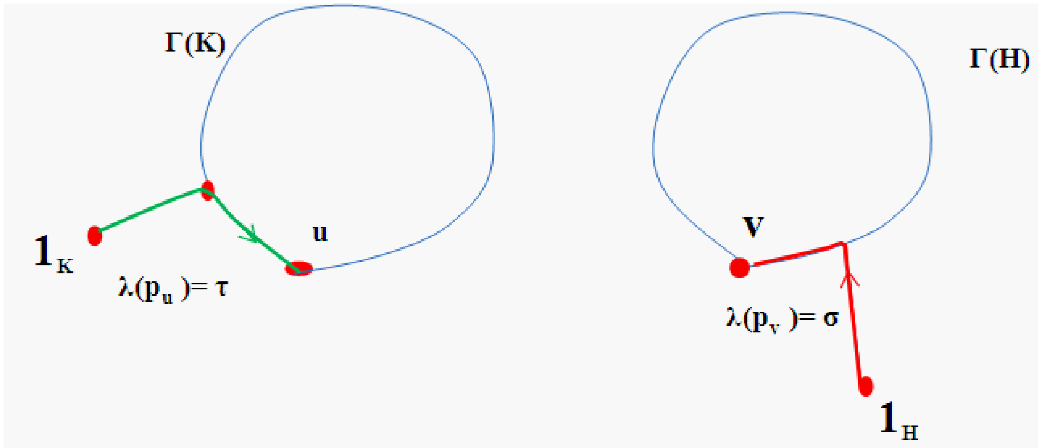
$$\begin{aligned} g &= y \bullet z = y \bullet \alpha^{-1}\alpha \bullet z \\ &= k \bullet h \in KH, \end{aligned}$$

contrary to our assumption that $KgH \neq KH$. $\square$

PROPOSITION 8. *Let $H, K \leq F(X)$ be two subgroups of $F(X)$. Then for any vertex $v \times u$ of $\Gamma(H) \times \Gamma(K)$ the subgroup $L(\Gamma(H) \times \Gamma(K), v \times u)$ is conjugate to a subgroup of the form $gHg^{-1} \cap K$ for some $g \in F(X)$.*

*Moreover, it $v \times u$ does not belong to the connected component of $1_H \times 1_K$, then the element $g$ can be chosen so that $KgH \neq KH$.*

PROOF. Consider




Let $p_v$ be reduced path in $\Gamma(H)$ from $1_H$ to $v$ with label $\sigma$. Similarly, let $p_u$ be a reduced path in $\Gamma(K)$ from $1_K$ to $u$ with label $\tau$. $L(\Gamma(H), v) = \sigma^{-1}H\sigma$ and $L(\Gamma(K), u) = \tau^{-1}K\tau$. Therefore

$$L(\Gamma(H) \times \Gamma(K), v \times u) = \sigma^{-1}H\sigma \cap \tau^{-1}K\tau,$$

is conjugate to

$$\tau\sigma^{-1}H\sigma\tau^{-1} \cap K,$$

and $g = \tau\sigma^{-1}$ satisfies the requirement of the proposition.

Suppose now that $v \times u$ does not belong to the connected component of $1_H \times 1_K$ in $\Gamma(H) \times \Gamma(K)$ but $g = \tau\sigma^{-1} \in KH$. Thus $\tau\sigma^{-1} = kh$ for some $k \in K$, $h \in H$ and therefore

$$k^{-1}\tau = h\sigma.$$

Let $\alpha$ be the freely reduced form of the element $k^{-1}\tau = h\sigma$. Recall that $k^{-1} \in K$ and so $k^{-1}$ is the label of a reduced $p_1$ in $\Gamma(K)$ from $1_K$ to $1_K$. Then $p_1p_u$ is a path in $\Gamma(K)$ whose label freely reduces to $\alpha$. Therefore there is a reduced path $p_1^{'}$ in $\Gamma(K)$ from $1_K$ to $u$ with label $\alpha$. Similarly, since $h \in H$, there is a path $p_2$ in $\Gamma(H)$ from $1_H$ to $1_H$ with label $h$. Hence $p_2p_v$ is a path in $\Gamma(H)$ from $1_H$ to $v$ whose label freely reduces to $\alpha$. Again, it follows that there is a reduced path $p_2^{'}$ in $\Gamma(H)$ from $1_H$ to $v$ with label $\alpha$. Now the definition of $\Gamma(H) \times \Gamma(K)$ implies that there is a path in $\Gamma(H) \times \Gamma(K)$ from $1_H \times 1_K$ to $v \times u$ with label $\alpha$.

However, this contradicts our assumption that $v \times u$ does not belong to the connected component of $1_H \times 1_K$. Thus $g \notin KH$ and $KgH \neq KH$, as required □

The reduced rank:= $\bar{r}(H) = \max\{rank(H) - 1, 0\}$. In 1956 Hanna Neumann asked:

The Hanna Neumann Conjecture $(HNC)$,

$$\bar{r}(H \cap K) \leq \bar{r}(H)\,\bar{r}(K).$$

In 1990 Walter Neumann formulated:

The Strengthened Hanna Neumann Conjecture $(SHNC)$,

$$\sum_{x \in H \setminus \frac{F}{K}} \bar{r}\left(x^{-1}Hx^{-1} \cap K\right) \leq \bar{r}(H)\,\bar{r}(K).$$

Proved by Mineeyev, July 2011 (see the slides about HNC ).

Definition 20. *Let $H$ be a subgroup of a group $G$. We say that $H$ is a malnormal subgroup of $G$ if for any $g \in G - H$*

$$gHg^{-1} \cap H = 1.$$

Definition 21. *We say that $H$ is cyclonormal if for any $g \in G - H$*

$$gHg^{-1} \cap H,$$

*is cyclic.*

Theorem 13. *Let $H \leq F(X)$ be a subgroup. Then $H$ is malnormal in $F(X)$ if and only if every component of $\Gamma(H) \times \Gamma(H)$, which does not contains $1_H \times 1_H$, is a tree.*

Remark 9. *If $H \leq F(X)$, $H$ is a subgroup. Ask for a subgroup of type $H \leq K \ni K * K_1 = F$ is a multiple. But not every subgroup of a free group is a multiple. Then $K$ is not represented, and $H$ has finite index. Which implies that it is an algebraic extension, and since the algebraic extension is algebraic. Now the following proposition is of this type.*

Proposition 9. *Let $\Gamma$ be a folded connected $X-$ digraph and let $\Gamma^{'}$ be a connected subgraph of $\Gamma$. Let $v$ be a vertex of $\Gamma^{'}$. Then $H = L\left(\Gamma^{'}, v\right)$ is a free factor of $G = L(\Gamma, v)$. Further, if $\Gamma^{'}$ does not contain $Core(\Gamma, v)$, then $H \neq G$. In particular, is both $\Gamma$ and $\Gamma^{'}$ are core graphs with respect to $v$ and $\Gamma^{'} \subsetneqq \Gamma$ then $H \neq G$.*

Proof. Let $T^{'}$ be a spanning tree in $\Gamma^{'}$. Then there exists a spanning tree $T$ of $\Gamma$ such that $T^{'}$ is a subgraph of $T$. $T^{'}$ has no cycles. Take

$$Y_T = \left\{S_e | e \in E^+(\Gamma - T)\right\},$$

as generating set of $G$. This was proved in lecture 2 as Claim 1. $G\left\langle S_e | e \in E^+(\Gamma - T)\right\rangle$

$$\begin{aligned} Y_T &= \left\{S_e | e \in E^+(\Gamma - T), e \in \Gamma^{'}\right\} \cup \left\{S_e | e \in E^+(\Gamma - T), e \notin E\Gamma^{'}\right\} \\ &= \left\{S_e | e \in E^+\left(\Gamma^{'} - T^{'}\right)\right\} \cup \left\{S_e | e \in E^+(\Gamma - T), e \notin E\Gamma^{'}\right\} \\ &= Y_{T'} \cup Z. \end{aligned}$$

Then, we have 2 disjoint subsets,

$$G = F(Y_T) = F(Y_{T'}) * F(Z) = H * F(Z),$$

and $H$ is a free factor of $G$ as required.

Suppose now that $\Gamma^{'}$ does not contain $Core(\Gamma, v)$. We claim that there is a positive edge $e$ of $\Gamma$ which does not belong to $\Gamma^{'}$ and is not in $T$. Assume this is not the case. Then all edges outside of $\Gamma^{'}$ lie in $T$. Hence $\Gamma - \Gamma^{'} \subset T$ is a union of disjoint trees. This implies that $Coore(\Gamma, v)$ is contained in $\Gamma^{'}$, contrary to our assumptions. Thus the claim holds and hence $Z \neq \varnothing$. Therefore, $H \neq G$ as required. $\square$

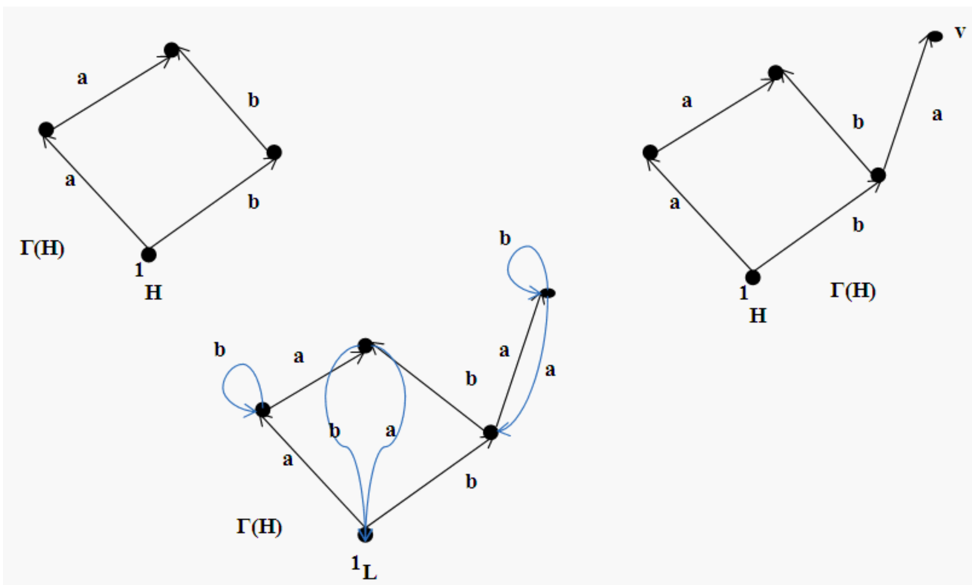
Remark 10. *If the graph is not complete add edges to make it complete.*

THEOREM 14. *(Marshall Hall's Theorem) Let $H$ be a finitely generated subgroup of $F(X)$. Let $g \in F(X)$ be such that $g \notin H$. Then there exists a finitely generated subgroup $K$ of $F(X)$ such that*

(1) $L = \langle H, K \rangle = H * K$,
(2) *$L$ has finite index in $F(X)$,*
(3) $g \notin L$

PROOF. Idea of the proof. Write $g$ as a reduced word in $X$. Add a path with label $g$ beginning at $1_H$ to $\Gamma(H)$. Fold the obtained graph (denote the result by $\Gamma$). Add edges to $\Gamma$ to obtain a finite folded complete graph $\Gamma'$. $L = L\left(\Gamma', 1_H\right)$.

$\Gamma_H$ is a subgraph of $\Gamma_L$




□

REMARK 11. *Hall's theorem implies that a free group $F(X)$ of a finite rank is subgroup separable , that is to say any finitely generated subgroup of $F(X)$ is equal to the intersection of finite index subgroup of $F(X)$ containing $H$. This is an important and nontrivial property of free groups. This can rephrase as follows:*

*Another definition $G$ is a subgroup separable if $\forall\ H \leq G$, $\forall g \in G - H\ \exists\ \overline{G}$ a finite qoutient of $G \ni \bar{g} \notin \overline{H}$ ,(where $\overline{H}$ is the image).*

REMARK 12. *A free group is subgroup separable. This implies residual finite, which follows from linearity.*

## 2. Normal Subgroups

THEOREM 15. *(Normal subgroup) Let $H \leq F(X)$ be a nontrivial subgroup of $F(X)$. Then $H$ is normal in $F(X)$ iff the following conditions are satisfied:*

(1) *The graph $\Gamma(H)$ is $X-$ digraph (so that there are no degree-one vertices in $\Gamma(H)$ and hence $\Gamma(H) = Type(\Gamma(H))$).*
(2) *For any vertex $v$ of $\Gamma(H)$ the based $X-$digraphs $(\Gamma(H), 1_H)$ and $(\Gamma(H), v)$ are isomorphic (that is $L(\Gamma(H), v) = H$).*

Definition 22. *Let $H$ be a subgroup of a group $G$. The commensurator $Comm_G(H)$ of $H$ in $G$ is defined as*

$$Comm_G(H) = \left\{g \in G \middle| \left|H : H \cap gHg^{-1}\right| < \infty, \text{ and } \left|gHg^{-1} : H \cap gHg^{-1}\right| < \infty\right\}.$$

Proposition 10. *The $Comm_G(H)$ is a subgroup of $G$ containing $H$.*

Proof. Let $g_1, g \in Comm_G(H)$, then $\left|H : H \cap gHg^{-1}\right| < \infty$, lets conjugate by $g_1$ so that

$$\left|g_1 H g_1^{-1} : g_1 H g_1^{-1} \cap g_1 g H g^{-1} g_1^{-1}\right| < \infty,$$

then the index of $H$ is finite $[G : H] < \infty$, then

$$\left|H \cap g_1 H g_1^{-1} : H \cap g_1 H g_1^{-1} \cap g_1 g H g^{-1} g_1^{-1}\right| < \infty,$$

this is a subgroup of finite index has finite index.

$$\left|H : H \cap g_1 g H g^{-1} g_1^{-1}\right| < \infty,$$

therefore, $g_1 g \in Comm_G(H)$.

Now we conjugate by $\left(g_1^{-1} g^{-1}\right)$ so that

$$\left|\left(g_1^{-1}g^{-1}\right) H \left(g_1^{-1}g^{-1}\right)^{-1} : \left(g_1^{-1}g^{-1}\right) H \left(g_1^{-1}g^{-1}\right)^{-1} \cap \left(g_1^{-1}g^{-1}\right) gHg^{-1} \left(g_1^{-1}g^{-1}\right)^{-1}\right| < \infty,$$

then

$$\left|H \cap \left(g_1^{-1}g^{-1}\right) H \left(g_1^{-1}g^{-1}\right)^{-1} : H \cap \left(g_1^{-1}g^{-1}\right) H \left(g_1^{-1}g^{-1}\right)^{-1} \cap \left(g_1^{-1}g^{-1}\right) gHg^{-1} \left(g_1^{-1}g^{-1}\right)^{-1}\right| < \infty,$$

so that

$$\left|H \ : H \ \cap g_1^{-1} H g_1\right| < \infty,$$

therefore $g_1^{-1} \in Comm_G(H)$. □

Lemma 7. *Let $H \leq F(X)$ be a nontrivial finitely generated subgroup. Then*

$$|F(X) : H| < \infty,$$

*iff* $F(X) = Comm_{F(X)}(H)$.

Proof. $(\Longrightarrow)$

It is obvious that

$$|F(X) : H| < \infty,$$

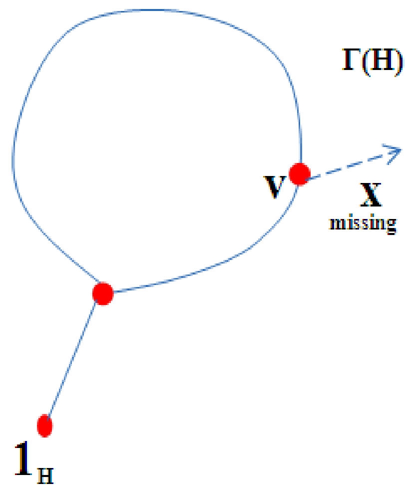
implies $F(X) = Comm_{F(X)}(H)$.

$(\Longleftarrow)$

Suppose now that $H$ is a finitely generated subgroup of $F(X)$ and $F(X) = Comm_{F(X)}(H)$. Assume that $|F(X) : H| = \infty$. Then the graph $\Gamma(H)$ is not $X-$regular, we want a contradiction. Thus there is a vertex $v \in \Gamma$ and a letter $x \in X \cup X^{-1}$ such that there is no edge labeled $x$ with origin $v$ in $\Gamma(H)$. Since $H$ is a nontrivial subgroup of infinite index in $F(X)$, the rank of $F(X)$, the rank of $F(X)$ is at least two and so

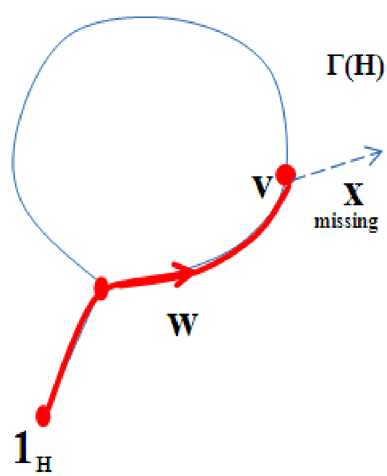
$$\#(X) \geq 2.$$

Let $a \in X$ be a letter such that $a \neq x^{\pm 1}$.




Since for any $g \in F(X)$ we have $\left|gHg^{-1} : H \cap gHg^{-1}\right| < \infty$, for any element of $gHg^{-1}$ some power of this element belongs to $H \cap gHg^{-1}$ and so to $H$. Hence for any $g \in F(X)$ and any $h \in H$ there is $n \geq 1$ such that $g^{-1}h^n g \in H$. Let $h \in H$ be a nontrivial element, so that $h$ is a freely reduced word in $X$. Let $y \in X \cup X^{-1}$ be that first letter of $h$ and let $z \in X \cup X^{-1}$ be the last letter of $h$. Then for any $m \geq 1$ the freely reduced form $h^n$ begins with $y$ and ends with $z$.

Let $w$ be the label of a reduced path in $\Gamma(H)$, from $1_H$ to $v$.




Since there is no edge labeled $x$ with origin $v$ in $\Gamma(H)$, any freely reduced word with initial segment $wx$ does not belong to $H$. Put $q = y$ if $y \neq z^{-1}$.

If $y = z^{-1}$ and $y \in \left\{x, x^{-1}\right\}$, put $q = a$. If $y = z^{-1}$ and $y \notin \left\{x, x^{-1}\right\}$, put $q = x$. Then for any $m \geq 1$ the word $q\overline{h^m}q^{-1}$ is freely reduced. (Recall that $\overline{h^m}$ is the freely reduced from of $h^m$.) Contradicts the assumption that the graph is not complete.

Choose a freely reduced word $w^{'}$ such that the word $wxw^{'}q$ is freely reduced. This is obviously possible since $X$ has at least two elements. Put $g = wxw^{'}q$. By our assumptions there is $n \geq 1$ such that the $gh^ng^{-1}$ is

$$wxw^{'}qy \cdots zq^{-1}\left(w^{'}\right)^{-1}x^{-1}w^{-1}. \tag{1-✠}$$

The word $(1-✠)$ has initial segment $wx$ and hence cannot represent an element of $H$. This yields a contradiction. □

Corollary 4. *(Greenberg-Stallings Theorem) Let $H, K$ be finitely generated subgroups of $F(X)$ such that $H \cap K$ has finite index in both $H$ and $K$. Then $H \cap K$ has finite index in the subgroup $\langle H \cup K \rangle$.*

## 3. Some other properties of free groups

Definition 23. *$G$ acts on a set if $\forall g \in G$, $s \in S$ and $gs$ is defined, and $gs \in S$*

$$(G, S) \to S$$

(1) $es = s, \forall s \in S$,
(2) $g_1(g_1 s) = g_1 g_2 s$.

Lemma 8. *(Ping-Pong lemma) Let a group $G$, generated by $a$ and $b$, act on a set $X$. Assume that there are two nonempty subsets $A$ and $B$ of $X$, so that $A \cap B = \varnothing$, and $a^n \bullet B \subseteq A$ and $b^n \bullet A \subseteq B$ for all integers $n \neq 0$. Then $G$ is freely generated by $a$ and $b$.*

Proof. Let $w$ be a nonempty reduced word in the alphabet $a^{\pm}, b^{\pm}$. W.l.og., we can assume that $w$ begins and ends with $a^{\pm}$, for if not then for $m$ large enough a conjugate $w_1 = a^m w a^{-m}$ of $w$ does, and $w = 1$ iff $w_1 = 1$. Let

$$w = a^{n_1}b^{m_1} \bullet a^{n_{k-1}}b^{m_{k-1}}a^{n_k},$$

with $n_i, m_i \neq 0$. Then

$$\begin{aligned} wB &= a^{n_1}b^{m_1} \bullet a^{n_{k-1}}b^{m_{k-1}}a^{n_k} \bullet B \subseteq a^{n_1}b^{m_1} \bullet a^{n_{k-1}}b^{m_{k-1}} \bullet A \subseteq \\ a^{n_1}b^{m_1} \bullet a^{n_{k-1}} \bullet B &\subseteq \cdots \subseteq a^{n_1} \bullet B \subseteq A. \end{aligned}$$

We have a reduced word. It follows that $w \neq 1$, and so $a$ and $b$ freely generated $G$. □

Corollary 5. *The matrices*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

*generate a free subgroup in $SL_2(\mathbb{Z})$.*

Proof. Denote by $G = \langle A, B \rangle$ the subgroup of $SL_2(\mathbb{Z})$ generated by $A$ and $B$. The group $G$ acts on $X = \mathbb{R}^2$ by left multiplication, and if we set

$$V = \left\{[x, y]^T \mid |x| < |y|\right\} \text{ and } W = \left\{[x, y]^T \mid |x| > |y|\right\},$$

then

$$A^n \bullet W \subseteq V, \text{ and } B^n \bullet V \subseteq W,$$

for all $n \neq 0$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in V$,

$$\begin{aligned} A \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x+2y \\ y \end{pmatrix} \in W \\ |x+2y| &> |y| . \end{aligned}$$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

And

$$B^n \begin{pmatrix} x \\ y \end{pmatrix} \in V.$$

By the Ping-pong lemma, $G$ is freely generated by $A$ and $B$. □

Definition 24. *A group $G$ is called linear if it can be embedded into a group of matrices $GL_n(\mathbb{P})$ for some integer $n \geq 1$ and some fields $\mathbb{P}$.*

Theorem 16. *A free group of countable rank is linear. In particular, any finitely generated free group is linear.*

Definition 25. *A group $G$ is called residually finite if for any nontrivial element $g \in G$ there exists a homomorphism*

$$\phi : G \to H,$$

*such that $G$ is map into a finite group $H$ so that $\phi(g) \neq 1$.*

Clearly, finite groups are residually finite, and subgroups of residually finite groups are residually finite.

Example 22. *Prove that $SL_n(\mathbb{Z})$ is residually finite.*

Proof. Let $M \in SL_n(\mathbb{Z}) \ni M \neq I$ then $M - 1 \neq 0$ $\exists$ $m_i = k$ we take a homomorphism

$$\phi_R : \mathbb{Z} \to \frac{\mathbb{Z}}{p\mathbb{Z}},$$

this defines a homomorphism

$$\phi_S : SL_n(\mathbb{Z}) \to SL_n\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right),$$

Take $p \ni$ if $p \nmid k$, then $\phi(M) \neq 1$. □

Example 23. $SL_n\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right) \cong$ *finite group.*

Theorem 17. *Free groups are residually finite.*

Theorem 18. *Every finitely generated subgroup of a linear group is residually finite.*

Example 24. $\mathbb{Q}$ *is not residually finite. Because it is a divisible group.*

Example 25. *Finite groups are not divisible. Finite Abelian groups are not divisible.*

Residually finite groups are finitely presented groups, i.e., $G = \langle G \mid R \rangle$ , where $G, R < \infty$, and $G$ has a solved word problem.

(1) We enumerate all consequences of relations if $w = 1$ in $G$ we obtain $w$ in this process.
(2) Enumerate all finite quotients $\phi(G)$ of $G$. If $w \neq 1$ in $G$, then $\phi(G) \neq 1$, in some finite quotient.

## 4. Homomorphisms of groups

The universal property of free groups allows one to describe arbitrary groups in terms of generators and relators. Let $G$ be a group with a generating set $S$. By the universal property of free groups there exists a homomorphism

$$\begin{aligned} \varphi &: F(S) \to G, \ni \\ \varphi(s) &= s, \ \forall s \in S. \end{aligned}$$

It follows that $\varphi$ is onto, so by the first isomorphism theorem

$$G \cong \frac{F(S)}{\ker(\varphi)}.$$

In this event $\ker(\varphi)$ is view as the set of relators of $G$, and a group word $w \in \ker(\varphi)$ is called a relator of $G$ in generators $S$. If a subset $R \subset \ker(\varphi)$ generates $\ker(\varphi)$ as a normal subgroup of $F(S)$ then it is termed a set of defining relations of $G$ relative to $S$. The pair $\langle S \mid R \rangle$ is called a presentation of $G$, it determines $G$ uniquely up to isomorphism. The presentation $\langle S \mid R \rangle$ is finite if both sets $S$ and $R$ are finite. A group is finitely presented if it has at least one finite presentation. Presentations provide a universal method to describe groups.

Example 26. *$G = \langle s_1 \cdots s_n | [s_i, s_j], \forall\ 1 \leq i < j \leq n \rangle$ is the free abelian group of rank $n$.*

Example 27. *$C_n = \langle s | s^n = 1 \rangle$ is the cyclic group of order $n$.*

Example 28. *Both presentations $\left\langle a, b | ba^2 b^{-1} a^{-3} \right\rangle$ and $\left\langle a, b | ba^2 b^{-1} a^{-3}, \left[bab^{-1}, a\right] \right\rangle$ define the Baumslag-Solitar group $BS(2,3)$.*

If a group $G$ is defined by a presentation, then one can try to find homomorphism from $G$ into other groups.

Lemma 9. *Let $G = \langle S \mid R \rangle$ be a group defined by a (finite) presentation with the set of relators*

$$R = \left\{ r_j = y_{i_1}^{(j)} \cdots y_{i_j}^{(j)} | y_i^{(j)} \in S^{\pm 1}, 1 \leq j \leq m \right\},$$

*and let $H$ be an arbitrary group. A map*

$$\psi : S^{\pm 1} \to H,$$

*extends to a homomorphism*

$$\widetilde{\psi} : G \to H,$$

*iff*

$$\psi(r_j) = \psi\left(y_{i_1}^{(j)}\right) \cdots \psi\left(y_{i_j}^{(j)}\right) = 1,$$

*in $H$ for all $r_j \in R$.*

Proof. Define the map

$$\begin{aligned}\widetilde{\psi} &: G \to H, \text{ by} \\ \widetilde{\psi}\left(y_{n_1} \cdots y_{n_t}\right) &= \psi\left(y_{n_1}\right) \cdots \psi\left(y_{n_t}\right),\end{aligned}$$

whenever $y_{n_j} \in S^{\pm 1}$. If $\widetilde{\psi}$ is a homomorphism, then obviously $\psi\left(r_j\right) = 1$, $\forall r_j \in R$.

The converse follows from

Lemma 10. *(Mapping property of qoutient groups) Let $N$ be a normal subgroup of $G$, let $\overline{G} = \frac{G}{N}$, and let*

$$\pi; G \to \overline{G},$$

*be the canonical map, $\pi\left(g\right) = \bar{g} = gN$. Let*

$$\phi : G \to G^{'},$$

*be a homomorphism such that $N \leq \ker\left(\phi\right)$. Then there is a unique homomorphism*

$$\overline{\phi} : \overline{G} \to G^{'},$$

*such that $\bar{\phi} \circ \pi = \phi$. This map is defined by the rule $\overline{\phi}\left(\bar{g}\right) = \phi\left(g\right)$.*

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \overline{G} \\ & \phi \searrow & \downarrow \overline{\phi} \\ & & G^{'} \end{array}$$

$\square$

## 5. Exercises

Exercise 6. *Prove the following theorem. (Normal subgroup) Let $H \leq F\left(X\right)$ be a nontrivial subgroup of $F\left(X\right)$. Then $H$ is normal in $F\left(X\right)$ iff the following conditions are satisfied:*

Theorem 19.
1. *The graph $\Gamma\left(H\right)$ is $X-$ digraph (so that there are no degree-one vertices in $\Gamma\left(H\right)$ and hence $\Gamma\left(H\right) = Type\left(\Gamma\left(H\right)\right)$).*
2. *For any vertex $v$ of $\Gamma\left(H\right)$ the based $X-$digraphs $\left(\Gamma\left(H\right), 1_H\right)$ and $\left(\Gamma\left(H\right), v\right)$ are isomorphic (that is $L\left(\Gamma\left(H\right), v\right) = H$).*

Exercise 7. *Prove the following theorem. (Greenberg-Stallings Theorem) Let $H, K$ be finitely generated subgroups of $F\left(X\right)$ such that $H \cap K$ has finite index in both $H$ and $K$. Then $H \cap K$ has finite index in the subgroup $\langle H \cup K\rangle$.*

Exercise 8. *Prove that the these presentations define isomorphic groups. Both presentations $\left\langle a, b | ba^2b^{-1} = a^3\right\rangle$ and $\left\langle a, b | ba^2b^{-1} = a^3, \left(\left[b, a\right]\right)^2 = \left(abb^{-1}a^{-1}\right)^2 = b\right\rangle$ define the Baumslag-Solitar group $BS\left(2, 3\right)$.*

# Chapter 4

Let $G$ be a group, by the commutant (or derived subgroup) $G^{'}$ of $G$ we mean the subgroup generated by all the commutators

$$[g,b] = gbg^{-1}b^{-1},$$

in $G$. Since $a\,[g,b]\,a^{-1} = \left[aga^{-1}, aba^{-1}\right]$, the commutant is a normal subgroup of $G$. The qoutient $\frac{G}{G'}$ is called the Abelianization of $G$. This name is given to this qoutient because $\frac{G}{G'}$ is an Abelian group.

Example 29. *The Abelianization of a free group $F_n$ is the free Abelian group of rank $n$. In general, if*

$$G = \langle s_1 \cdots s_n \mid r_1 \cdots r_m \rangle,$$

*then*

$$\frac{G}{G'} = \langle s_1 \cdots s_n \mid r_1 \cdots r_m,\ \ [s_i, s_j]\,(1 \leq i < j \leq n)\rangle$$

As the following corollary shows, the abelianization $\frac{G}{G'}$ is the largest Abelian quotient of $G$, in a sense.

Corollary 6. *Let $H$ be an Abelian quotient of $G$, and let*

$$v : G \to \frac{G}{G'},$$

*and*

$$\psi : G \to H,$$

*be the natural homomorphism. Then there is a homomorphism*

$$\varphi : \frac{G}{G'} \to H,$$

*so that the following diagram commutes:*

$$\begin{array}{ccc} G & \to & \frac{G}{G'} \\ & \psi \searrow & \downarrow \varphi \\ & & H \end{array}$$

Proof. Let $G$ be generated by $S = \{s_1, \cdots, s_n\}$, then $\frac{G}{G'}$ is generated by $v(S) = \{v(s_1), \cdots, v(s_n)\}$. We still denote $v(s_i)$ by $s_i$, since we want to fix the alphabet $S^{\pm}$ for both $G$ and $\frac{G}{G'}$. Hence, $\frac{G}{G'}$ has the presentation above. Define a map

$$\begin{aligned} \varphi^{'} &: v(S) \to H, \text{ by} \\ \varphi^{'}(s_i) &= \psi(s_i),\ \forall i. \end{aligned}$$

Observed that

$$\varphi^{'}(r_j) = \psi(r_j) = 1,$$

in $H$, since $\psi$ is a homomorphism and $r_j = 1$ in $G$. Also,

$$\begin{aligned} \varphi^{'}([s_i, s_j]) &= \psi([s_i, s_j]) \\ &= [\psi(s_i), \psi(s_j)] = 1, \end{aligned}$$

since $H$ is Abelian. It follows now from the previous lemma that the map $\varphi^{'}$ extends to a homomorphism from $\frac{G}{G'}$ to $H$. □

## 1. Generators and relations

The free Burnside group of exponent $n$ with two generators is given by the presentation

$$\langle a, b | u^n = 1\rangle,$$

for all words $u$ in the alphabet $a, b$. The fundamental group of the orientable surface of genus $n$ is given by the presentation

$$\langle a_1, b_1, \cdots, a_n, b_n | [a_1, b_1] \cdots, [a_n, b_n] = 1\rangle.$$

## 2. Geometric Group theory

The object of study in Geometric Group Theory finitely generated groups given by presentations

$$\langle a_1, \cdots, a_n | r_1, r_2, \cdots\rangle,$$

where $r_i$ is a word in $a_1, \cdots, a_n$. That is groups generated by $a_1, \cdots, a_n$ with relations $r_1 = 1, r_2 = 1, \cdots$ imposed. Some classical results

Theorem 20. *(Boone-Novikov's solution of Dehn's problem) There exists a finite presented group with undecidable word problem.*

Theorem 21. *(Higman) A group has recursively enumerable (= we can enumerate all the words) word problem iff it is a subgroup of a finitely presented group.*

Theorem 22. *(Adian- Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.*

It is still unknown if such a group is infinite for $n = 5, 8$ etc.

The growth rate of a group is a well defined notion from asymtotic analysis. To say that a finitely generated group has polynomial growth means the number of elements of length (relative to a symmetric generating set) at most $n$ is bounded above by a polynomial function $p(n)$. The order of growth is then the least degree of any such polynomial function $p$.

Definition 26. *A nilpotent group $G$ is a group with a lower central series terminating in the identity subgroup.*

Theorem 23. *(Gromov's solution of Milnor's problem) Any group polynomial growth has a nilpotent subgroup of finite index.*

(Converse) If the group has a nilpotent subgroup of finite index then it has polynomial growth.

DEFINITION 27. *A group $G$ acts on a set $X$ if for each $g \in G$ there is a bijection $x \to gx$ defined on $X$ such that*

$$ex = x, \ \ (g_1 (g_2 (x))) = (g_1 g_2)(x).$$

DEFINITION 28. *$f$ is an isometry if there exist a map,*

$$f : Y \to X,$$

*and*

$$|f(y_1), f(y_2)|_X = |y_1, y_2|, \ \ (\forall y_1, y_2 \in Y).$$

EXAMPLE 30. *$G$ acts on itself from the left by isometry,*

$$\begin{aligned} \left|g_1^{-1} g_2\right| &= \left|g g_1^{-1} g g_2\right| \\ &= \left|g_1^{-1} g^{-1} g g_2\right|. \end{aligned}$$

## 3. Finitely generated groups viewed as metric spaces

Let $G$ be a group given as a quotient

$$\pi : F(S) \to G,$$

of the free group on a set $S$. Therefore $G = \langle S|R\rangle$. The word length $|g|$ of an element $g \in G$ is the smallest integer $n$ for which there exists a sequence $s_1 . \cdots s_n$ of elements in $S \cup S^{-1}$ such that

$$g = \pi(s_1 . \cdots s_n).$$

The word metric $d_S(g_1, g_1)$ is defined on $G$ by

$$d_S(g_1, g_2) = \left|g_1^{-1} g_2\right|.$$

$G$ acts on itself from the left by isometries.

DEFINITION 29. *(Cayley graph) Note, that if $S$ and $\overline{S}$ are two finite generating sets of $G$ then $d_S$ and $d_{\overline{S}}$ are bi-Lipschitz equivalent, namely $\exists\ C\ \forall\ g_1, g_2 \in G$,*

$$\frac{1}{C} d_S(g_1, g_2) \leq d_{\overline{S}}(g_1, g_2) \leq C\ d_S(g_1, g_2).$$

DEFINITION 30. *(Polynomial growth) A ball of radius $n$ is $Cay(G, S)$ is*

$$B_n = \{g \in G| \ \ |g| \leq n\}.$$

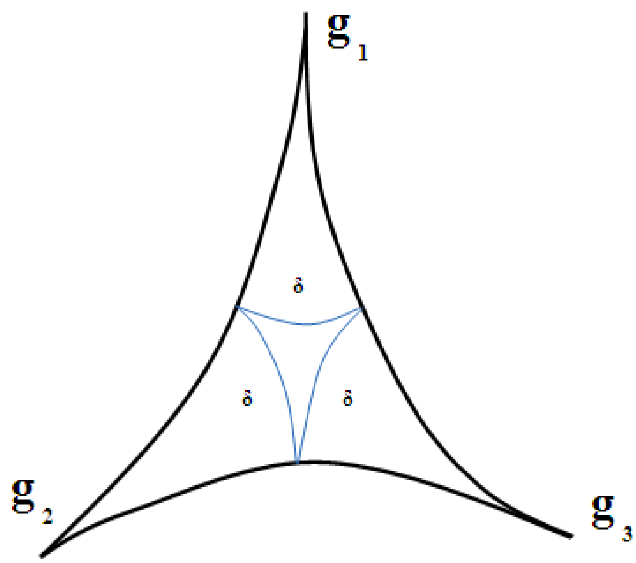
DEFINITION 31. *A group $G$ has polynomial growth iff the number of elements in $B_n$ is bounded by a polynomial $p(n)$.*

## 4. Hyperbolic groups

DEFINITION 32. *A geodesic metric space is called $\delta-$hyperbolic if for every geodesic triangle, each edge is contained in the $\delta$ neighborhood of the union of the other two edges.*

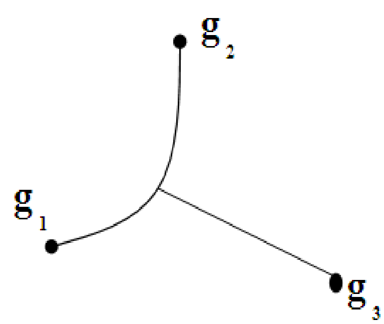
DEFINITION 33. *If $\delta = 0$ the space is called a real tree, or $\mathbb{R}-$ tree.*

Example 31. *A group $G$ is hyperbolic $Cay\,(G, X)$ is hyperbolic (= It looks like a tree)*




Example 32. *Geodesic triangles are $\delta-$thin hyperbolic.*

Example 33. *$F\,(S)$ is hyperbolic so is a $0-$ hyperbolic, so $\delta = 0$.*




## 5. Quasi-isometry

Definition 34. *Let $(X, d_X)$ and $(Y, d_Y)$ be metrics spaces. Given real numbers $k \geq 1$ and $C \geq 0$, a map*

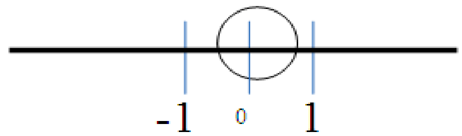
$$f : X \to Y,$$

*is called a $(k, C)-$ quasi-isometry if*

(1) $\frac{1}{k} d_X\,(x_1, x_2) - C \leq d_Y\,(f\,(x_1)\,, f\,(x_2)) \leq k d_X\,(x_1, x_2) + C$, $\forall x_1, x_2 \in X$.
(2) *the $C$ neighborhood of $f\,(X)$ is all of $Y$.*

**Examples of quasi-isometries**

EXAMPLE 34. *$(\mathbb{R}; d)$ and $(\mathbb{Z}; d)$ are quasi-isometric. The natural embedding of $\mathbb{Z}$ in $\mathbb{R}$ is isometry. It is not surjective, but each point of $\mathbb{R}$ is at most $\frac{1}{2}$ away from $\mathbb{Z}$.*



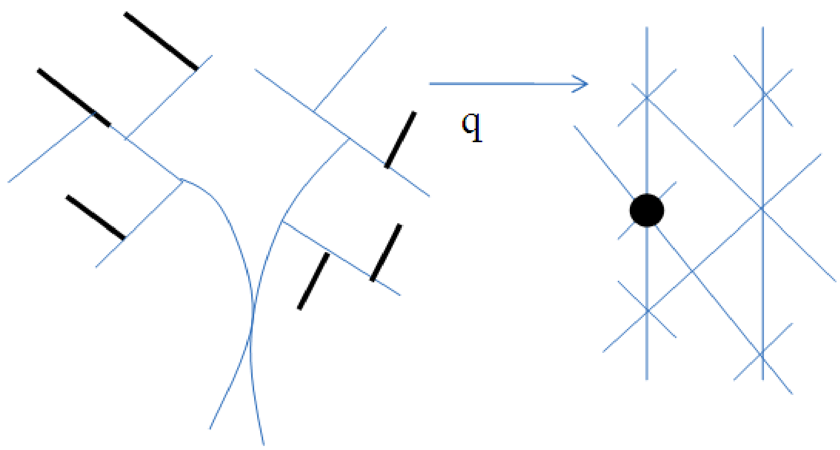

*$\mathbb{R}$ is at most $\frac{1}{2}$ away from $\mathbb{Z}$.*

EXAMPLE 35. *All regular trees of valence at least 3 are quasi-isometric. We denote by $T_k$ the regular tree of valence $k$ and we show that $T_3$ is quasi-isometric to $T_k$ for every $k \geq 4$. We define that map*

$$q : T_3 \to T_k,$$

*sending all edges drawn in thin lines isometrically onto edges and all paths of length $k-3$ drawn in thick lines onto one vertex. The map $q$ thus defined is surjective and it satisfies the inequality*

$$\frac{1}{k-2} dist\,(x, y) - 1 \leq dist\,(q\,(x)\,, q\,(y)) \leq dist\,(x, y)\,.$$




Free groups of different rank are quasi- isometric

EXAMPLE 36. *All non-Abelian free groups of finite rank are quasi-isometric to each other. The Cayley graph of the free group or rank $n$ with respect to a set of $n$ generators and their inverses is the regular simplicial tree valence $2n$.*

EXAMPLE 37. *Let $G$ be a group with a finite generating set $S$, and let $Cay\,(G, S)$ be the corresponding Cayley graph. We can make $Cay\,(G, S)$ into a metric space by identifying each edge with a unit interval $[0, 1]$ in $\mathbb{R}$ and defining $d\,(x, y)$ to be the length of the shortest path joining $x$, and $y$. This coincides with the path length metric when $x$ and $y$ are vertices. Since every point of $Cay\,(G, S)$ is in the $\frac{1}{2}-$neighborhood of some vertex, we see that $G$ and $Cay\,(G, S)$ are quasi-isometric for this choice of $d$.*

EXAMPLE 38. *Every bounded metric space is quasi-isometric to a point.*

EXAMPLE 39. *If $S$ and $T$ are finite generating sets for a group $G$, then $(G, d_S)$ and $(G, d_T)$ are quasi-isometric.*

EXAMPLE 40. *The main example, which partly justifies the interest in quasi-isometries, is the following. Given $M$ a compact Riemannian manifold, let $\widetilde{M}$ be its universal covering and let $\pi_1(M)$ be its fundamental group. The group $\pi_1(M)$ is finitely generated, in fact even finitely presented. The metric space $\widetilde{M}$ with the Riemannian metric is quasi-isometric to $\pi_1(M)$ with some word metric.*

EXAMPLE 41. *If $G_1$ is a finite index subgroup of $G$, then $G$ and $G_1$ are quasi-isometrically equivalent.*

## 6. Quasi-isometries Rigidity

We consider the question of Gromov: Characterize all classes of groups $\mathcal{K}$ complete with respect to quasi-isometries (every group quasi-isometric to a group from $\mathcal{K}$ has a finite index subgroup in $\mathcal{K}$). The following list give us groups which are in the rigid class.

(1) Finitely presented groups,
(2) Nilpotent groups,
(3) Abelian groups,
(4) Hyperbolic groups,
(5) nonabelian free groups of finite rank (follows from the fact that their Cayley graphs are trees).
(6) Amenable group (see below).

REMARK 13. *Solvable groups $\cong$ quasi-isometric groups.*

EXAMPLE 42.

$$Sol := \left\{ \left( \begin{array}{ccc} e^{\frac{z}{2}} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-\frac{z}{2}} \end{array} \right) : (x, y, z) \in \mathbb{R}^3 \right\}.$$

(Eskin, Fisher, Whyte) obtained first results on quasi-isometric rigidity of non-nilpotent polycyclic groups.

DEFINITION 35. *A group $G$ is polycyclic if we have*

$$G \trianglerighteq G_1 \trianglerighteq G_2 \cdots \trianglerighteq G_k = \{1\},$$

*where $\frac{G_i}{G_{I+1}}$ are cyclic $i = 0, \cdots, k-1$.*

REMARK 14. *Polycyclic groups are close to hyperbolic groups.*

THEOREM 24. *Any group quasi-isometric to the three dimensional solvable Lie group Sol is virtually a lattice in Sol.*

That completed the classification of three dimensional manifolds up to quasi-isometry started by Thusston, Schwartz and other.

**Conjecture**

Let $G$ be a solvable Lie group, and $\Gamma$ a lattice in $G$. Any finitely generated group $\Gamma^{'}$ quais-isometric to $\Gamma$ is virtually a lattice in a solvable Lie group $G^{'}$.

Equivalently, any finitely generate group quasi-isometric to a polycyclic group is virtually polycyclic.

## 7. Limit groups (fully residually free groups)

DEFINITION 36. *A marked group $(G, S)$ is a group $G$ with a prescribed family of generators $S = (s_1, \cdots, s_n)$.*

Two marked groups $(G, (s_1, \cdots, s_n))$ and $\left(G', \left(s_1', \cdots, s_n'\right)\right)$ are isomorphic as marked groups if the bijection

$$s_i \longleftrightarrow s_i'$$

extends to an isomorphism.

EXAMPLE 43. *$(\langle a \rangle, (1, a))$ and $(\langle a \rangle, (a, 1))$ are not isomorphic as marked groups.*

Denote $\mathcal{G}_n$ the set of groups marked by $n$ elements up to isomorphism of marked groups. One can define a metric on $\mathcal{G}_n$ by setting

$$(\maltese\spadesuit) \qquad dist\left((G, S), \left(G', S'\right)\right) = e^{-N}.$$

the distance between two marked groups $(G, S)$, and $\left(G', S'\right)$ to be $e^{-N}$ if they have exactly the same relations of length at most $N$ (under the bijection $S \longleftrightarrow S'$)(Grigorchuk, Gromov's metric).

DEFINITION 37. *A limit group is a limit (with respect to the metric above) of marked free groups in $\mathcal{G}_n$.*

REMARK 15. *We see that $(\maltese\spadesuit)$ is a metric:*

(1) If $G \cong G'$ this
$$dist\left((G, S), \left(G, S'\right)\right) = 0.$$
(2) Symmetric,
$$dist\left((G, S), \left(G', S'\right)\right) = e^{-N} = dist\left(\left(G', S'\right), (G, S)\right).$$
(3) Triangle inequality,
$$\begin{aligned} dist\left((G, S), \left(G', S'\right)\right) &\leq dist\left((G, S), \left(G'', S''\right)\right) + dist\left(\left(G'', S''\right), \left(G', S'\right)\right) \\ &= \frac{e^{-N}}{2} + \frac{e^{-N}}{2} = e^{-N}.. \end{aligned}$$
Therefore it is a metric.

EXAMPLE 44. *A free Abelian group of rank 2 is a limit of a sequence of cyclic groups with marking*

$$(\langle a \rangle, (a, a^n)), \ n \to \infty.$$

*this is*

$$\begin{aligned} &\left(\langle a \rangle, \left(\underset{=x}{a}, \underset{=y}{a^n}\right)\right), \\ y &= x^n, \end{aligned}$$

*and*

$$\begin{aligned} &\left(\langle a \rangle, \left(\underset{=x}{a}, \underset{=y}{a^m}\right)\right), \\ y &= x^m. \end{aligned}$$

*The relation is given by $[x, y] = 1$.*

THEOREM 25. *Let $G$ be a finitely generated group. Then the following conditions are equivalent:*

(1) $G$ is fully residually free (that is for finitely many non-trivial elements $g_1, \cdots g_n \in G$ there exists a homomorphism $\phi$ from $G$ to a free group such that $\phi(g_i) \neq 1$ for $i = 1, \cdots, n$).
(2) (Champetier and Guirardel) $G$ is a limit of free groups in Gromov-Grigorchuk metric.
(3) (Remeslennikov) $G$ is universally equivalent to $F$ (in the language without constants).

REMARK 16. *The property* (3) *implies that $G$ is a limit of free groups.*

REMARK 17. *Fully residually free=Residually finite.*

REMARK 18. *Fully residually free$\neq$ Residually free.*

EXAMPLE 45. *To see that Fully residually free$\neq$ Residually free, consider $F \times F$ is residually free but not fully residually free.*

EXAMPLE 46. *Universal sentence that is true in $F$, $\forall x, y, z$*

$$([x, y] = 1 \text{ and } [x, z] = 1 \text{ and } x \neq y \rightarrow [y, z] = 1).$$

## 8. Free actions on metric spaces

THEOREM 26. *A group $G$ is free iff it acts freely by isometries or by automorphism on a tree.*

Free action= no inversion of edges and stabilizers of vertices are trivial

EXAMPLE 47. $\left|g_1^{-1} g_2\right| = \left|(gg_1)^{-1} (gg_2)\right|$.

DEFINITION 38. *Stabilizers of vertices: If $G$ acts on a set $V$, $v \in V$. $G_v = \{g \in G | gv = v\}$ is the stabilizer of $v$.*

EXAMPLE 48. *Every free group acts trivially on a Cayley graph.*

## 9. $\mathbb{R}-$ Trees

DEFINITION 39. *A $\mathbb{R}-$ tree is a metric space $(X, \rho)$ where*

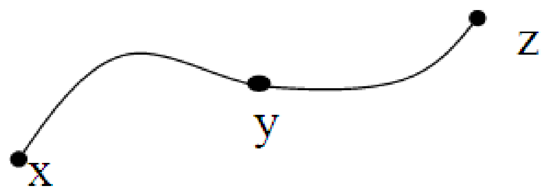
$$\rho : X \times X \rightarrow \mathbb{R},$$

*which satisfies the following properties:*

(1) *$(X, \rho)$ is geodesic,*
(2) *if two segments of $(X, \rho)$ intersect in a single point, which is an endpoint of both, then their union is a segment,*
(3) *the intersection of two segments with a common endpoint is also a segment.*

REMARK 19. *An $\mathbb{R}-$ tree or $0-$hyperbolic metric space, no triangle just tripods.*

EXAMPLE 49. *of property* (2)

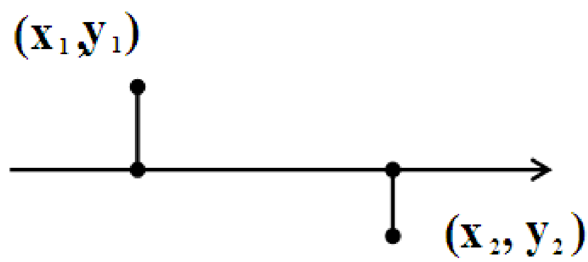



EXAMPLE 50. *Consider $X = \mathbb{R}$ with usual metric.*

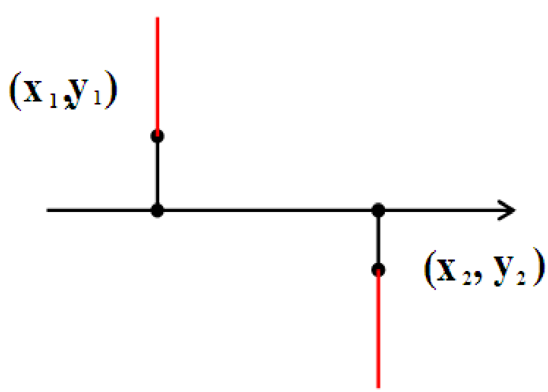
EXAMPLE 51. *A geometric realization of a simplicial tree.*

EXAMPLE 52. *Consider $X = \mathbb{R}^2$ with metric $d$ defined by*

$$d\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \left\{ \begin{array}{c} \left|y_1\right| + \left|y_2\right| + \left|x_1 - x_2\right|, \text{ if } x_1 \neq x_2, \\ \left|y_1 - y_2\right|, \text{ if } x_1 = x_2. \end{array} \right\}$$

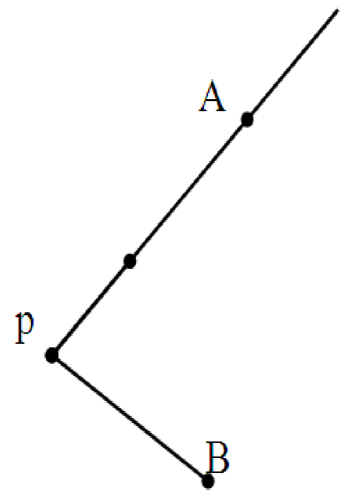



EXAMPLE 53. *To consider the property* (2) *for the figure above we just add lines as the figure shows:*



EXAMPLE 54. *Consider $X = \mathbb{R}^2$ with SNCF metric (French Railway System).*

$$d_p(x,y) \quad : \quad = \left\{ \begin{array}{c} \|x-y\|_{L^2}\,,\ \text{if } x \text{ and } y \text{ lie on the same ray} \\ \text{from the origin,} \\ \|x\|_{L^2} + \|y\|_{L^2}\,,\ \text{otherwise.} \end{array} \right\}$$




*unique geodesic*

## 10. Finitely generated $\mathbb{R}-$ free groups

THEOREM 27. *(Rips', 1991 not published) A finitely generated group is $\mathbb{R}-$ free (acts freely on an $\mathbb{R}-$tree by isometries) iff it is a free product of surface groups (except for the non-orientable surfaces of genus 1,2,3) and free Abelian groups of finite rank.*

REMARK 20. *Gaboriau, Levitt, Paulin (1994) gave a complete proof of Rips' theorem.*

REMARK 21. *Bestvina, Feighn (1995) gave another proof of Rips' theorem proving a more general result for stable actions on $\mathbb{R}-$ trees.*

## 11. Bass-Serre

DEFINITION 40. *Free product with amalgamation. Let $G$ be a free product*

$$G = A \underset{u=v}{*} B,$$

*(free product of $A$ and $B$ amalgamated over $u = v$). If $U \leq A$, $V \leq B$ consider*

$$\varphi : U \underset{\cong}{\rightarrow} V,$$

$$G = \left\langle gen.\ of\ A,\ gen.\ of\ B \middle| \begin{array}{c} Rel.\ of\ A \\ Rel.\ of\ B \end{array} \forall u \in U,\ \varphi(u) = u \right\rangle.$$

DEFINITION 41. *$G = A\underset{u}{*}$ is HNN extension if $\varphi(U) \leq A$ is an isometry.*

$$G = \left\langle gen.\ of\ A, \underset{stable\ letter}{t} \middle| Rel.\ of\ A,\ \varphi(u) = tut^{-1}, \forall u \in U \right\rangle.$$

Example 55. *Consider $G = B(2,3)$ is a HNN extension generated by a cyclic group this is*

$$\begin{aligned} A &= \langle a \rangle,\ t = b \\ U &= \left\langle a^2 \right\rangle, \\ V &= \left\langle a^3 \right\rangle, \\ \varphi &: a^2 \to a^3. \end{aligned}$$

Remark 22. $B(2,3) = \left\langle a, b | ba^2b^{-1} = a^3 \right\rangle$ *is non Hopfian. This is there exist an epimorphism*

$$\begin{aligned} \theta &: a \to a^2, \\ \theta &: b \to b, \end{aligned}$$

*with nontrivial kernel. This is* $\left[bab^{-1}, a^3\right] \in \ker\theta$. *To see this we have*

$$\begin{aligned} \left[bab^{-1}, a^3\right] &= bab^{-1}a^3 \left(bab^{-1}\right)^{-1} \left(a^3\right)^{-1} \\ &= bab^{-1} \left(ba^2b^{-1}\right) \left(bab^{-1}\right)^{-1} \left(ba^2b^{-1}\right)^{-1} \\ &= ba^3a^{-1}a^{-2}b^{-1} = 1. \end{aligned}$$

Definition (40) and (41) both are called splittings of $G$.

Definition 42. *A directed graph consists of a set of vertices $V(X)$ and a set of edges $E(X)$ together with two functions*

$$\begin{aligned} \sigma &: E(X) \to V(X), \\ \tau &: E(X) \to V(X). \end{aligned}$$

*For an edge $e \in E(X)$ the vertices $\sigma(e)$ and $\tau(e)$ are called the origin and the terminus of $e$.*

Definition 43. *A non-oriented graph is a directed graph $X$ with involution*

$$- : E(X) \to E(X),$$

*which satisfies the following conditions:*

(1) $\overline{\overline{e}} = e$,
(2) $e \neq \overline{e}$,
(3) $\sigma(\overline{e}) = \tau(e)$.

We refer to a pair $\{e, \overline{e}\}$ as a non-oriented edge.

Definition 44. *A path $p$ in a graph $X$ is a sequence if edges $e_1, \cdots, e_n$ such that $\sigma(e_i) = \tau(e_{i+1})$, $i \in \{1, \cdots, n-1\}$. Put $\sigma(p) = \sigma(e_1)$, $\tau(p) = \tau(e_n)$.*

Definition 45. *A path $p = e_1, \cdots, e_n$ is reduced if $e_{i+1} \neq \overline{e}_i$ for each $i$.*

Definition 46. *A path $p$ is closed (or a loop) if $\sigma(p) = \tau(p)$.*

Let $X$ and $Y$ be graphs. A morphism of graphs

$$\phi : X \to Y,$$

consists of two maps

$$\begin{aligned} \phi_V &: V(X) \to V(Y), \\ \phi_E &: E(X) \to E(Y), \end{aligned}$$

such that

$$\begin{aligned} \phi_V(\sigma(e)) &= \sigma\phi_E(e), \\ \phi_V(\tau(e)) &= \tau\phi_E(e), \\ \phi_V(\overline{e}) &= \overline{\phi_E(e)}. \end{aligned}$$

We say that a graph $X$ is a subgraph of a graph $Y$ if

$$V(X) \subseteq V(Y),\ E(X) \subseteq E(Y),$$

and the inclusion maps

$$V(X) \hookrightarrow V(Y),\ E(X) \hookrightarrow E(Y),$$

form a morphism of graphs

$$X \to Y.$$

Definition 47. *A graph of groups $\Gamma(\mathcal{G}, X)$ consists of*

(1) *a connected graph $X$;*
(2) *a function $\mathcal{G}$ which for every vertex $v \in V(X)$ assigns a group $G_v$, and for each edge $e \in E(X)$ assigns a group $G_e$ such that*

$$G_{\overline{e}} = G_e.$$

(3) *For each edge $e \in E(X)$ there exists a monomorphism*

$$\sigma : G_e \to G_{\sigma e}.$$

Let $\Gamma(\mathcal{G}, X)$ be a graph of groups. Since

$$G_{\overline{e}} = G_e.$$

then there exists a monomorphism

$$\sigma : G_e \to G_{\sigma e} = G_{\tau e}.$$

which we denote by

$$\tau : G_e \to G_{\tau e}.$$

Let $\Gamma = \Gamma(\mathcal{G}, X)$ be a graph of groups, and let $T$ be a maximal subtree of $X$. Suppose the groups $G_v$ are given by presentations $G_v = \langle X_v | R_v \rangle$, $v \in V(X)$. We define a fundamental group $\pi(\Gamma)$ of the graph of groups $\Gamma$ by generators and relations:

- Generators of $\pi(\Gamma)$ :

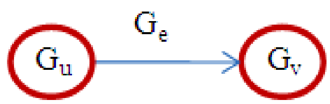
$$\bigcup_{v \in V(X)} X_v \bigcup \{t_e | e \in E(X)\}.$$

- Relations of $\pi(\Gamma)$ :

$$\bigcup_{v \in V(X)} R_v \bigcup \left\{t_e^{-1}\sigma g t_e = \tau g | g \in G_e, e \in E(X)\right\} \cup \left\{t_{\overline{e}} = t_e^{-1} | e \in E(X)\right\} \cup \{t_e = 1 | e \in T\}.$$

We assume here that $\sigma(g)$ and $\tau(g)$ are words in generators $X_{\sigma e}$ and $X_{\tau e}$, correspondingly.

Example 56. *Let* $\Gamma$ *be*

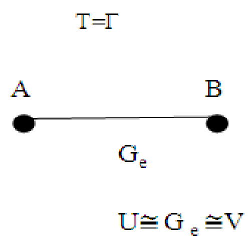



*then*

$$\pi\left(\Gamma\right) = G_u \underset{G_e}{*} G_v,$$

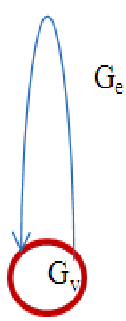
*free product with amalgamation.*

Example 57. *Consider*




$$\begin{aligned} G_e &\hookrightarrow A, \\ G_e\left(U\right) &= B. \\ \pi\left(\Gamma\right) &= G. \end{aligned}$$

Example 58. *If* $\Gamma$ *is*



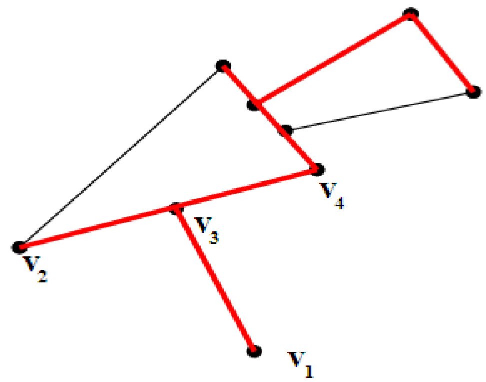

*then*

$$\pi\left(\Gamma\right) = G_v \underset{G_e}{*}$$

*-HNN extension of* $G_v$*. Were we have*

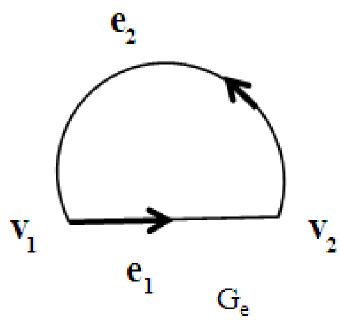
$$\begin{aligned} G_v &= A, \\ \sigma G_e &= U, \\ \tau G_e &= V. \end{aligned}$$

Example 59. *Think of the following graph*




*where the maximum subtree is in red. Amalgamation of the product along the maximum subtree is HNN=*$\pi(\Gamma)$.

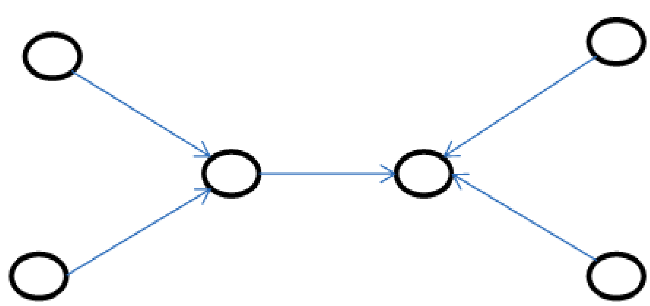
Example 60. *Consider the following graph*




*we have generators of* $G_{v_1}$ *and* $G_{v_2}$. *Now the relators are* $\sigma g = \tau g$, $\forall g \in G_{e_1}$. *Then*

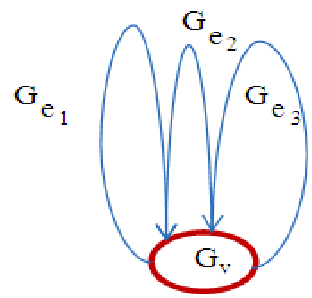
$$\begin{aligned} H &= G_{v_1} \underset{G_{e_1}}{*} G_{v_2}. \\ K &= H_{\sigma(G_e)} * \\ \pi(\Gamma) &= K. \end{aligned}$$

Example 61. *If* $\Gamma$ *is*



*then* $\pi(\Gamma)$ *is called a tree product.*

EXAMPLE 62. *If* $\Gamma$ *is*




*then* $\pi(\Gamma)$ *is a generalized HNN extension.*

EXERCISE 9. *Prove that if* $G_1$ *is a finite index subgroup of* $G$*, then* $G$ *and* $G_1$ *are quasi-isometrically equivalent.*

# Chapter 5

Remark 23. *Bass- Serre was to understand the structure of group $SL_2(K)$. When $K$ is a field, Serre wanted to split $K$. Suppose $SL_2(K)$ acts on a tree then $SL_2(K)$ split.*

## 1. Amalgamated free products

Definition 48. *Let $A, B, C$ be groups and let*

$$\begin{aligned} \sigma &: C \to A, \text{ and} \\ \tau &: C \to B, \end{aligned}$$

*be injective homomorphisms. If the diagram below is a push out then we write*

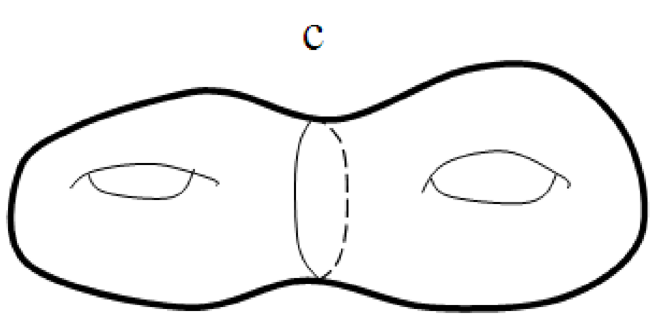
$$G = A \underset{C}{*} B$$

*and we say that $G$ is the amalgamated (free) product of $A$ and $B$ over $C$.*

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d] \\ B \arrow[r] & G \end{tikzcd}$$

Example 63. $F(a,b) * F(c,d) = \langle a,b,c,d | [ab][dc] = 1 \rangle = S_2$. *There is no relations. Let $[a,b] = [c,d]$, then $S_2$ is an orientable surface of genus 2.*

$$\begin{aligned} \sigma c &= \langle [a,b] \rangle, \\ \tau c &= \langle [c,d] \rangle, \\ c &= \langle c \rangle, \\ \sigma c &= [a,b], \\ \tau c &= [c,d]. \end{aligned}$$

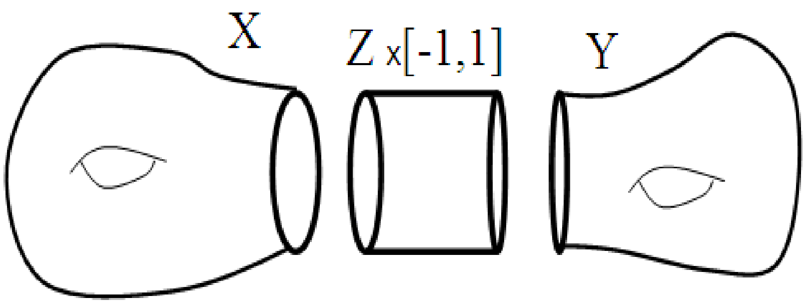


If $G$ is a group, then there exists a connected CW-complex $K(G,1)$ (the Eilenberg-MacLane space) such that $\pi_1(K(G,1)) \cong G$. For $A, B, C, \sigma, \tau$ as above, let $X = K(A,1)$, $Y = K(B,1)$, $Z = K(C,,1)$ be three spaces and realize $\sigma$ and $\tau$ as maps

$$\begin{aligned} \delta_+ &: Z \to X, \\ \delta_- &: Z \to Y. \end{aligned}$$

Now let

$$W = X \bigsqcup (Z \times [-1,1]) \bigsqcup Y \cap /\tilde{},$$




$$W = X \bigsqcup (Z \times [-1,1]) \bigsqcup Y \cap /\tilde{}$$

where $(z, \pm 1)\,\tilde{}\,\delta_{\pm 1}(z)$. By the Seifert-Van Kampen theorem,

$$\pi_1(W) \cong A \underset{C}{*} B.$$

Suppose that $A = \langle S_1 | R_1 \rangle$, $B = \langle S_2 | R_2 \rangle$, then

$$A \underset{C}{*} B = \left\langle S_1 \bigsqcup S_2 | R_1, R_2, \{\sigma(c) = \tau(c), c \in C\} \right\rangle.$$

Example 64. *Let $\Sigma$ be a connected surface and let $\gamma$ be a separating, simple closed curve. Let $\frac{\Sigma}{\gamma} = \Sigma_- \bigsqcup \Sigma_+$. Then*

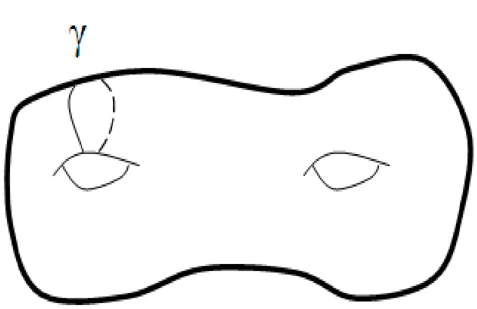
$$\pi_1(\Sigma) \cong \pi_1(\Sigma_-) \underset{\langle \gamma \rangle}{*} \pi_1(\Sigma_+).$$

*If $\gamma$ is non-separating (but still 2-sided), then there are two natural maps*

$$\delta_{\pm 1} : \mathbb{S}^1 \to \Sigma_\circ,$$

*representing $\gamma$, where $\Sigma_\circ = \Sigma - im(\gamma)$.*

Example 65. *The HNN extension of $\Sigma_\circ = \Sigma - im(\gamma)$ is:*



Let $G = \pi_1(X, \Gamma)$, be the fundamental group of the graph of groups, with $G_v, G_e$ and with subgroups

$$\begin{aligned} \sigma(G_e) &\leq G_{\sigma(e)}, \\ \tau(G_e) &\leq G_{\tau(e)}. \end{aligned}$$

Consider

$$X = \coprod X_v \bigsqcup \cup (X_e \times [-1, 1]),$$

identify

$$(X_e \times \{-1\}) \longleftrightarrow \bar{\sigma}(X_e),$$

where $\bar{\sigma}$ is the embedding that realized $\sigma$. The same holds for

$$(X_e \times \{1\}) \longleftrightarrow \bar{\tau}(X_e),$$

where $\bar{\tau}$ is the embedding that realized $\tau$. This is called graphs of spaces. This definition does not depends on the tree. We just construct this space with out the assumption that groups act on trees. Let $G = \langle A * B | c = \phi(c), c \in C \rangle$. Choose a system of right coset representatives $T_C$ and $T_D$, where $D = \phi(C)$.

Definition 49. *A $C-$ normal form is a sequence $(x_\circ, x_1, \cdots, x_n)$ such that*
1. $x_\circ \in C$,
2. $x_i \in T_C - \{1\}$ *or* $x_i \in T_D - \{1\}$, *and the consecutive terms $x_i$ and $x_{i+1}$ lie in distinct systems of representatives. Similarly one can define a $D-$ normal form.*

Theorem 28. *Any element $g \in G = A \underset{C=D}{*} B$ can be uniquely written in the form $g = x_\circ x_1 \cdots x_n$, where $(x_\circ, x_1, \cdots, x_n)$ is a $C-$ normal form.*

Proof. Let $g \in G$. If $g \in C$, then $g = x_\circ$ which is a normal form.

$$g = a_1 b_1 \cdots a_k b_k \notin C, \tag{1.1}$$

and $a_i, b_i \notin C, D$, for $i \neq 1$. Where $b$ can be represented as $b = \widetilde{b}\overline{b}$, $\overline{b} \in T_B$ and $\widetilde{b} \in D$ and $\widetilde{b} \in \sigma^{-1}(\widetilde{a}), \widetilde{a} \in C$ and $\widetilde{a} = \widetilde{b}$. Then

$$\begin{aligned} g &= a_1 b_1 \cdots a_k \left(\widetilde{b}\overline{b}\right)_k \\ &= a_1 b_1 \cdots \underbrace{a_k \widetilde{b_k}}_{a_k'} \overline{b_k} \\ &= a_1 b_1 \cdots a_k' \overline{b_k}. \end{aligned}$$

By induction suppose that we have (1.1) $\ni$we can apply induction to $g = a_1 b_1 \cdots a_k' \overline{b_k}$. Therefore

$$a_1 b_1 \cdots a_k' \overline{b_k},$$

is a normal form.

(Uniqueness). We make $G$ act on the set $W_C$ of $C-$ normal form (or $D-$normal form). First $A$ acts on $W_C$, let $\tau \in W_C$. If $\tau \in C$ then let $\tau = x_\circ$, so that

$$\begin{aligned} g\tau &= g x_\circ \in C, \text{ if } g \in C, \\ g\tau &= g\widetilde{x_\circ} g \overline{x_\circ}, \text{ if } g \notin C \end{aligned}$$

where $g\overline{x_\circ}$ is the right coset representative. If $\tau \notin C$, then

$$g\tau = \left\{ \begin{array}{c} \left(gx_\circ, x_1, \cdots, x_n\right), \text{ if } g \in C, \\ \left(\widetilde{gx_\circ} \overline{gx_\circ}, x_1, \cdots, x_n\right), \text{ if } x_1 \in B, g \notin C, \\ \left(\widetilde{gx_\circ x_1} \overline{gx_\circ x_1}, x_2, \cdots, x_n\right), \text{ if } x_1 \in A, \text{ and } gx_\circ x_1 \notin C \\ \left(gx_\circ x_1, x_2, \cdots, x_n\right), \text{ if } x_1 \in A, \ gx_\circ x_1 \in C. \end{array} \right\}$$

Similarly, $B$ acts on $W_D$. Let

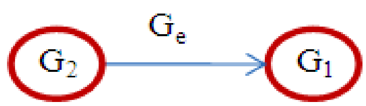
$$\begin{aligned} \psi &: W_C \to W_D, \ni \\ \psi\left(x_\circ, x_1, \cdots, x_n\right) &= \left(\phi\left(x_\circ\right), x_1, \cdots, x_n\right) \\ b\left(\tau\right) &= \psi^{-1}\left(b\psi\left(\tau\right)\right). \end{aligned}$$

Free products acts also on normal forms. Extends the action of $A$, $B$ on $W_C$ to $A * B$. Check that $c\left(\phi\left(c\right)\right)^{-1} \in Ker\left(\text{ of the action}\right)$. Then $A \underset{C}{*} B$ acts on $W_C$. Take $g \in G$

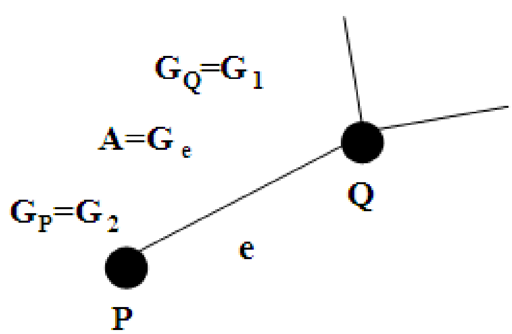
$$g\left(1\right) = x_\circ x_1 \cdots x_n.$$

Therefore, $\exists$ a unique normal form. $\square$

This graph is called a segment




THEOREM 29. *Let $G_1 \underset{A}{*} G_2$. Then there exist a tree $X$, on which $G$ acts without inversion on edges such that the factor graph $G\backslash X$, (=graph of orbits or vertices of the trees $X$) is a segment. Moreover this segment can be lifted to a segment in $X$ with the property that the stabilizers in $G$ of its vertices and edges are equal to $G_1, G_2$ and $A$ respectively.*

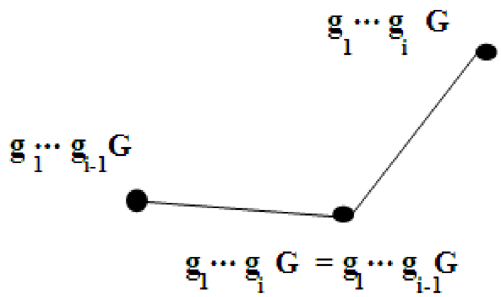



PROOF. Let $X^\circ = \frac{G}{G_1} \cup \frac{G}{G_2}$, (union of left cosets) and $X_+^1 = \frac{G}{A}$. Put $\sigma\left(gA\right) = gG_1$, $\tau\left(gA\right) = gG_2$, and let $\widetilde{T}$ be the segment in $X$ with the vertices $G_1, G_2$ and the positively oriented edge $A$. $G$ acts on $X$ by left multiplication.

$X$ is connected. Indeed, let $g = g_1 \cdots g_n$ with $g_i \in G_1$ or $g_i \in G_2$ (alternating product) depending on the parity of $i$. Then $gG_1$ is connected by a path to $G_1$. If $g_i \in G_1$, then

$$g_1 \cdots g_{i-1} G_1 = g_1 \cdots g_i G_1,$$

if $g_i \in G_2$, then $g_1 \cdots g_{i-1}G_1$ and $g_1 \cdots g_i G_1$ are connected by the edges to $g_1 \cdots g_{i-1}G_2 = g_1 \cdots g_i G_2$. Now, the connectivity follows by induction on $n$.




Suppose there is a reduced loop $e_1 \cdots e_n$. With out lost of generality

$$\sigma(e_1) = G_1.$$

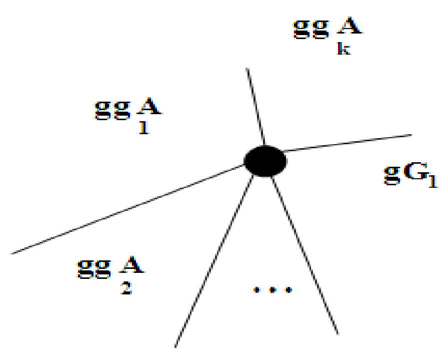
Since adjacent vertices are cosets of different subgroups, $n$ is even and there exists $x_i \in G_1 - A$, $y_i \in G_2 - A$ such that

$$\begin{aligned} \sigma(e_2) &= x_1 G_2, \\ \sigma(e_3) &= x_1 y_1 G_1, \cdots, \tau(e_n) = x_1 y_1 \cdots x_{\frac{n}{2}} y_{\frac{n}{2}} G_1. \end{aligned}$$

Since

$$\tau(e_n) = \sigma(e_1) = G_1,$$

this contradicts uniqueness of normal form. Also we have no loops because the vertices are alternating.




We have that $g_1 \cdots g_n$ are left coset representatives of $A$ in $G_1$ different edges are the different cosets so we obtain the tree. The quotient is the segment. $\square$

REMARK 24. *In $X$, all edges with initial vertex $gG_1$ have the form $gg_1A$, where $g_1$ runs over the set of representatives of the left cosets of $A$ in $G_1$. The degree of $gG_1$ is $|G_1 : A|$. The stabilizer of $gG_1$ is $gG_1g^{-1}$.*

For the converse of the above theorem if we have a group that acts on trees.

THEOREM 30. *Let $G$ act without inversions on edges on a tree $X$ and suppose that the factor graph $G \backslash X$ is a segment. Let $\widetilde{T}$ be an arbitrary lift of this segment* *in $X$. Denote its vertices by $P$, $Q$, and the edge by $e$, and let $G_p$, $G_q$, $G_e$ be their stabilizers. Then the homomorphism*

$$\phi : G_P \underset{G_e}{*} G_Q \to G,$$

*which is the identity on $G_P$ and $G_Q$ is an isomorphism.*

PROOF. Write $G^{'} = \langle G_p, G_q \rangle$ and prove that $G = G^{'}$. If $G^{'} < G$ then the graph $G^{'}\ \widetilde{T}$ and $\left(G - G^{'}\right)\widetilde{T}$ are disjoint: Suppose $g^{'} \in G^{'}$ and $g \in \left(G - G^{'}\right)$ and suppose they are not disjoint. Let $gG_p = g^{'}G_p$ this cannot be since this implies that $g \in g^{'}G_p \subseteq G^{'}$, but $g \in \left(G - G^{'}\right) \Longrightarrow\Longleftarrow$. Also $gG_P = g^{'}G_Q$ cannot happen since $P$ and $Q$ are different orbits. Then $G^{'}\ \widetilde{T}$ and $\left(G - G^{'}\right)\widetilde{T}$ are disjoint.

But $G\,\widetilde{T} = X$ is connected, which is a contradiction.

Injection of $\phi$. Let $\widetilde{G} = G_P \underset{G_e}{*} G_Q$ and let $\widetilde{X}$ be the tree constructed from $\widetilde{G}$ as in the proof of the previous theorem (29). Define a morphism

$$\begin{array}{rcl} \psi & : & \widetilde{X} \to X, \text{ by} \\ gG_r & \to & \phi(g)\, r, \end{array}$$

where $r \in \{P, Q, e\}$. It is surjective because $X = G\widetilde{T}$ and $G = \langle G_P, G_Q \rangle$, and is locally injective morphism from a tree to a tree, therefore injective. Let $g \in \widetilde{G} - G_P$. Then the vertices $G_P$ and $gG_P$ of the tree $\widetilde{X}$ are distinct, therefore vertices $P$ and $\phi(g)\, P$ of the tree $X$ are also distinct. Hence $\phi(g) \neq 1$. Where

$$\phi : G_P \underset{G_e}{*} G_Q \to G,$$

□

## 2. Action of $SL_2(\mathbb{Z})$ on the hyperbolic plane

$\mathbb{H}^2 = \{z \in \mathbb{C} | im(z) > 0\}$.

DEFINITION 50. *A hyperbolic line is an open half circle or an open half line (in the Euclidean sense) in $\mathbb{H}^2$ such that its closure meets the real axis at right angles.*

DEFINITION 51. *A Möbius transformation of $\mathbb{H}^2$ is a map*

$$z \to \frac{az + b}{cz + d},$$

*where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$.*

$SL_2(\mathbb{R})$ acts on $\mathbb{H}^2$ by the rule

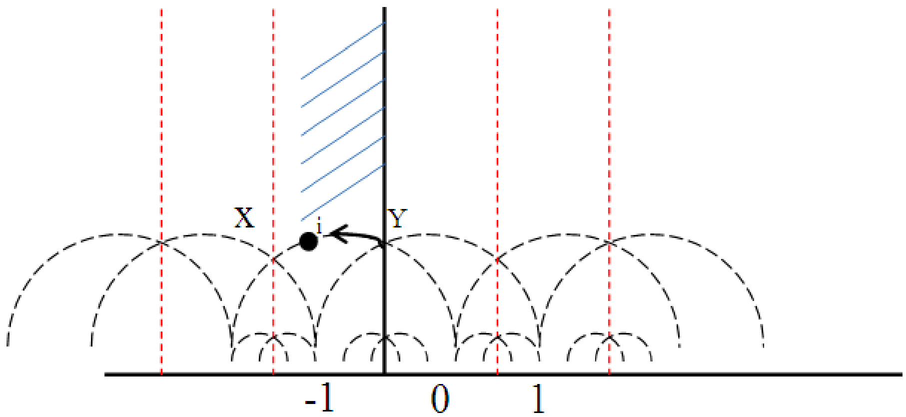
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az + b}{cz + d},$$

The kernel is $\{\pm 1\}$.

Let

$$\mathcal{M} = \left\{ z | 1 < |z|, -\frac{1}{2} < \operatorname{Re}(z) \leq \frac{1}{2} \right\} \cup \left\{ e^{i\alpha} | \frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2} \right\}.$$

THEOREM 31. *The set $\mathcal{M}$ is the fundamental domain for the action of $PL_2(\mathbb{Z})$ on $\mathbb{H}^2$.*




## 3. Action of $SL_2(\mathbb{Z})$ on a tree

THEOREM 32. *The union of the images of the arc*

$$T = \left\{ e^{i\alpha} | \frac{\pi}{3} \le \alpha \le \frac{\pi}{2} \right\},$$

*under the action of the group $SL_2(\mathbb{Z})$ is a tree. $SL_2(\mathbb{Z})$ acts on this tree without inversion on edges and so that distinct points of the arc are in equivalent. The stabilizers of endpoints are generated by the matrices*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

*and*

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

*of orders 4 and 6. The stabilizer of the arc is generated by*

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

*of order 2. In particular*

$$SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_6.$$

EXAMPLE 66. *Let $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\langle A, C \rangle \cong D_4$, $\langle B, C \rangle \cong D_6$. Then $GL_2(\mathbb{Z}) \cong D_4 \underset{D_2}{*} D_6$.*

THEOREM 33. [*Serre*] *For $n \ge 3$ the groups $SL_2(\mathbb{Z})$ and $GL_2(\mathbb{Z})$ cannot be represented as nontrivial amalgamated products. (They do not act on trees).*

## 4. Trees and HNN extensions

Let $G = \left\langle H, t | t^{-1} a t = \phi(a), a \in A, \phi(A) = B \right\rangle$.

DEFINITION 52. *A normal form is a sequence $(g_\circ, t^{\epsilon_1}, g_1, \cdots, t^{\epsilon_n}, g_n)$ such that*

1. *$g_\circ$ is an arbitrary element of $H$,*
2. *if $\epsilon_i = -1$, then $g_i \in T_A$ (right coset representative),*

(3) *if* $\epsilon_i = 1$, *then* $g_i \in T_B$,
(4) *there is no consecutive subsequence* $t^{\epsilon}, 1, t^{-\epsilon}$.

REMARK 25. $at = tb$, $tg_i \in T_B$ *and* $g \in B$. *Similarly* $bt = t^{-1}a$.

THEOREM 34. *(Britton's Lemma):*

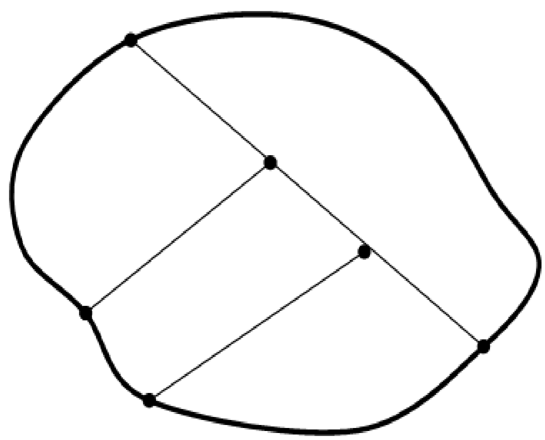
(1) *Every element* $x \in G$ *has a unique representation*

$$x = g_\circ t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n,$$

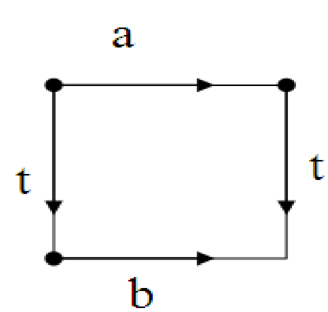
*where* $(g_\circ, t^{\epsilon_1}, g_1, \cdots, t^{\epsilon_n}, g_n)$ *is a normal form.*

(2) $H$ *is embedded into* $G$ *by the map* $h \to h$. *If* $w = g_\circ t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$, *and this expression does not contain subwords* $t^{-1} g_i t$ *with* $g_i \in A$ *or* $t g_i t^{-1}$ *with* $g_i \in B$, *then* $w \neq 1$ *in* $G$.

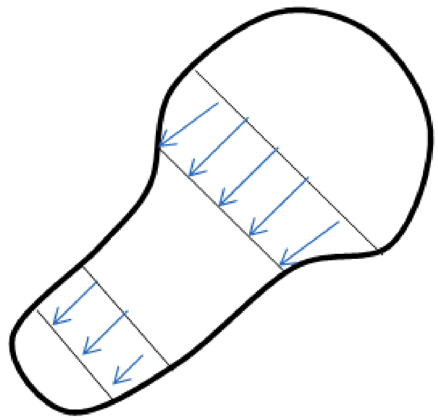
EXAMPLE 67. *Consider the Van-Kampen diagram for* $G = \langle X | R \rangle$ *labeled planar graphs. Simple loops are labeled by relations. We can read a word* $w$ *on the contour (boundary) of the diagrams. Reduced cycled reduced* $w = 1$ *in* $G$ *iff* $\exists$ *a diagram with boundary.*



EXAMPLE 68. *Consider the extension of HNN of*



*The Van-Kampen looks like,*



THEOREM 35. *Let* $G = \left\langle H, t | t^{-1} a t = \phi(a), a \in A, \phi(A) = B \right\rangle$. *Then there exist a tree* $X$ *on which* $G$ *acts without inversion of edges such that the factor graph* $G \backslash X$ *is a loop. Moreover, there is a segment* $\widetilde{Y}$ *in* $X$ *such that the stabilizers of its vertices and edges in the group* $G$ *are equal to* $H$, $tHt^{-1}$ *and* $A$ *respectively.*

PROOF. Set $X^{\circ} = \frac{G}{H}$, $X_{+}^{1} = \frac{G}{A}$ (all cosets are left), $\sigma(gA) = gH$, $\tau(gA) = gtH$, and let $\widetilde{Y}$ be the segment in $X$ with vertices $H$, $tH$. $G$ acts on $X$ by left multiplication. This implies that there is uniqueness of normal form. $\square$

THEOREM 36. *Let* $G$ *act without inversions on edges on a tree* $X$ *and suppose that the factor graph* $Y = G \backslash X$ *is a loop. Let* $\widetilde{Y}$ *be an arbitrary segment in* $X$. *Denote its vertices by* $P, Q$, *and the edge by* $e$, *and let* $G_p$, $G_q$, $G_e = G_{\bar{e}}$ *be their stabilizers. Let* $x$ *be an arbitrary element such that* $Q = xP$. *Put* $G_e^{'} = x^{-1} G_e x$ *and let*

$$\phi : G_e \to G_e^{'},$$

*be the isomorphism induced by conjugation by* $x$. *Then* $G_e^{'} \leq G_P$ *and the homomorphism*

$$\left\langle G_P, t | t^{-1} a t = \phi(a), a \in G_e \right\rangle \to G,$$

*which is the identity on* $G_P$ *and sends* $t$ *to* $x$ *is an isomorphism.*

PROOF. Sketch of proof:
Because all vertices are the same order we have,

$$\begin{aligned} Q &= xP,\ G_Q = xG_P, \\ G_e &\leq G_P,\ \ G_e \leq G_Q, \text{ so} \\ G_e &\leq xG_P x^{-1} \text{ and } x^{-1} G_e x \leq G_P. \end{aligned}$$

$\square$

DEFINITION 53. *A graph of groups* $\Gamma(\mathcal{G}, X)$ *consists of*

(1) *a connected graph* $X$;
(2) *a function* $\mathcal{G}$ *which for every vertex* $v \in V(X)$ *assigns a group* $G_v$, *and for each edge* $e \in E(X)$ *assigns a group* $G_e$ *such that*

$$G_{\bar{e}} = G_e.$$

(3) *For each edge $e \in E(X)$ there exists a monomorphism*

$$\sigma : G_e \to G_{\sigma e}.$$

Let $\Gamma(\mathcal{G}, X)$ be a graph of groups. Since

$$G_{\overline{e}} = G_e.$$

then there exists a monomorphism

$$\sigma : G_e \to G_{\sigma e} = G_{\tau e}.$$

which we denote by

$$\tau : G_e \to G_{\tau e}.$$

Let $\Gamma = \Gamma(\mathcal{G}, X)$ be a graph of groups, and let $T$ be a maximal subtree of $X$. Suppose the groups $G_v$ are given by presentations $G_v = \langle X_v | R_v \rangle$, $v \in V(X)$. We define a fundamental group $\pi(\Gamma)$ of the graph of groups $\Gamma$ by generators and relations:

- Generators of $\pi(\Gamma)$ :

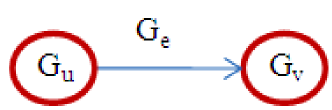
$$\bigcup_{v \in V(X)} X_v \bigcup \{t_e | e \in E(X)\}.$$

- Relations of $\pi(\Gamma)$ :

$$\bigcup_{v \in V(X)} R_v \bigcup \left\{t_e^{-1} \sigma g t_e = \tau g | g \in G_e, e \in E(X)\right\} \cup \left\{t_{\overline{e}} = t_e^{-1} | e \in E(X)\right\} \cup \{t_e = 1 | e \in T\}.$$

We assume here that $\sigma(g)$ and $\tau(g)$ are words in generators $X_{\sigma e}$ and $X_{\tau e}$, correspondingly.

Example 69. *Let $\Gamma$ be*

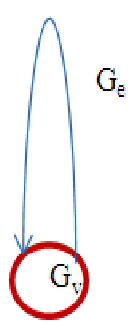



*then*

$$\pi(\Gamma) = G_u \underset{G_e}{*} G_v,$$

*free product with amalgamation.*

Example 70. *If $\Gamma$ is*



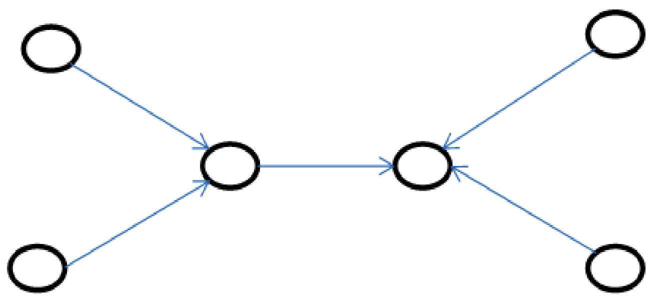

*then*

$$\pi(\Gamma) = G_v \underset{G_e}{*}$$

*-HNN extension of $G_v$. Were we have*

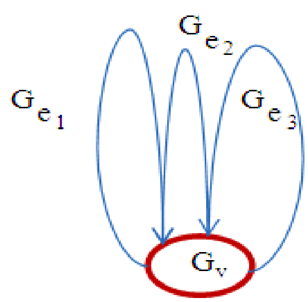
$$\begin{aligned} G_v &= A, \\ \sigma G_e &= U, \\ \tau G_e &= V. \end{aligned}$$

Example 71. *If $\Gamma$ is*



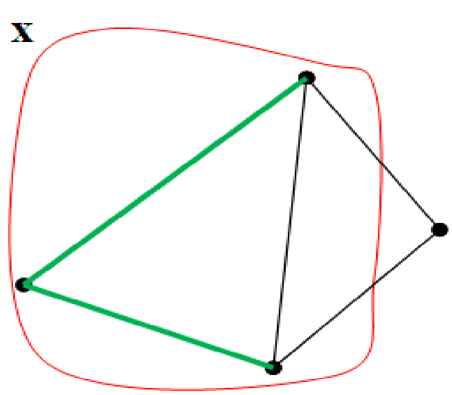
*then $\pi(\Gamma)$ is called a tree product.*

Example 72. *If $\Gamma$ is*




*then $\pi(\Gamma)$ is a generalized HNN extension.*

Let $\pi(\Gamma|_Y, S)$ be the fundamental group of a graph. Let $S$ be a maximal subtree of $Y$, and $T$ a max subtree of $X$.



Let $\Gamma = (G, X)$ be a graph of groups and $Y \subseteq X$ be a connected subgraph. Then one can define a subgraph of groups

$$\Gamma|_Y = (G_Y, Y),$$

where $G_Y = G|_Y$ is the restriction of $G$ on $Y$, i.e., every vertex and every edge from $Y$ has the same associated groups as in $\Gamma$. Every maximal subtree $S$ of $Y$ can be extended to a maximal subtree $T$ of $X$ with $S \subseteq T$. The identical map

$$\left\{ \begin{array}{c} g_v \in G_v \to g_v \in G_v, \ \ (v \in Y) \\ t_e \to t_e, \ \ (e \in Y), \end{array} \right\}$$

gives rise to a homomorphism of the free product

$$\phi_\circ : \mathop{*}_{v \in V(Y)} G_v * F(E(Y)) \to \pi(\Gamma, T),$$

where $F(E(Y))$ is a free group with basis $E(Y)$. Clearly, $\phi_\circ$ sends all defining relations of $\pi(\Gamma|_Y, S)$ into identity. Hence it induces a homomorphism

$$\phi_Y : \pi(\Gamma|_Y, S) \to \pi(\Gamma, T).$$

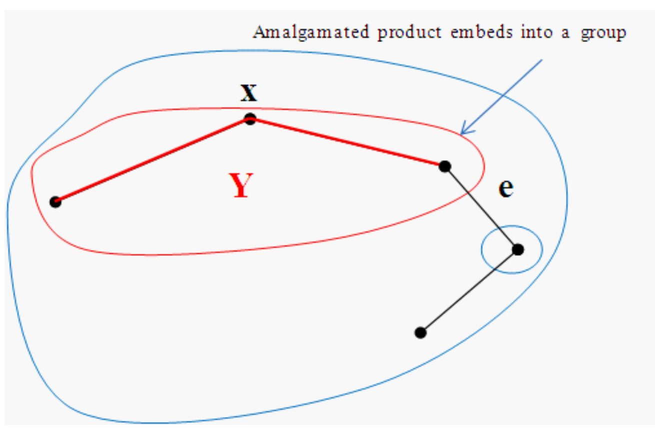
We call $\phi_Y$ the canonical homomorphism.

THEOREM 37. *The canonical homomorphism*

$$\phi_Y : \pi(\Gamma|_Y, S) \to \pi(\Gamma, T).$$

*is a monomorphism.*

PROOF.




So we have a connected graph. Amalgamated product $Y$ embeds into a group. Now we can add and edge and extend the amalgamated product to the blue graph, and we repeat this process. Then we do induction on the number of edges.

$\boxed{\text{Case 1}}$ Let $X$ be a finite tree, so $S \subseteq T = X$. If $S = T$ then there is nothing to prove. If $S \neq T$ then there exists an edge $e \in T$ and a subtree $T_1$ of $T$ such that $T = T_1 \cup \{e\}$ and $S \subseteq T_1$. By induction on $|V(T)|$ the canonical homomorphism

$$\phi_1 : \pi(\Gamma|_S, S) \to \pi(\Gamma|_{T_1}, T_1),$$

is a monomorphism. Observe that also by induction we have canonical monomorphism

$$G_{\sigma(e)} \overset{\phi_{\sigma(e)}}{\hookrightarrow} \pi(\Gamma|_S, S) \overset{\phi}{\hookrightarrow} \pi(\Gamma|_{T_1}, T_1).$$

In particular

$$G_e \overset{\sigma}{\hookrightarrow} G_{\sigma(e)} \overset{\phi_{\sigma(e)}}{\hookrightarrow} \pi\left(\Gamma|_{T_1}, T_1\right),$$
$$G_e \overset{\tau}{\hookrightarrow} G_{\tau(e)},$$

are monomorphism. This shows that the representation of $\pi\left(\Gamma|_T, T\right)$ via generators and relations is a presentation of a free product with amalgamation:

$$\pi\left(\Gamma|_T, T\right) = \pi\left(\Gamma|_{T_1}, T_1\right) \underset{G_e}{*} G_{\tau(e)}.$$

Hence

$$\pi\left(\Gamma|_{T_1}, T_1\right) \overset{\phi_{T_1}}{\hookrightarrow} \pi\left(\Gamma|_T, T\right),$$

is a monomorphism as well as

$$\pi\left(\Gamma|_S, S\right) \overset{\phi}{\hookrightarrow} \pi\left(\Gamma|_{T_1}, T_1\right) \overset{\phi_{T_1}}{\hookrightarrow} \pi\left(\Gamma|_T, T\right),$$

as required.

$\boxed{\text{Case 2}}$ Suppose now that $T$ is an infinite tree and $S$ is finite. Then there exists an increasing chain of finite trees

$$S = T_\circ \subseteq T_1 \subseteq \cdots \subseteq T_i \subseteq \cdots$$

such that

$$T = \cup T_i.$$

Then the canonical monomorphism

$$\pi\left(\Gamma|_{T_\circ}, T_\circ\right) \hookrightarrow \pi\left(\Gamma|_{T_1}, T_1\right) \hookrightarrow \cdots,$$

provided an increasing chain of groups. Clearly,

$$\pi\left(\Gamma, T\right) = \lim_{i\to\infty} \pi\left(\Gamma|_{T_i}, T_i\right)$$ [1]

and for each $i$ there exists an embedding

$$\pi\left(\Gamma|_{T_i}, T_i\right) \hookrightarrow \pi\left(\Gamma, T\right).$$

In particular,

$$\pi\left(\Gamma|_S, S\right) \hookrightarrow \pi\left(\Gamma, T\right).$$

$\boxed{\text{Case 3}}$ Let $S \subset T$ be infinite trees. Then

$$\pi\left(\Gamma|_S, S\right) = \lim_{i\to\infty} \pi\left(\Gamma|_{S_i}, S_i\right),$$

for some infinite chain of finite subtrees

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots,$$

such that $S = \bigcup_i S_i$. By case 2

$$\phi_{S_I} : \pi\left(\Gamma|_{S_i}, S_i\right) \hookrightarrow \pi\left(\Gamma, T\right),$$

is a monomorphism for each $i$. Therefore, the canonical homomorphism

$$\pi\left(\Gamma|_S, S\right) = \lim_{i\to\infty} \pi\left(\Gamma|_{S_i}, S_i\right) \hookrightarrow \pi\left(\Gamma, T\right),$$

is a monomorphism.

[1] Notice that direct limit hare is just a union of the increasing chain of groups.

$\boxed{\text{Case 4}}$ Let $X$ be an arbitrary graph and $X - T$ be finite. Then $\pi(\Gamma, T)$, is an HNN extension of $\pi(\Gamma|_T, T)$ (see example 72 above). Case 3 implies that

$$\pi(\Gamma|_S, S) \overset{\phi}{\hookrightarrow} \pi(\Gamma|_T, T) \overset{\phi_T}{\hookrightarrow} \pi(\Gamma, T),$$

is a monomorphism. It follows from the properties of HNN extensions that the canonical map

$$\pi(\Gamma|_T, T) \overset{\phi_T}{\hookrightarrow} \pi\left(\Gamma|_{T\cup(Y-S)}, T\right),$$

is a monomorphism.

$\boxed{\text{Case 5}}$ Let now $X$ be an arbitrary graph. Then $X = \bigcup\limits_i X_i$ such that $T \subseteq X_i$ and $X_i - T$ is finite for every $i$. Then

$$\pi(\Gamma, T) = \lim_{i\to\infty} \pi(\Gamma|_{X_i}, T),$$

by Case 4,

$$\pi(\Gamma|_{Y\cap X_i}, S) \overset{\phi_{Y\cap X_i}}{\hookrightarrow} \pi(\Gamma|_{X_i}, T),$$

is monic, as well as

$$\pi(\Gamma|_Y, S) = \lim_{i\to\infty} \pi(\Gamma|_{Y\cap X_i}, S) \hookrightarrow \lim_{i\to\infty} \pi(\Gamma|_{X_i}, T) = \pi(\Gamma, T)$$

□

## 5. Exercises

Exercise 10. *(Britton's Lemma):*

Theorem 38. (1) *Every element $x \in G$ has a unique representation*

$$x = g_\circ t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n,$$

*where $(g_\circ, t^{\epsilon_1}, g_1, \cdots, t^{\epsilon_n}, g_n)$ is a normal form.*

(2) *$H$ is embedded into $G$ by the map $h \to h$. If $w = g_\circ t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$, and this expression does not contain subwords $t^{-1} g_i t$ with $g_i \in A$ or $t g_i t^{-1}$ with $g_i \in B$, then $w \neq 1$ in $G$.*

Exercise 11. *Let*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

*and*

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

*of orders* 4 *and* 6. *Let*

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

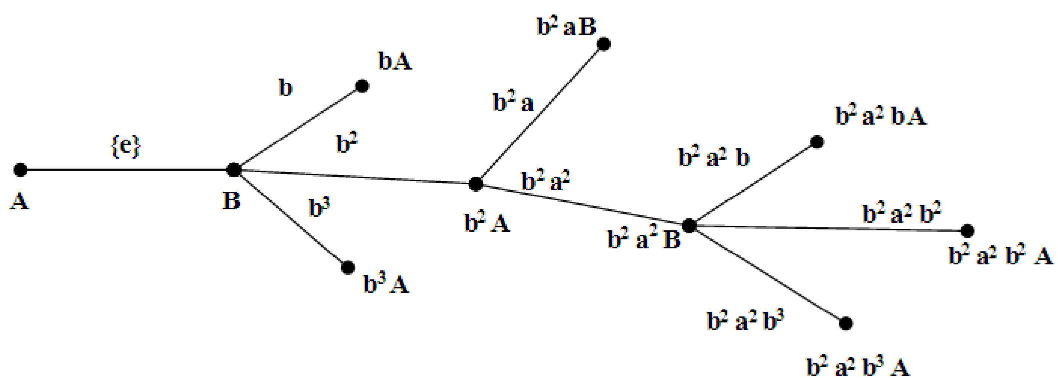
*The stabilizer of the arc is generated by*

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

*of order* 2. *Prove that $\langle A, C\rangle \cong D_4$, $\langle B, C\rangle \cong D_6$. Deduce that $GL_2(\mathbb{Z}) \cong D_4 \underset{D_2}{*} D_6$.*

# Chapter 6

EXAMPLE 73. *Consider $G = A * B$, where $A = \langle a | a^3 = 1 \rangle$ and $B = \langle b | b^4 = 1 \rangle$, construct a Bass-Serre tree*

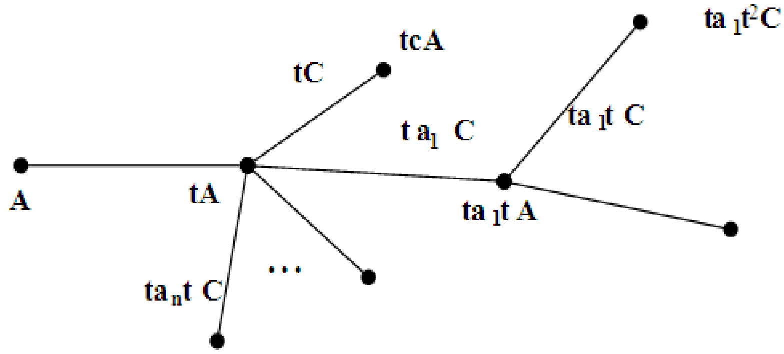

*where $b^2a^2B = b^2a^2bB$, this is*

$$\begin{aligned} \sigma(gE) &= gA, \\ \tau(gE) &= gB. \end{aligned}$$

EXAMPLE 74. *Consider $G = \langle a, bt | b^{-1}a^2b^2t = a^3b^3 \rangle$ and $G = A_t *$ we constructed a HNN extension of $G$. Let $C = \langle a^2b^2 \rangle$, and $D = \langle a^3b^3 \rangle$, where the order*

$$\begin{aligned} |C| &= \infty, \\ |D| &= \infty. \end{aligned}$$

*Then the index of* $\left[A : \left\langle a^2b^2\right\rangle\right] = \infty$.




*Which is an infinite tree, the degree is infinite since both groups have infinite order.*

$$\begin{aligned}\sigma\left(ta_1tC\right) &= ta_1tA,\\ \tau\left(gE\right) &= gtA,\\ \tau\left(ta_1tC\right) &= ta_1t^2A.\end{aligned}$$

Condition 1. *Suppose we have a free product* $A * B = G$*. If* $g \in G$, $|g| < \infty$, *then* $g$ *is conjugate into* $A$ *or* $B$*.*

Proof.
(1) Take the normal form $x_\circ x_1$ then if we take the square, i.e. $x_\circ x_1 x_\circ x_1$ is also a normal form $\neq e$.
(2) Suppose the path starts $A$ and ends at $A$, this is,

$$\begin{aligned} g &= a_1ba_1,\\ g^2 &= a_1b_1a_2a_1b_1a_2,\end{aligned}$$

by (1) this is a normal form. Unless $a_1$ is the inverse of $a_2$ we have normal forms for higher power $\neq e$, this is

$$g^3 = a_1b_1a_2a_1b_1a_2a_1b_1a_2 \neq e$$

by induction on the length we the result follows.

□

Let $\Gamma = \Gamma\left(G, X\right)$ be a graph of groups, and let $T$ be a maximal subtree of $X$. Suppose the groups $G_v$ are given by presentations $G_v = \left\langle X_v | R_v\right\rangle$, $v \in V\left(X\right)$. We define a fundamental group $\pi\left(\Gamma, T\right)$ of the graph of groups $\Gamma$ by generators and relations:

- Generators of $\pi\left(\Gamma\right)$ :

$$\bigcup_{v\in V(X)} gen\left(G_v\right) \bigcup \left\{t_e | e \in E\left(X\right)\right\}.$$

- Relations of $\pi\left(\Gamma\right)$ :

$$\bigcup_{v\in V(X)} gen\left(G_v\right) \bigcup \left\{t_e^{-1}\sigma g t_e = \tau g | g \in G_e, e \in E\left(X\right)\right\} \cup \left\{t_{\overline{e}} = t_e^{-1} | e \in E\left(X\right)\right\} \cup \left\{t_e = 1 | e \in T\right\}.$$

We assume here that $\sigma\left(g\right)$ and $\tau\left(g\right)$ are words in generators $X_{\sigma e}$ and $X_{\tau e}$, correspondingly.

Now we will construct $\pi(\Gamma, T)$ in a different way, which does not depend on a choice of the tree $T$.

PROOF. Let the free product

$$\begin{aligned} G &= \mathop{*}_{v\in V(X)} G_v * F(E(X)) \\ &= \left\langle \bigcup_{v\in V(X)} gen(G_v) \bigcup E(X) \mid \bigcup_{v\in V(X)} gen(G_v) \right\rangle . \end{aligned}$$

Put (here by $ncl_G(X)$ we denote the normal subgroup generated by $X$):

$$N = ncl_G \left\langle e^{-1}\sigma(g)\, e = \tau(g), e\overline{e} = 1 | e \in E(X), g \in G_e \right\rangle ,$$

where we use the following notation $F(e_1, \overline{e}_1, e_2, \overline{e}_2)$ for $e_1^{-1} = \overline{e}_1$ and denote $F(\Gamma) = \frac{G}{N}$. If we set

$$M_T = ncl_{F(\Gamma)}(eN | e \in T),$$

then

$$\frac{F(\Gamma)}{M_T} \cong \pi(\Gamma, T).$$

So it suffices to show that $\frac{F(\Gamma)}{M_T}$ does not depend on $T$.

$$\begin{array}{c} G \\ \downarrow \\ F(\Gamma) \\ \downarrow \mu \\ \pi_1(\Gamma, T), \end{array}$$

where $\pi_1(\Gamma, T)$ depends on $T$.Fix a point $v_\circ \in V(X)$ and define a subgroup $H \leq G$ which consists of all elements of the free product $G$ that can be presented in the form

$$g_\circ e_1 g_1 e_2 g_2 \cdots e_n g_n,$$

where $g_\circ \in G_{v_\circ}$, $g_i \in G_{\tau e_i}$, $e_i \in E(X)$ and $e_1, \cdots, e_n$ is a closed path in $X$ at $v_\circ$.

REMARK 26. *$H$ is a subgroup of $G$.*

For a vertex $v \in V(X)$ denote by $p_v$ the unique path in $T$ from $v_\circ$ to $v$. Let $\phi : G \to H$ be the homomorphism which is defined on free factors of $G$ as follows:

$$\left\{ \begin{array}{c} \phi(g_v) = p_v g_v p_v^{-1}, \\ \phi(e) = p_{\sigma e} e p_{\tau e}^{-1}, \end{array} \right\}$$

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \lambda \downarrow & & \\ F(\Gamma) & & \\ \mu \downarrow & & \\ \pi_1(\Gamma, T) & & \end{array}$$

REMARK 27. *$\phi$ is a homomorphism.*

It suffices to show that $\phi(\mathrm{Rel}(G_v)) = 1$.

Remark 28. *$\phi$ is a retract on $H$, i.e., $\phi(G) = H$ and $\phi_{|H} = id_{|H}$. This is $h \in H$, $g(h) = h$, then*

$$h = g_\circ e_1 g_1 e_2 g_2 \cdots e_n g_n,$$

*then*

$$\begin{aligned} & p_{v_\circ} g_\circ p_{v_\circ}^{-1} p_{v_\circ} e_1 p_{\tau e_1}^{-1} p_{\tau e_1} g_1 \cdots p_{v_n}^{-1} p_{v_n} e_1 p_{\tau e_n}^{-1} p_{\tau e_n} g_n, \\ = \; & g_\circ e_1 g_1 e_2 g_2 \cdots e_n g_n \end{aligned}$$

Denote by $\lambda : G \to F(\Gamma)$ the canonical epimorphism and put

$$\pi_1(\Gamma, v_\circ) = \lambda(H) \leq F(\Gamma).$$

Clearly, $\pi_1(\Gamma, v_\circ)$ does not depend on $T$.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\lambda"'] \arrow[dr, "\phi_1"] & H \arrow[d, "\lambda_H"] \\ F(\Gamma) \arrow[r, dashed, "\phi_2"'] & \pi_1(\Gamma, v_\circ) \end{tikzcd}$$

The composition $\phi_1 = \lambda_{|H} \circ \phi$ gives a homomorphism.

$\boxed{\text{Claim 1}}$ $\phi_1(N) = 1$. Indeed,

$$\begin{aligned} \phi_1(e\bar{e}) &= \left(p_{\sigma e} e p_{\tau e}^{-1}\right)\left(p_{\tau e} \bar{e} p_{\sigma e}^{-1}\right) = p_{\sigma e} e \bar{e} p_{\sigma e}^{-1} \\ &= p_{\sigma e} p_{\sigma e}^{-1} = 1. \end{aligned}$$

Similarly,

$$\phi_1\left(e^{-1}\sigma(g)\, e\tau\left(g^{-1}\right)\right) = 1.$$

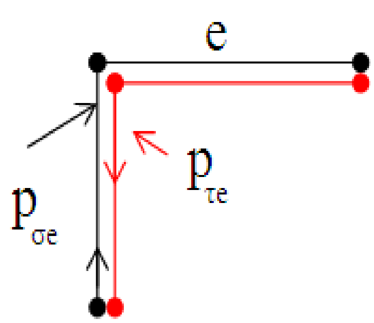
Therefore $\phi_1$ induces a homomorphism

$$\phi_2 : F(\Gamma) \to \pi(\Gamma, v_\circ).$$

$\boxed{\text{Claim 2}}$ $\phi_2(M_T) = 1$. Indeed,

$$\phi_2(eN) = \lambda(\phi(e)) = \lambda\left(p_{\sigma e} e p_{\sigma e}^{-1}\right).$$

If $e \in T$ then $p_{\sigma e} e p_{\tau e}^{-1} = 1$ hence $\phi_2(eN) = 1$.




$p_{\sigma e} e p_{\tau e}^{-1} = 1$

Therefore $\phi_2$ induces a homomorphism

$$\phi_3 : \frac{F(\Gamma)}{M_T} \to \pi(\Gamma, v_\circ),$$

and we have the following commutative diagram where

$$\mu : F(\Gamma) \to \frac{F(\Gamma)}{M_T},$$

is the canonical epimorphism,

$$\begin{tikzcd}
G \arrow[r, "\phi"] \arrow[d, "\lambda"'] \arrow[dr, "\phi_1"] & H \arrow[d, "\lambda_H"] \\
F(\Gamma) \arrow[r, "\phi_2"] \arrow[d, "\mu"'] & \pi_1(\Gamma, V_\circ) \arrow[dl, shift left, "\overline{\mu}"] \\
\pi_1(\Gamma, T) \arrow[ur, shift left, "\phi_3"] &
\end{tikzcd}$$

On the other hand the restriction $\overline{\mu}$ of $\mu$ on $\pi(\Gamma, v_\circ)$ gives a homomorphism

$$\overline{\mu} : \pi(\Gamma, v_\circ) \to \pi(\Gamma, T).$$

We claim that $\overline{\mu} \circ \phi_3 = \phi_3 \circ \overline{\mu} = id$, i.e. $\overline{\mu}$ and $\phi_3$ are isomorphisms. Indeed, since $p_v = e_1 \cdots e_n$, $e_i \in T$, and $\mu(\lambda(e_i)) = 1$ we have the following chain of equalities

$$\begin{aligned} \phi_3(g_v M_T) &= \phi_2(g_v N) = \lambda(\phi(g_v)) \\ &= \lambda\left(p_v g_v p_v^{-1}\right); \end{aligned}$$

$$\mu\left(\lambda\left(p_v g_v p_v^{-1}\right)\right) = \mu(\lambda(g_v)).$$

We have

$$\phi_3(eM_T) = \phi_2(eN) = \phi(e) = p_{\sigma e} e p_{\tau e}^{-1}.$$

Therefore,

$$\overline{\mu}(\phi_3(eM_T)) = \mu\left(\lambda\left(p_{\sigma e} e p_{\tau e}^{-1}\right)\right) = eM_T.$$

□

## 1. Graphs of spaces

Let $G$ be a fundamental group of a graph of groups with underlying graph $Y$. For each edge or vertex $a$ of $Y$ choose $X_a$ to be a connected CW-complex with fundamental group $G_a$. We can do this so that there is an embedding

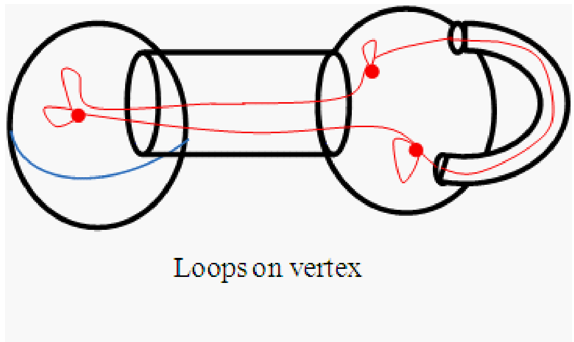
$$X_e \hookrightarrow X_v,$$

realizing the inclusion of groups. We form the topological space $\Gamma(\mathcal{X}, Y)$ by beginning with the disjoint union

$$\coprod_{v \in V} X_v \coprod \coprod_{e \in E} X_e \times I,$$

and

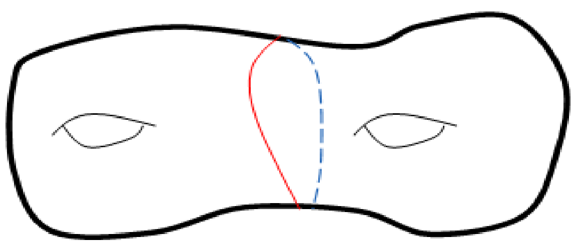
(1) identifying $X_e \times I$ with $X_{\overline{e}} \times I$ via $(x, t) \equiv (x, 1-t)$ and
(2) gluing $X_e \times \{0\}$ to $X_{\sigma(e)}$ via the given inclusion. The resulting topological space has fundamental group $G$.




EXAMPLE 75. *Consider* $G = A \underset{C=D}{*} B$, *the automorphism, and* $C$ *is Abelian. Consider the homomorphism* $\varphi$

$$\varphi : \left\{ \begin{array}{c} a \to a, a \in A \\ b \to cdc^{-1}, \begin{array}{c} c \in C \\ b \in B \end{array} \end{array} \right\}$$

*and* $c_1 = d_1 \Longrightarrow c = cd_1c^{-1}$ *since* $C$ *is Abelian. Therefore* $\varphi$ *is an automorphism and generalize Dehn twist* $:= \left\{ \begin{array}{c} Cut \\ Paste \\ Twist \end{array} \right\}$. *The above automorphism corresponds to the following group:*



LEMMA 11. *Let* $p : X \to Y$ *be a locally surjective morphism of graphs and* $T$ *be a subtree of* $Y$. *Then there exists a morphism of graphs* $j : T \to X \ni p \circ j = id_T$. *In particular,* $j$ *is injective.*

PROOF. Consider all the pairs $\left(T^{'}, j^{'}\right)$ with $T^{'} \subseteq T$,

$$j^{'} : T^{'} \to X,$$

$p \circ j^{'} = id_{T^{'}}$. By Zorn's lemma there exists a maximal pair $(T_\circ, j_\circ)$ with respect to inclusion. We claim that $T_\circ = T$. Suppose not then there exists an edge $e \in T$ with $\sigma e \in T_\circ$, $\tau e \notin T_\circ$. Since $p$ is locally surjective there exists $e^{'} \in V(X)$ with $pe^{'} = e, \sigma e^{'} = j\sigma e$. Put $T^{'} = T_\circ \cup \{e\}, j^{'} = j \cup \left\{\left(e, e^{'}\right)\right\}$. Then

$$j^{'} : T^{'} \to X,$$

is a graph morphism and $p \circ j^{'} = id_{T^{'}}$. But $T^{'} \supsetneq T_\circ$, this contradicts the maximality of $T_\circ$. $\square$

Suppose now that $G$ acts on $X$ without inversions,

$$p : X \to \frac{X}{G},$$

is the canonical projection, and $T \subseteq \frac{X}{G}$ is a maximal subtree of $\frac{X}{G}$. By the lemma (11) above there exists a morphism

$$j : T \to X,$$

with

$$p \circ j = id_T.$$

In particular, $jT$ is a tree, and

$$j : T \to jT,$$

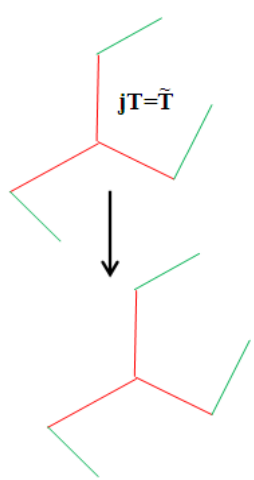
is an isomorphisms of graphs. We call $jT$ a representative tree of the action of $G$ on $X$.


Lemma 12. *Let $G$ act on $X$ without inversions and*

$$p : X \to \frac{X}{G},$$

*be the canonical projection. If $Y^1_{\pm}$ is an orientation of $Y = \frac{X}{G}$, then $X^1_+ = p^{-1}Y^1_+$ is an orientation of $X$.*

Theorem 39. *Let $G$ be a group which acts freely (on a set acts faithfully) and without inversions on a tree $X$. Then $G$ is free.*



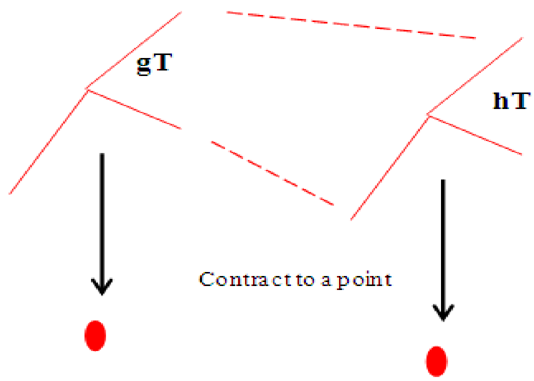

The trees upstairs get map to different trees downstairs

Proof. Let $T$ be a representative tree for the action of $G$ on $X$. Then

$$gT \cap hT = \varnothing, \;\; g \neq h.$$

Indeed, if for some vertices $v, w \in T$ we have $gv = hw$ then $p(v) = p(w)$ contradicting the fact that $p|_T$ is injective, unless $v = w$. But $gv = hv$ implies $h^{-1}gv = v$ and hence $h^{-1}g = 1$, i.e. $h = g$. Contradiction.

Observe that every vertex of $X$ belongs to $gT$ for some $g \in G$. It follows that $X$ consists of subgraphs $gT$, $g \in G$, and some other edges. Since $X$ is a tree there exists at most one edge connecting $gT$ and $hT$ for $g \neq h$. Denote by $Y$ the graph which is obtained from $X$ by contacting every subtree $gT$, $g \in G$, to a single vertex which we denote by $gT$.Clearly, $Y$ is a tree.




Put

$$S = \left\{ g \in G | g \neq 1 \exists e \in X_+^1 \ni \sigma e \in T \text{ and } \tau e \in gT \right\}.$$

We label edges from $Y$ by elements from $S^{\pm} = S \cup S^{-1}$ as follows. Let $e \in X_+^1$, $e$ connects $gT$ to $hT$. Then the edge $g^{-1}e$ connects $T$ to $g^{-1}hT$, and $g^{-1}e \in X_+^1$. Hence $g^{-1}h \in S$. We label the edge $e$ by $g^{-1}h$, and the edge $\overline{e}$ by $h^{-1}g$. How $S$ generates $G$. This is $G = \langle S \rangle$ take a path in $S$ and apply induction on the number of paths.

$\boxed{\text{Claim.}}$ The labeled graph $Y$ is isomorphic to the Cayley graph $\Gamma\left(G, S^{\pm 1}\right)$.

We define a map

$$\phi : \Gamma\left(G, S^{\pm 1}\right) \to Y,$$

in the following way:

(1) $\phi(g) = gT$ (vertex), $g \in G$.
(2) If $(g, s)$ is an edge in $\Gamma\left(G, S^{\pm 1}\right) \ni s \in S$ then there exists an edge $e'$ in $Y$ connecting $T$ to $sT$. Define

$$\phi : (g, s) \to ge'.$$

Also, if $(g, s) \in E\left(\Gamma\left(G, S^{\pm 1}\right)\right)$ and $s \notin S$, then $s^{-1} \in S$ and we define

$$\phi : (g, s) \to \overline{\phi\left(gs, s^{-1}\right)}.$$

It is not hard to check that $\phi$ is a morphism of graphs. Indeed, if $e \in E\left(\Gamma\left(G, S^{\pm 1}\right)\right)$ and $e = (g, s)$ then $\sigma e = g$, $\tau e = gs$. Now $\phi(e) = ge'$, where $\sigma e' = T$, $\tau e' = sT$.

Hence

$$\begin{aligned} \sigma\phi(e) &= \sigma\left(ge^{'}\right) = g\sigma e^{'} \\ &= gT = \phi(\sigma e), \end{aligned}$$

$$\begin{aligned} \tau\phi(e) &= \tau\left(ge^{'}\right) = g\tau e^{'} \\ &= gsT = \phi(\tau e). \end{aligned}$$

It is left to verify that $\phi$ is bijective on edges. □

Let $G$ be a group acting on a connected graph $X$, $Y = \frac{X}{G}$,

$$p : X \to \frac{X}{G},$$

be the canonical projection, $T \subseteq Y$ be a maximal subtree of $Y$, and

$$j : T \to X,$$

be a morphism of graphs with $p \circ j = id_T$. We want to make $Y$ into a graph of groups by assigning to its vertices and edges certain groups which are stabilizers of some vertices and edges under the action of $G$ on $X$. To this end we want to extend

$$j : T \to X,$$

to a map (not a graph morphism, in general)

$$\begin{aligned} j &: Y \to X, \ni \\ p \circ j &= id_Y. \end{aligned}$$

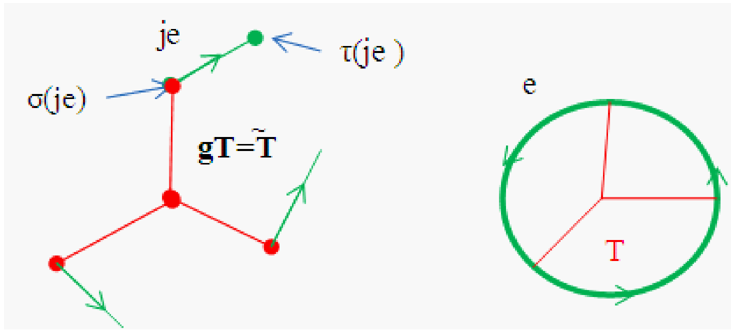
Since $V(T) = V(Y)$ we need to define $j$ on edges from $Y - T$. We will define $j$ on $Y_+^1 - T$ and put $j(e) = \overline{j(\overline{e})}$ for $e \notin A$. Let $e \in Y_+^1 - T$. Since $p$ is locally surjective there exists an edge $e^{'} \in E(X)$ with $p\sigma e^{'} = \sigma e$ and $pe^{'} = e$. Then we define

$$\begin{aligned} j(e) &= e^{'} \text{ and} \\ j(\overline{e}) &= \overline{e}^{'}. \end{aligned}$$

Clearly, $p \circ j = id_Y$. Observed that $p(\tau je) = \tau e = p(j\tau e)$ hence $G\tau je = Gj\tau e$ so $\exists \gamma_e \in G$ such that $\gamma_e j\tau e = \tau je$. For uniformity, if $e \in Y_+^1 \cap T$ then we define $\gamma_e = 1$, and for arbitrary $e$ we put $\gamma_{\overline{e}} = \gamma_e^{-1}$. Recall that for $x \in X$, $Stab_G(x) = \{g \in G | gx = x\}$. Now we define a graph of groups on $Y$ as follows: $y \in Y \Longrightarrow G_y = Stab_G(jy)$. It is left to define a monomorphism

$$\begin{aligned} \sigma &: G_e \to G_{\sigma e}, \\ \tau &: G_e \to G_{\tau e}. \end{aligned}$$

It suffices to do it for each $e \in Y_+^1$. Let $e \in Y_+^1$, then $\sigma je = j\sigma e$.




Hence $Stab_G(je) \leq Stab_G(j\sigma e)$, so $G_e \leq G_{\sigma e}$, and in this case $\sigma$ is the inclusion map. Also

$$\begin{aligned} Stab_G(je) &\leq Stab_G(\tau je) = Stab_G(\gamma_e j\tau e) \\ &= \gamma_e Stab_G(j\tau e)\gamma_e^{-1}. \end{aligned}$$

We define

$$\tau : G_e \to G_{\tau e},$$

as

$$g \mapsto \gamma_e g \gamma_e^{-1}.$$

This defines a graph of groups $\Gamma = (G, Y)$. It is easy to check that maps

$$\begin{aligned} G_v &\to G_v, \\ t_e &\to \gamma_e. \end{aligned}$$

extend to a homomorphism $\phi$,

$$\phi : \pi(\Gamma) \to G.$$

Remark 29. *$\phi$ is a homomorphism, you only need to check for the relations.*

Now we would like to reconstruct (as much as possible) the original graph $X$ and the action of $G$ on $X$ from the quotient graph of groups $\Gamma = \Gamma\left(G, \frac{X}{G}\right)$. We can do this when $X$ is a tree.

Theorem 40. *Let $G = \pi(Y, T)$. Then there exists a tree $X$, on which $G$ acts without inversion on edges such that the factor graph $G \backslash X$ is isomorphic to $Y$ and stabilizers of the vertices and edges in $X$ are conjugate to the canonical images in $G$ of the groups $G_v$ and $\sigma(G_e)$. Moreover for the projection*

$$p : X \to Y,$$

*there exists a lift $\left(\widetilde{T}, \widetilde{Y}\right)$ of the pair $(T, Y)$ such that $p$ maps $\widetilde{T}$ isomorphically onto $T$ and $\widetilde{Y}^1 - \widetilde{T}^1$ bijectively onto $Y^1 - T^1$ and each edge from $\widetilde{Y}^1 - \widetilde{T}^1$ has the initial or terminal vertex in $\widetilde{T}$.*

Proof. Let $X^\circ = \bigcup_{v \in Y_\circ} \frac{G}{G_v}$ (union of the cosets) and $X_+^1 = \bigcup_{v \in Y_+^1} \frac{G}{G_v}$. Put

$$\begin{aligned} \sigma(gG_e) &= gG_{\sigma(e)}, \\ \tau(gG_e) &= gt_e G_{\tau(e)}, \end{aligned}$$

$g \in G, e \in Y^1_+$ and let $\widetilde{T}$ be the lift of the tree $T$ in $X$ with

$$\widetilde{T}^{\circ} = \bigcup_{v \in T^{\circ}} \{G_v\}$$

which are left cosets and stabilizers as well. Set

$$\widetilde{T}^1_+ = \bigcup_{v \in T^1_+} \{G_v\}.$$

$G$ acts on $X$ by left multiplication. □

EXAMPLE 76. *Let* $H \leq G = A * B$, *then* $G_{e_H} = H \cap \underbrace{G_e}_{=\{e\}} = \{e\}$, *which is a free product. Then,*

$$H \cap A^g, \text{ or } H \cap B^g,$$

*where* $A^g$ *is conjugate to* $A$, *and* $B^g$ *is conjugate to* $B$. *So that*

$$H = H_1 * \cdots * H_K * F.$$

*Which is Kurosh theorem.*

Let $\Gamma = \Gamma(\mathcal{X}, Y)$ be the graph of spaces. Take a closed subset $Z = \coprod_e X_e \times \frac{1}{2} \subset \Gamma$ which has a collar neighborhood in $\Gamma$. Let $\widetilde{\Gamma}$ be the universal covering of $\Gamma$ and $\widetilde{Z} \subset \widetilde{\Gamma}$ be the preimage of $Z$. We define the dual tree $X$ to $\widetilde{Z} \subset \widetilde{\Gamma}$ .Its vertices are the components of $\widetilde{\Gamma} - \widetilde{Z}$. Its unoriented edges are the components of $\widetilde{Z}$. The vertices of an edge given by a component $\widetilde{Z_\circ}$ of $\widetilde{Z}$ are the two components of $\widetilde{\Gamma} - \widetilde{Z}$ which have $\widetilde{Z_\circ}$ in their closure. The fact that $\widetilde{\Gamma}$ is simply connected (as the universal cover) implies that $X$ is contractible and hence is a tree. The natural action of $G$ on $\widetilde{\Gamma}$ leaves $\widetilde{Z}$ invariant, therefore induces action on $X$.

REMARK 30. *Given a universal covering space*

$$p : \widetilde{X} \to X,$$

*and a connected, locally path-connected subspace* $A \subset X$ *such that the inclusion is injective on* $\pi_1$, *then each component* $\widetilde{A}$ *of* $p^{-1}(A)$ *is a universal cover of* $A$. *To see this, note that*

$$p : \widetilde{A} \to A,$$

*is a covering space, so we have injective maps*

$$\pi_1\left(\widetilde{A}\right) \to \pi_1(A) \to \pi_1(X),$$

*whose composition factors through* $\pi_1(X) = 1$, *hence* $\pi_1\left(\widetilde{A}\right) = 1$.

LEMMA 13. *Let* $\Gamma = \Gamma(\mathcal{X}, Y)$ *be the graph of spaces.* $\widetilde{\Gamma}$ *is itself a graph of spaces. Also, each vertex space (respectively edge space) is the universal cover of a vertex space of* $\Gamma$ *(respectively edge space).*

PROOF. For any $v \in Y$, let $L_v = X_v \coprod \left( \coprod_e X_e \times \frac{1}{2} \right)$, where $E$ is the set of edges that are the edges incident to $v$. It should be noted that $X_v$ is a deformation retract of $L_v$. Also, recall that the edge maps are $\pi_1-$ injective. It follows that

$$\widetilde{X}_v \hookrightarrow \widetilde{L}_v.$$

So $\widetilde{L}_v$ is built from $\widetilde{X}_v$ by attaching covering spaces of edge spaces. Because $\sigma, \tau$ are $\pi_1-$ injective, these covering spaces of edge spaces really are universal covers. By iteratively gluing together copies of the $\widetilde{L}_v$ we can construct a simply connected cover of $\Gamma$. $\square$

LEMMA 14. *Let $\Gamma$ be a graph of groups and let $X$ be the underlying graph of the universal cover $\widetilde{\Gamma}$ . For any vertex $v \in Y$ the set of vertices of $X$ lying above $v$ is in bijection with $\frac{G}{G_v}$ and $G$ acts by left translation.*

PROOF. Let $\Gamma$ be a graph of groups and let $G = \pi_1(\Gamma)$. Fix a base point $u \in X_v$ and a choice of lift $\widetilde{u} \in \widetilde{X}$. Let $g \in G_v$. Then space $\widetilde{X}_v$ is a universal cover of $X_v$, and so the lift of $g$ to the universal cover $\widetilde{\Gamma}$ at $\widetilde{u}$ is contained in $\widetilde{X}_v$. Therefore the preimage of $u$ that are contained in $\widetilde{X}_v$ corresponds to the elements of $G_v$.

Now consider $g \in G \backslash G_v$. If $g$ is lifted at $\widetilde{u}$, then the terminus of this lift is not in $\widetilde{X}_v$, but in some other component of the preimage of $\widetilde{X}_v$. Call the component where lift terminates $\widetilde{X}_{\tilde{v}_1}$. If $g, h$ are such that both have lifts that terminate in $\widetilde{X}_{\tilde{v}_1}$ then $h^{-1}g \in G_v$. Completing the proof. $\square$

## 2. Exercises

EXERCISE 12. *Fix a point $v_\circ \in V(X)$ and define a subgroup $H \leq G$ which consists of all elements of the free product $G$ that can be presented in the form*

$$g_\circ e_1 g_1 e_2 g_2 \cdots e_n g_n,$$

*where $g_\circ \in G_{v_\circ}$, $g_i \in G_{\tau e_i}$, $e_i \in E(X)$ and $e_1, \cdots, e_n$ is a closed path in $X$ at $v_\circ$. Prove that $H$ is a subgroup.*

EXERCISE 13. *Let $\phi : G \to H$ be the homomorphism which is defined on free factors of $G$ as follows:*

$$\left\{ \begin{array}{c} \phi(g_v) = p_v g_v p_v^{-1}, \\ \phi(e) = p_{\sigma e} e p_{\tau e}^{-1}, \end{array} \right\}$$

*Prove that $\phi$ is a homomorphism. (Hint. It suffices to show that $\phi(Rel(G_v)) = 1$).*

EXERCISE 14. *Prove that $\phi$ is a retract on $H$, i.e. $\phi(G) = H$, and $\phi_{|H} = id_H$.*

EXERCISE 15. *Define a graph of groups $\Gamma = (G, Y)$. Consider the maps*

$$\begin{array}{ccc} G_v & \to & G_v, \\ t_e & \to & \gamma_e. \end{array}$$

*extend to a homomorphism $\phi$,*

$$\phi : \pi(\Gamma) \to G.$$

*Prove that $\phi$ is a homomorphism.*

# Chapter 7

Recall

THEOREM 41. *Let* $G = \pi(Y,T)$. *Then there exists a tree* $X$, *on which* $G$ *acts without inversion on edges such that the factor graph* $G\backslash X$ *is isomorphic to* $Y$ *and stabilizers of the vertices and edges in* $X$ *are conjugate to the canonical images in* $G$ *of the groups* $G_v$ *and* $\sigma(G_e)$. *Moreover for the projection*

$$p : X \to Y,$$

*there exists a lift* $\left(\widetilde{T}, \widetilde{Y}\right)$ *of the pair* $(T,Y)$ *such that* $p$ *maps* $\widetilde{T}$ *isomorphically onto* $T$ *and* $\widetilde{Y}^1 - \widetilde{T}^1$ *bijectively onto* $Y^1 - T^1$ *and each edge from* $\widetilde{Y}^1 - \widetilde{T}^1$ *has the initial or terminal vertex in* $\widetilde{T}$.

PROOF. Let $X^\circ = \bigcup\limits_{v\in Y_\circ} \frac{G}{G_v}$ (union of the cosets) and

$$X^1_+ = \bigcup_{v\in Y^1_+} \frac{G}{G_v}.$$

Put

$$\begin{aligned} \sigma(gG_e) &= gG_{\sigma(e)}, \\ \tau(gG_e) &= gt_eG_{\tau(e)}, \end{aligned}$$

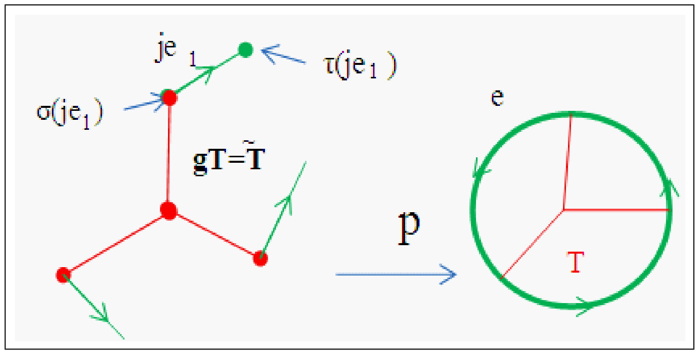
$g \in G, e \in Y^1_+$ and let $\widetilde{T}$ be the lift of the tree $T$ in $X$ with

$$\widetilde{T}^\circ = \bigcup_{v\in T^\circ} \{G_v\}$$

which are left cosets and stabilizers as well. Set

$$\widetilde{T}^1_+ = \bigcup_{e\in T^1_+} \{G_e\}.$$

$G$ acts on $X$ by left multiplication. $\square$




Let $G$ be a fundamental group of a graph of groups with underlying graph $Y$. For each edge or vertex $a$ of $Y$ choose $X_a$ to be a connected CW-complex with fundamental group $G_a$. We can do this so that there is an embedding

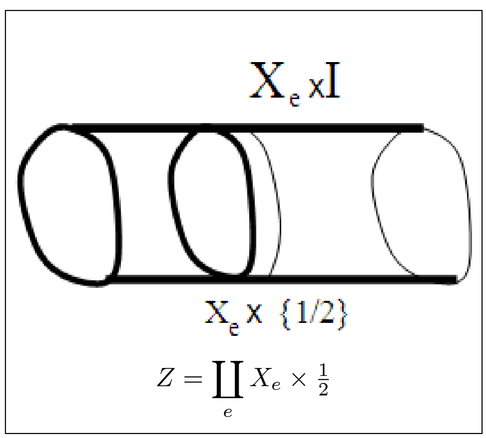
$$X_e \hookrightarrow X_v,$$

realizing the inclusion of groups. We form the topological space $\Gamma(\mathcal{X}, Y)$ by beginning with the disjoint union

$$\coprod_{v\in V} X_v \coprod \coprod_{e\in E} X_e \times I,$$

and

(1) identifying $X_e \times I$ with $X_{\overline{e}} \times I$ via $(x,t) \equiv (x, 1-t)$ and
(2) gluing $X_e \times \{0\}$ to $X_{\sigma(e)}$ via the given inclusion. The resulting topological space has fundamental group $G$.




To prove that $G$ acts on a tree. We do the following.

Proof. Let $\Gamma = \Gamma(\mathcal{X}, Y)$ be the graph of spaces. Take a closed subset $Z = \coprod_e X_e \times \frac{1}{2} \subset \Gamma$ which has a collar neighborhood in $\Gamma$. Let $\widetilde{\Gamma}$ be the universal covering of $\Gamma$ and $\widetilde{Z} \subset \widetilde{\Gamma}$ be the preimage of $Z$. We define the dual tree $X$ to $\widetilde{Z} \subset \widetilde{\Gamma}$ .Its vertices are the components of $\widetilde{\Gamma} - \widetilde{Z}$. Its unoriented edges are the components of $\widetilde{Z}$. The vertices of an edge given by a component $\widetilde{Z_\circ}$ of $\widetilde{Z}$ are the two components of $\widetilde{\Gamma} - \widetilde{Z}$ which have $\widetilde{Z_\circ}$ in their closure. The fact that $\widetilde{\Gamma}$ is simply connected (as the universal cover) implies that $X$ is contractible and hence is a tree. The natural action of $G$ on $\widetilde{\Gamma}$ (acts on the universal cover of $\widetilde{\Gamma}$ ) leaves $\widetilde{Z}$ invariant, therefore induces action on $X$. $\square$

REMARK 31. *Given a universal covering space*

$$p : \widetilde{X} \to X,$$

*and a connected, locally path connected subspace $A \subset X$ such that the inclusion is injective on $\pi_1$, then each component $\widetilde{A}$ of $p^{-1}(A)$ is a universal cover of $A$. To see this, note that*

$$p : \widetilde{A} \to A,$$

*is a covering space,*

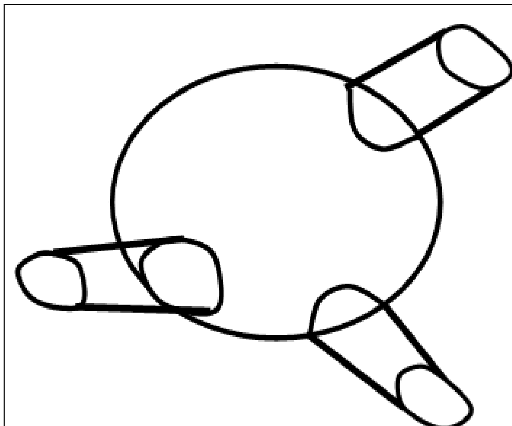
$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{p'} & A \\ i_{\widetilde{A}} \downarrow & & \downarrow i_A \\ \widetilde{X} & \xrightarrow{p} & X \end{array}$$

*so we have injective maps*

$$\pi_1\left(\widetilde{A}\right) \to \pi_1(A) \to \pi_1(X),$$

*whose composition factors through $\pi_1\left(\widetilde{X}\right) = 1$, hence $\pi_1\left(\widetilde{A}\right) = 1$.*

LEMMA 15. *Let $\Gamma(\mathcal{X}, Y)$ a graph of spaces. $\widetilde{\Gamma}$ is itself a graph of spaces. Also, each vertex space (respectively edge space) is the universal cover of a vertex space of $\Gamma$ (respectively edge space).*



$L_v$

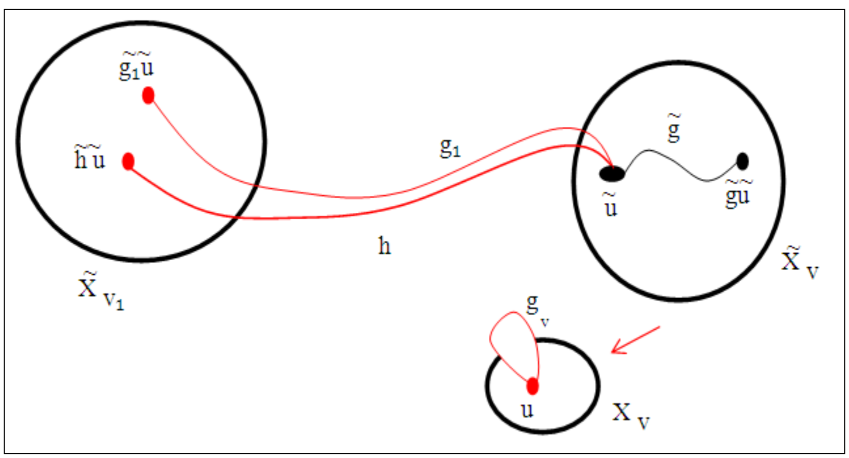
PROOF. For any $v \in Y$, let $L_v = X_v \coprod \left( \coprod_{e \in E} X_e \times \frac{1}{2} \right)$, where $E$ is the set of edges which are incident to $v$. It should be noted that $X_v$ is a deformation retract of $L_v$. Also, recall that the edge maps are $\pi_1$ injective. It follows that

$$\widetilde{X_v} \hookrightarrow \widetilde{L_v}.$$

So $\widetilde{L_v}$ is built from $\widetilde{X_v}$ by attaching covering spaces of edges spaces. Because $\sigma, \tau$ are $\pi_1$ injective, these covering spaces of edges spaces really are universal covers. By iteratively gluing together copies of the $\widetilde{L_v}$ we can construct a simply connected cover of $\Gamma$. $\square$

Lemma 16. *Let $\Gamma$ be a graph of groups and let $X$ be the underlying graph of the universal cover $\widetilde{\Gamma}$. For any vertex $v \in Y$ the set of vertices of $X$ lying above $v$ is a bijection with $\frac{G}{G_v}$ and $G$ acts by left translation.*

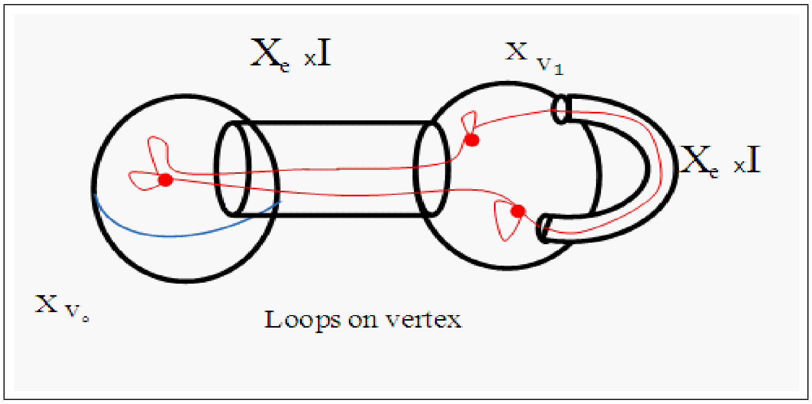
Proof. Let $\Gamma$ be a graph of groups and let $G = \pi_1(\Gamma)$. Fix a base point $u \in X_v$ and a choice of lift $\widetilde{u} \in \widetilde{X}$. Let $g \in G_v$. The space $\widetilde{X}_v$ is a universal cover of $X_v$, and so the lift of $g$ to the universal cover $\widetilde{\Gamma}$ at $\widetilde{u}$ is contained in $\widetilde{X}_v$. Therefore the preimages of $u$ that are contained in $\widetilde{X}_v$ correspond to the elements of $G_v$.




Now consider $g \in G \backslash G_v$. If $g$ is lifted at $\widetilde{u}$, then the terminus of this lift is not in $\widetilde{X}_v$, but in some other component of the preimage of $X_v$. Call the component where the lift terminates $\widetilde{X}_{\widetilde{v_1}}$. If $g, h$ are such that both have lifts that terminate in $\widetilde{X}_{\widetilde{v_1}}$ then $h^{-1}g \in G_v$. To see this

(1) If $g \in G_v$, then $\widetilde{g}\widetilde{u} \in \widetilde{X}_v$, $G_v \leq Stab\left(\widetilde{X}_v\right)$,
(2) If $gG_v \neq G_v$, then $h^{-1}g \in G_v$,
(3) If $\widetilde{X}_{v_1} \sim gG_v$, then $Stab\left(\widetilde{X}_{v_1}\right) = gG_vg^{-1}$.

$\square$




## 1. The Normal form theorem

THEOREM 42. *Let $\Gamma$ be a graph of groups and $G = \pi_1(\Gamma)$.*

(1) *Any $g \in G$ can be written as $g = g_\circ t_{e_1}^{\epsilon_1} g_1 \cdots t_{e_n}^{\epsilon_n} g_n$ as before.*
(2) *If $g = 1$, this expression includes "backtracking", meaning that for some $i$, $e_i = e_{i+1}$ with $\epsilon_i = \epsilon_{i+1}$, and furthermore that if $\epsilon_i = -1$, then $g_i \in \sigma(G_{e_i})$ and if $\epsilon_i = 1$, then $g_i \in \tau(G_{e_i})$.*

REMARK 32. *"backtracking" $= t_{e_1}^{-1} g_i t_{e_1} \in G_{v_i}$. Suppose that $e_i \notin T$ this implies that $\exists$ a relation $g_i \in G_{v_i} = G_{\sigma(e_i)}$. Where we have*

$$t_{e_i}^{-1} \sigma(g_i) t_{e_i} = g_{\tau(e_i)},$$

*the word is reduced.*

THEOREM 43. *(Kusosh's theorem) Let $H$ be a free product of the groups $H_i$, $i \in I$ amalgamated over a common subgroup $A$. Let $G$ be a subgroup of $H$ such that*

$$G \cap xAx^{-1} = \{1\},$$

*for all $x \in H$. Then there exists a free group $F$ and a system of representatives $X_i$ of double cosets $G \backslash \frac{H}{H_i}$ such that $G$ is a free product of the group $F$ and the groups $G \cap xH_ix^{-1}$ for $i \in I, x \in X_i$.*

PROOF. Let $X$ be a tree on which $H$ acts.

$$X^\circ = \left(\frac{H}{A}\right) \cup \left(\bigcup_{i \in I} \frac{H}{H_i}\right),$$

and

$$X_+^1 = \bigcup_{i \in I} \left(\frac{H}{A} \times \{i\}\right).$$

The initial and terminal vertices of an edge $(hA, i)$ are $hA$ and $hH_i$. $G$ acts on $X$. Let $Y = G \backslash X$,

$$p : X \to Y,$$

the canonical projection, $T$ a maximal subtree of $Y$, $\left(\widetilde{T}, \widetilde{Y}\right)$ a lift. $V\widetilde{T}$ is maximal set of left cosets $xA$ and $xH_i$ with the property that they are not $G$ equivalent, i.e.,

$$\left\{ \begin{array}{c} hH_i, \text{ is equivalent} \\ g\widetilde{h}H_i. \end{array} \right\}$$

Thus, there exists systems of representatives $X_A$ and $X_i$ of double cosets $G\backslash\frac{H}{A}$ and $G\backslash\frac{H}{H_i}$ such that

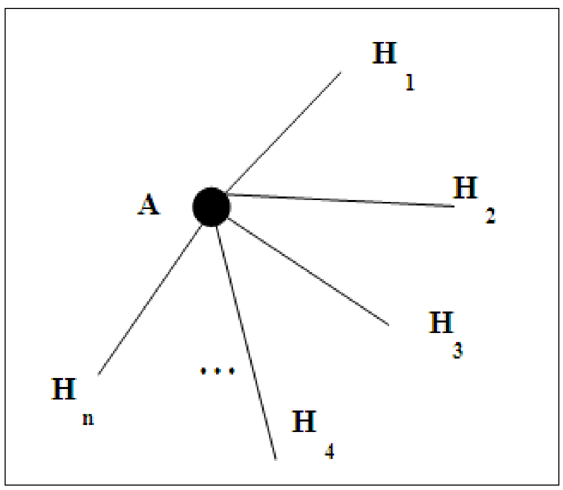
$$\widetilde{T}^{\circ} = \{xA | x \in X_A\} \cup \bigcup_{i \in I} \{xH_i | x \in X_i\}.$$

$$\begin{aligned} Stab\,(xA) &= G \cap xAx^{-1} = \{1\}, \\ Stab\,(xH_i) &= G \cap xH_ix^{-1}, \end{aligned}$$

the stabilizers of edges are trivial. Downstairs we get s free product. Recall that

$$\begin{aligned} Stab\,(S) &= H, \\ G &\leq H, H = \underset{A}{*} H_i. \\ Stab_G\,(S) &= G \cap Stab_H\,(S). \end{aligned}$$

For each edge $\widetilde{e} \in \widetilde{Y}^1$ with terminal vertex outside $\widetilde{T}^{\circ}$, let $t_e^{-1}$ be an element carrying this vertex into $\widetilde{T}^{\circ}$. Thus $F$ has a basis consisting of all these elements $t_e$




□

EXAMPLE 77. $G = \left\langle G_v, t_e | \underbrace{t_{e_i}^{-1} \sigma\left(g_i\right) t_{e_i} = g_{\tau(e_i)}}_{Can\ be\ remove} g \in G_e,\quad \underbrace{t_e = 1}_{stable\ letters}, i \quad fe \in I, v \in Y \right\rangle$ *then it is a free product generated by* $t_e$.

THEOREM 44. *((˜1949) Higman- Neumann theorem) If* $G$ *is a countable group, then* $G$ *can be embedded in a 2 generator group.*

PROOF. Let $x_1, x_2, \cdots$ be a generating set for $G$. Then $G < G_1 = G * \mathbb{Z}$. Let $t$ be a generator of $\mathbb{Z}$, and $y_i = x_i t$, $y_{\circ} = t$. Then $G_1$ is generated by $y_{\circ}, y_1, \cdots$, and $|y_i| = \infty$. Let

$$G_2 = \left\langle G_1, t_{\circ}, t_1, \cdots | t_i^{-1} y_i t_i = y_{i+1} \right\rangle$$

Then $G_1 < G_2$ because $G_2$ is obtained from $G_1$ by an infinite sequence of $HNN$ extensions. The subgroup $K$ of $G_2$ generated by the $t_i'$s is free and has the $t_i'$s as basis. Embed $K$ into $F_2 = \langle a, b \rangle$ by setting $b^{-i}ab^i = t_i$.

$$G_3 = G_2 \underset{K}{*} F_2.$$

Then $G_2 < G_3$ and $G_3$ is generated by $y_\circ, a, b$. The subgroup $H$ of $G_3$ generated by $y_\circ, b$ is free of rank 2 because $H \cap G_2$ and $H \cap F_2$ are infinite cyclic, and $H \cap K = \{1\}$.

Let

$$G_4 = \left\langle G_3, s | s^{-1}as = b,\ s^{-1}bs = y_\circ \right\rangle = G_3 \underset{F_2}{*},$$

where the two inclusions of $F_2$ in $G_3$ have images $H$ and $F_2$. $G_4$ is generated by $a$ and $s$. □

EXAMPLE 78. *$Hg = \left\langle G, t | t^{-1}gt = g^2 \right\rangle$,*

(1) *If $g = 1$, then $Hg \cong G * \langle t \rangle$.*
(2) *If $g \neq 1$, then $Hg \ncong G * \langle t \rangle$. Under the word problem.*

$\boxed{\text{Open problem}}$ Construct a finitely presented infinite group of bounded exponent. Where we mean by bounded exponent=($g^n = 1$ for any $g \in G$).

THEOREM 45. *Let $H$ be a fundamental group of a graph of groups $\Gamma(H, X)$ and $G \leq H$. Then $G$ is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of groups $H_v$, $v \in X^\circ$ and edge groups are subgroups of conjugates of $H_e, e \in X^1$.*

DEFINITION 54. *We say that $G$ splits over a subgroup $C$ if $G = A\underset{C}{*}$ or $G = A\underset{C}{*}B$ with $A \neq C \neq B$. If $G$ splits over some subgroup we say that $G$ is splittable.*

DEFINITION 55. *If $C$ is abelian (respectively cyclic) we say that $G$ has abelian (respectively cyclic) splitting.*

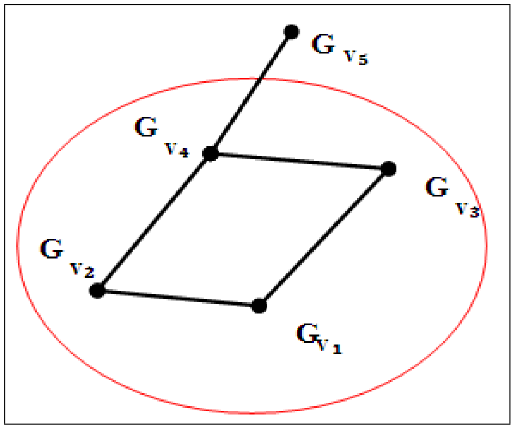
COROLLARY 7. *If $H = A\underset{C}{*}$ or $H = A \underset{C}{*} B$ and $G$ is a finitely generated non splittable subgroup of $H$, then $G$ lies in a conjugate of $A$ or $B$.*

PROOF. $G$ is a subgroup which acts on the tree $T$. $G$ is finitely generated. Why? Because the number of edges in the tree are finite. The number of stable letters are finite. Now add all the edges. Therefore

$$G = \bigcup_{i=1}^{\infty} \text{edges},$$

$\Longrightarrow\Longleftarrow$ since $G$ is finitely generated. □

Example 79. *Suppose you have a finite graphs of groups*




*Two vertex groups are not trivial, this example is for the corollary.*

Example 80. *If $G$ is not finitely generated. Take*

$$H = \mathbb{Z} \underset{\mathbb{Z}}{*} = \left\langle a, t | t^{-1}at = a^2 \right\rangle,$$

*has infinite order.*

$$G = \left\langle t^{-i}at^i, i = 1, 2, \cdots \right\rangle$$

Example 81. *Consider the homomorphism $SL_2(\mathbb{Z}) = \mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_6 \to \mathbb{Z}_{12}$ given by the natural embeddings of the factors in the group $\mathbb{Z}_{12}$. Prove that its kernel is a free group of rank 2.*

Proof. $SL_2(\mathbb{Z}) = \left\langle a | a^4 = 1 \right\rangle \underset{a^2=b^3}{*} \left\langle b | b^6 = 1 \right\rangle.$

$$\mathbb{Z}_{12} = \left\langle c, | c^{12} = 1 \right\rangle.$$

A natural homomorphism

$$\begin{aligned} \varphi &: a \to c^3, \\ \varphi &: b \to c^2, \\ \varphi &: ab^{-1} \to c. \end{aligned}$$

We need to show that $[a, b], [a, b^2]$ belong to the kernel. Then

$$\ker \varphi \cong F_2$$

where $F_2$ is a free group of rank 2. Let

$$N = \left\langle \underbrace{[a, b]}_{=x}, \underbrace{[a, b^2]}_{=y} \right\rangle \in \ker \varphi$$

Then we show that $N \trianglelefteq G$.

$$\begin{aligned} axa^{-1} &= a^2ba^{-1}b^{-1}a^{-1} \\ &= bab^{-1}a^{-1} = x^{-1}. \end{aligned}$$

$$\begin{aligned} bxb^{-1} &= baba^{-1}b^{-2} \\ &= bab^{-1}a^{-1}ab^2a^{-1}b^{-2} = x^{-1}y. \end{aligned}$$

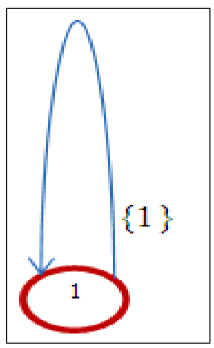
$$aya^{-1} = a^2b^2a^{-1}b^{-2}a^{-2}.$$
$$byb^{-1} = bab^2a^{-1}b^{-2}$$

Then $\frac{G}{N}$ has at most 12 elements such that $\bar{a}^m\bar{b}^k$ where $\left\{ \begin{array}{c} m = 0, \cdots, 3, \\ k = 0, \cdots, 5. \end{array} \right\}$. Therefore, $x, y$ freely generates the $\ker \varphi$. Consider all 2 letter products elements of $G$ which have the form

$$a^{\epsilon_i} b^{\alpha_I} \cdots b^{\alpha_I} a b^{\alpha_I} a, \tag{1.1}$$

for $\left\{ \begin{array}{c} \epsilon = 0, 1, 2 \\ \alpha_i = 1, 2. \end{array} \right\}$. Then (1.1) are not trivial elements. We need to consider $xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}$. Therefore $N$ is free. □

EXAMPLE 82. *HNN extension*




*Which is a free group corresponding to stable letters.*

THEOREM 46. *A group $G$ is virtually free of finite rank iff $G$ is the fundamental group of a finite graph of finite groups.*

The proof uses the notion of ends of a f.g. group.

DEFINITION 56. *For each finite subgraph $K$ of Cayley $(G)$ ($G$ is f.g.), the number of connected components of Cayley $(G)$. $K$ is finite; denote by $n(K)$ the number of infinite ones. Now define the number of ends $e(G) = \sup n(K)$.*

## 2. Exercises

EXERCISE 16. *Prove that if $G = A * B$ where $A, B$ are non trivial and $H$ is a finitely generated normal subgroup of $G$, then $H$ is trivial or has finite index in $G$.*

# Chapter 8

## 1. Fully residually free groups

Definition 57. *A group $G$ is residually free if for any non-trivial $g \in G$ there exists $\phi \in Hom(G, F)$, where $F$ is a free group, such that $g^{\phi} \neq 1$.*

Definition 58. *A group $G$ is **fully residually free** or **discriminated** by $F$ if for any finite subset $M \subseteq G$ with $1 \notin M$ there exists an $F$homomorphism*

$$\phi : G \to F,$$

*such that $1 \notin \phi(M)$.*

Definition 59. *$Fg$ such groups are called limit groups.*

Example 83. *$Fg$ free groups, free abelian groups, surface groups, extensions of centralizers of fully residually free groups, subgroups of fully residually free groups.*

Example 84. *Easy exercise:*

(1) *Residually free groups are torsion free. If $G$ is fully residually free group, then*

$$x^n = 1,$$

*in $G$, $x \neq 1$ in $G$, gives*

$$\phi : G \to F,$$

*with $x^{-1} \neq 1$ implies torsion free.*

(2) *$Fg$ free groups and free abelian groups are fully residually free.*
(3) *$Fg$ subgroups of fully residually free groups are fully residually free.*
(4) *Fully residually free groups are commutative transitive.*
(5) *$F \times F$ is NOT fully residually free. Consider $F \times \mathbb{Z}$ which not commutative transitive. Why? Well take*

$$[y, z] = 1,$$

$$\begin{array}{ccc} (y,1) & \longleftrightarrow & (1,x) \\ (z,1) & \longleftrightarrow & (1,x). \end{array}$$

Definition 60. *$G$ is an **extension of centralizer** of $H$ if*

$$G = \langle H, t | \left[C_H(u), t\right] = 1 \rangle .$$

Remark 33. *Definition (60) is an $HNN$ extension and also a normal form.*

Lemma 17. *If $H$ is a limit group, then $G$ is a limit group.*

PROOF.

$$\phi : G \to H$$

is identical on $H$ and $t^{\phi} = u^{n}$, for a large $n$.

$$\begin{aligned} t^{\phi} &\to u^{n}, \\ h &\in H \to h \in H \end{aligned}$$

elements of $H$ get send to elements of $H$. Elements in $G$ have canonical form

$$g_1 t^{m_1} g_2 t^{m_2} \cdots t^{m_k} g_{k+1},$$

where $g_i \notin C_H(u)$ ,(because it would not be a formal form) $i = 1, \cdots, k$. They are mapped into

$$g_1 u^{nm_1} g_2 u^{nm_2} \cdots u^{nm_k} g_{k+1}.$$

$\square$

EXAMPLE 85. *$F = G_\circ \le G_1 \le \cdots \le G_n$, then*

$$G_n = \langle G_{n-1}, t_n | [C(u_n), t_n] = 1 \rangle .$$

*Nonabelian , the limit has a splitting as a fundamental group.*

EXAMPLE 86. *Take*

$$G = \langle F(a,b), t | [[a,b], t] = 1 \rangle ,$$

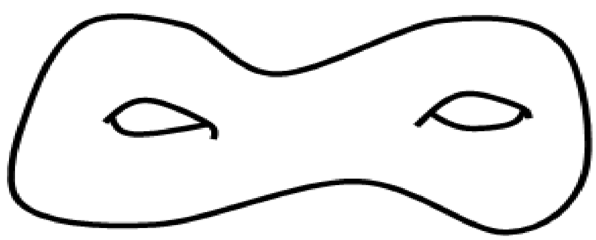
*this is freely residually free. Where*

$$H = \langle a, b, a^t, b^t \rangle ,$$

*also we see that $[a,b] = [a^t, b^t]$ , then*

$$H \cong S_2.$$

*Then the surface group is fully residually free.*



$H \cong S_2$

THEOREM 47. *[Kharlampovich, Miasnikov, 96] That is it. Every f.g. fully residually free group is a subgroup of a group obtained from a free group as a finite series of extensions of centralizers.*

Moreover, there is an algorithm to find this embedding.

EXAMPLE 87. *Examples of sentences in the theory of $F$: Only for free groups.*

(1) *( Vaught's identity)*$\forall x \forall y \forall z \left( x^2 y^2 z^2 = 1 \to ([x,y] = 1 \& [x,z] = 1 \& [y,z] = 1) \right)$. *Which a nontrivial sentence because of the universal quantifier* $\forall x \forall y \forall z$
(2) *(Torsion free)*$\forall x (x^n = 1 \to x = 1)$. *There are infinite universal sentences.*
(3) *(Commutation transitivity)*$\forall x \forall y \forall z (x \ne 1 \to ([x,y] = 1 \& [x,z] = 1) \to [y,z] = 1)$.

(4) *(Commutation transitivity) doesn't hold in $F_2 \times F_2$. Because the elements from the left term do not commute with elements from the right term*
(5) (*Separate conjugate Abelian*) $\forall x \forall y \left([x, x^y] = 1 \to [x, y] = 1\right)$
(6) $\underbrace{\forall x, y \exists z}_{\text{Not universal sentence}} \quad \left(xy = yx \to \left(x = z^2 \vee y = z^2 \vee xy = z^2\right)\right)$, *not true in a free abelian group of rank* 2.

EXAMPLE 88. *For* (6) *we take $xy = yx$ in $F$. Then $v = u^k$, and $y = u^m$ for some $u \in F$ and $k, m \in \mathbb{Z}$. If*

$$\begin{array}{rcl} \text{If } k & \in & 2\mathbb{Z},\ \text{then } z = u^{\frac{k}{2}}, \\ \text{If } m & \in & 2\mathbb{Z},\ \text{then } z = u^{\frac{m}{2}}, \\ \text{otherwise } k+m & \in & 2\mathbb{Z},\ \text{then } xy = u^{k+m}, z = u^{\frac{m+k}{2}}. \end{array}$$

*And we have*

$$\forall x, y \exists z \left(xy = yx \to \left(x = z^2 \underset{or}{\vee} y = z^2 \vee xy = z^2\right)\right)$$

(Separate conjugate Abelian)=maxAbelian subgroups are malnormal.

DEFINITION 61. *Malnormal. $A \leq G$, $A$ is malnormal if $A^g \cap A \neq 1 \to g \in A$.*

EXAMPLE 89. *In $F$ take $A = \langle a \rangle$, then $A^g \cap A \neq 1$ implies that we can represent*

$$g^{-1} u^n g = u^m \to g = u^k.$$

*Then maxAbelian groups are malnormal.*

EXAMPLE 90. $\forall x \forall y \exists z \left([x, y] = 1 \to \left(x = z^2 \vee y = z^2 \vee xy = z^2\right)\right)$. *This implies that if a group $G$ is $\forall\exists$ ( not an universal sentence) equivalent to $F$, then it does not have noncyclic abelian subgroups.*

EXAMPLE 91. *$F$ has Magnus' properties, namely, for $n, m$ the following sentence is true:*

$$\forall x \forall y \left(\exists z_1, \cdots, z_{m+n} \left(x = \prod_{i=1}^{n} z_i^{-1} y^{\pm 1} z_i \wedge y = \prod_{i=n+1}^{m+n} z_i^{-1} x^{\pm 1} z_i\right) \to \exists z \left(x = z^{-1} y^{\pm 1} z\right)\right).$$

In $F$ if two elements has the same normal closures then,

$$x = g^{-1} y g, \text{ or } x = g^{-1} y g,$$

for some $g \in F$. If they generate same normal subgroup then

$$ncl(x) = ncl(y),$$

where $ncl(x)$ is the minimal normal subgroup containing $x$.

## 2. Elementary theory

DEFINITION 62. *The elementary theory $Th(G)$ of a group $G$ is the set of all first order sentences in the language of group theory which are true in $G$.*

CONDITION 2. *Recall, that the group theory language consists of* ***multiplication*** , ***inversion*** $^{-1}$*, and the* ***identity symbol*** 1.

CONDITION 3. *Every group theory sentence is equivalent to one of the type:*

$$\Phi = \forall X_1 \exists Y_1 \cdots \forall X_k \exists Y_k \bigvee_{p=1}^{r} \left( \bigwedge_{i=1}^{s} u_{pi} \left(X_1, Y_1, \cdots, X_k, Y_k\right) = 1, \bigwedge_{j=1}^{t} v_{pj} \left(X_1, Y_1, \cdots, X_k, Y_k\right) \neq 1 \right).$$

*where $u_{pi}, v_{pj}$ are group words (products of variables (tuples) and their inverses).*

EXAMPLE 92. *Consider $X_1, Y_1, \cdots, X_n, Y_n$,*

$$u\left(X_1, Y_1, \cdots\right) = x_1^2 y^{-1} x x_2 y_1.$$

$\boxed{\text{Informally:}}$

$Th\,(G) =$ all the information about $G$ that can be expressed in the first order logic of group theory. Hence:

$$Th\,(G) = Th\,(H) \Leftrightarrow G,$$

and $H$ are indistinguishable in the first order logic (elementary equivalent) $Th\,(G)$ is decidable $\Leftrightarrow$ the first order information about $G$ is available (in principle) to us.

## 3. Tarski's Problems

In 1945 Alfred Tarski posed the following problems.

(1) Do the elementary theories of free non abelian groups $F_n$ and $F_m$ coincide?
(2) Is the elementary theory of a free non-abelian group $F_n$ decidable?

THEOREM 48. *[Kharlampovich and Myasnikov (1998-2006), independently Sela (2001-2006)] $Th\,(F_n) = Th\,(F_m)\,,\ m, n > 1$.*

THEOREM 49. *[Kharlampovich and Myasnikov] The elementary theory $Th(F)$ of a free group $F$ even with constants from $F$ in the language is decidable.*

REMARK 34. *Long history of Tarski's type problems in algebra. Crucial results on fields, groups, boolean algebras, etc.*

(3) Complex numbers $\mathbb{C}$,
(a). $Th\,(\mathbb{C}) = Th\,(F)$ iff $F$ is an algebraically closed field. This says that every equation has a solution in this field.
(b). $Th\,(\mathbb{C})$ is decidable.
This led to development of the theory of algebraically closed fields. Elimination of quantifiers: every formula is logically equivalent (in the theory ACF) to a boolean combination of quantifier free formulas (something about systems of equations).
(4) Reals $\mathbb{R}$
(a). $Th\,(\mathbb{R}) = Th\,(F)$ iff $F$ is an algebraically closed field. This says that every equation has a solution in this field.
(b). $Th\,(\mathbb{R})$ is decidable.
A real closed field = an ordered field where every odd degree polynomial has a root and every element or its negative is a square. Theory of real closed fields (Artin, Schreier), 17th Hilbert Problem (Artin).

Elimination of quantifiers (to equations): every formula is logically equivalent (in the theory RCF) to a boolean combination of quantifier free formulas.

(5) $p-adics$ $\mathbb{Q}_p$

Ax-Kochen, Ershov

(a). $Th(\mathbb{Q}_p) = Th(F)$ iff $F$ is $p-$adically closed field. This says that every equation has a solution in this field.

(b). $Th(\mathbb{Q}_p)$ is decidable.

Existence of roots of odd degree polynomials in $\mathbb{R} \approx$ Hensel's lemma in $\mathbb{Q}_p$.

Elimination of quantifiers (to equations).

(6) Tarski's problems are solved for abelian groups (Tarski, Szmielew).

(7) For non-abelian groups results are sporadic.

Theorem 50. *Novosibirsk Theorem [Malcev, Ershov, Romanovskii, Noskov] Let $G$ be a finitely generated solvable group. Then $Th(G)$ is decidable iff $G$ is virtually abelian ( finite extension of an abelian group).*

Example 93. *Free nilpotent groups are undecidable.*

(8) Elementary theories of free semigroups of different ranks are different and undecidable

(9) Interpretation of arithmetic.

(10) Elementary classification.

Remeslennikov, Myasnikov, Oger: elementary classification of nilpotent groups (not finished yet). Typical results:

Theorem 51. *[Myasnikov, Oger] Finitely generated non abelian nilpotent groups $G$ and $H$ are elementarily equivalent iff*

$$G \times \mathbb{Z} \simeq H \times \mathbb{Z}.$$

Many subgroups are definable by first order formulas, many algebraic invariants are definable. Where definable means:

$$f(x) = (\exists y_1 \cdots \phi(x_1, y_1 \cdots)).$$

(11) Free groups.

Nothing like that in free groups: no visible logical invariants. Ranks of free non-abelian groups are not definable. Only maximal cyclic subgroups are definable.

Elimination of quantifiers (as we know now): to boolean combinations of $\forall\exists$ formulas!

New methods appeared. It seems these methods allow one to deal with a wide class of groups which are somewhat like free groups: hyperbolic, relatively hyperbolic, acting nicely on $\Lambda$ hyperbolic spaces, etc.

## 4. Algebraic sets

$G$ - a group generated by $A$. $F(X)$ free group on $X = \{x_1, x_2, \cdots, x_n\}$.

Definition 63. *A system of equations $S(X, A) = 1$ in variables $X$ and coefficients from $G$ (viewed as a subset of $G * F(X)$).*

Definition 64. *A solution of* $S(X,A)=1$ *in* $G$ *is a tuple* $(g_1,\cdots,g_n)\in G^n$ *such that* $S(g_1,\cdots,g_n)=1$ *in* $G$.

Definition 65. $V_G(S)$, *the set of all solutions of* $S=1$ *in* $G$, *is called an* ***algebraic set*** *defined by* $S$.

Definition 66. *System of equations* $S_1(x)=1$ *and* $S_2(X)=1$ *are equivalent if they have the same solution set.*

| | |
|---|---|
| $G$ | $\mathbb{C}$ |
| $G(A)=G*F(x_1,x_2,\cdots,x_n)$ | $\mathbb{C}[x_1,x_2,\cdots,x_n]$ |
| $a_1x_1^2x_2^{-1}a_2=1,\ a's$ are constant | $c_1x_1x_2+c_2x_2x_2$ |

## 5. Radicals and coordinates groups

The maximal subset

$$R(S)\subseteq G*F(X)$$

with

$$V_G(R(S))=V_G(S)$$

is the radical of $S=1$ in $G$. The quotient group

$$G_{R(S)}=\frac{G[X]}{R(S)}=\frac{G*F(X)}{R(S)}.$$

is the coordinate group of $S=1$. Solutions of $S(X)=1$ in $G\Leftrightarrow G$ homomorphisms $G_{R(S)}\to G$.

| | |
|---|---|
| Radical in $G$ | Radical in $\mathbb{C}$ |
| $R(S)\geq ncl(S)$ | $Rad(S)$ |
| $\frac{G[X]}{R(S)}$ $G*F(X)$ | $\frac{\mathbb{C}[x_1,x_2,\cdots,x_n]}{Rad(S)}$ |

## 6. Zariski topology

Remark 35. *Zariski topology on* $G^n$ *formed by algebraic sets in* $G^n$ *as a subbasis for the closed sets.*

Remark 36. *Zariski topology on* $F$ *with constants in the language: closed sets= algebraic sets.*

Remark 37. *Coordinate groups of algebraic sets over* $F$ *are residually free.* $G$ *is residually free if for any non-trivial* $g\in G$ *there exists a*

homomorphism $\phi\in Hom(G,F)$ such that $g^{\phi}\neq 1$.

Lemma 18. *If* $[a,b]\neq 1$ *in* $F$ *(free group) then for all solution of the system*

$$[x,y^a]=1\wedge\left[x,y^b\right]=1\wedge\rightarrow\left[x,y^{ab}\right]=1,$$

*has either* $x=1$ *or* $y=1$.

Remark 38. *Finite disjunction is equivalent to a finite system of equations. All follows from the* $(CSA)$ *properties.*

PROOF. Suppose $x, y \neq 1$ and solve

$$[x, y^{ba}] = 1$$

and

$$[x, y^{ab}] = 1.$$

To obtain a contradiction of $[a, b] \neq 1$.

Since

$$\left\{ \begin{array}{c} [x^b, y^{ab}] = 1, \\ [x, y^{ab}] = 1, \end{array} \right\} \left\{ \begin{array}{c} \text{commutative and} \\ \text{transitive.} \end{array} \right\}$$

This implies that

$$[x, x^b] = 1 \rightarrow [x, b] = 1.$$

Similar we can show that

$$[x, a] = 1, \text{ and } [x, b] = 1,$$

implies that

$$[a, b] = 1$$

which is a contradiction. □

## 7. Noetherian groups

**Equationally Noetherian groups:** The following conditions are equivalent:

(1) $G$ is equationally Noetherian, i.e., every system $S(X) = 1$ over $G$ is equivalent to some finite part of itself.
(2) the Zariski topology over $G^n$ is Noetherian for every $n$, i.e., every proper descending chain of closed sets in $G^n$ is finite.
(3) Every chain of proper epimorphisms of coordinate groups over $G$ is finite.

PROOF. (1) $\Longrightarrow$ (2)

The varieties of systems

$$V(S_1) > V(S_2) > \cdots > V(S_n) = V(S_{n+1}).$$

If $V(S_2)$ is smaller than $V(S_1)$ then it has more equations. This implies that $\exists$ infinite equations. Then $\exists$ an infinite chain of varieties$\Longrightarrow\Longleftarrow$ . Since $S_n = S_{n+1}$ for some $n$.

(2) $\Longrightarrow$ (1) Consider the radical

$$\frac{G}{R(S_1)} \underset{\neq}{\rightarrow} \frac{G}{R(S_2)}, \text{ then}$$

$$S_1 \quad \nsim \quad S_2.$$

Then for the finite ascending chain of radicals, i.e.,

$$\frac{G}{R(S_1)} \underset{\neq}{\rightarrow} \frac{G}{R(S_2)} \quad \rightarrow \quad \cdots \frac{G}{R(S_{n-1})} \underset{\neq}{\rightarrow} \frac{G}{R(S_n)}$$

$$S_1 \quad \nsim \quad S_2, \cdots, S_{n-1} \nsim S_n.$$

If $\frac{G}{R(S_2)}$ has more solutions than $\frac{G}{R(S_1)}$, then we would obtain and infinite ascending chain which impossible. □

THEOREM 52. *[R. Bryant (1977), V.Guba (1986)]: Free groups are equationally Noetherian.*

PROOF. Let

$$H_\circ \to H_1 \to \cdots$$

be a sequence of epimorphisms between $fg$ groups. Then the sequence

$$Hom(H_\circ, F) \hookleftarrow Hom(H_1, F) \hookleftarrow \cdots$$

eventually stabilizes because

$$\begin{array}{ccccc} H_\circ & \to & H_1 & \to & H_2 \\ Hom(H_\circ, F) & \searrow & \downarrow & \swarrow & Hom(H_1, F) \\ & & F & & \end{array}$$

we embed $F$ in $SL_2(\mathbb{Q})$ and the sequence of algebraic varieties

$$Hom(H_\circ, SL_2(\mathbb{Q})) \hookleftarrow Hom(H_1, SL_2(\mathbb{Q})) \hookleftarrow \cdots$$

eventually stabilizes. □

EXAMPLE 94. *$F \leq GL_2(\mathbb{Z})$, for*

$$\begin{aligned} x_i &\to \begin{pmatrix} x_{11} & x_{12} \\ x_{13} & x_{14} \end{pmatrix}, \\ a_i &\to \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix}, \end{aligned}$$

*where $x_i a_i$. We have a system of equations on one side then on the other side we are going to have a system of polynomial equations. Let $S$ over $F$, and $F \to \bar{S}$ over $\mathbb{Z}$*

$$S_1 = 1 \to \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}.$$

$$S = 1 \to \left\{ \begin{array}{l} f_1 = 0 \\ f_2 = 0 \\ f_3 = 0 \\ f_4 = 1 \end{array} \right\}.$$

THEOREM 53. *Linear groups over a commutative, Noetherian, unitary ring are equationally Noetherian.*

THEOREM 54. *[Dahmani, Groves, 2006] Torsion free relatively hyperbolic groups with abelian parabolics are equationally Noetherian (for torsion free hyperbolic groups proved by Sela).*

## 8. Irreducible components

If the Zariski topology is Noetherian then every algebraic set can be uniquely presented as a finite union of its irreducible components:

$$V = V_1 \cup \cdots \cup V_k.$$

Recall, that a closed subset $V$ is irreducible if it is not a union of two proper closed (in the induced topology) subsets.

The following is an immediate corollary of the decomposition of algebraic sets into their irreducible components.

THEOREM 55. *(Embedding theorem) Let $G$ be equationally Noetherian. Then for every system of equations $S(X) = 1$ over $G$ there are finitely many irreducible systems $S_1(X) = 1, \cdots, S_m(X) = 1$ (that determine the irreducible components of the algebraic set $V(S)$) such that*

$$G_{R(S)} \hookrightarrow G_{R(S_1)} \times G_{R(S_2)} \times \cdots \times G_{R(S_m)}.$$

THEOREM 56. *$V_F(S)$ is irreducible $\Leftrightarrow$ $F_{R(S)}$ is discriminated by $F$.*

Recall, that a group $G$ is discriminated by a subgroup $H$ if for any finite subset $M \subseteq G$with $1 \notin M$ there exists an $H$ homomorphism:

$$\phi : G \to H$$

such that $1 \notin \phi(M)$.

PROOF. ($\Longrightarrow$)We will prove that $V(S)$ is irreducible if and only if $F_{R(S)}$ is discriminated in $F$ (by $F$ homomorphisms).

Suppose $V(S)$ is not irreducible and

$$V(S) = \bigcup_{i=1}^{n} V(S_i)$$

is its decomposition into irreducible components. Then

$$R(S) = \bigcap_{i=1}^{n} R(S_i),$$

and hence there exist

$$s_i \in R(S_i) \setminus \{R(S), R(S_j), j \neq i\}.$$

The set $\{s_i, i = 1, \cdots, n\}$cannot discriminated in $F$ (by $F$ homomorphisms)

($\Longleftarrow$) Suppose now $s_1, \cdots, s_n$ are elements such that for any retract

$$f : F_{R(S)} \to F$$

there exists $i$ such that $f(s_i) = 1$

$$\begin{tikzcd} & G_{R(S)} \arrow[dl] \arrow[dr] & \\ G_{R(S_1)} \arrow[dr] & & G_{R(S_2)} \arrow[dl] \\ & F & \end{tikzcd}$$

then $\exists s_i \in S_i$, it kills one of the elements. If is not irreducible then is not residually free. Then

$$V(S) = \bigcup_{i=1}^{m} V(S_i \cup s_i). \tag{8.1}$$

Equation (8.1) implies that every solution of $S$ is also a solution of $S_i$ but not all of then. This implies it is not residually free, which implies not reducible. $\square$

# Chapter 9

We recap the previous chapter

THEOREM 57. *[Kharlampovich, Miasnikov, 96] That is it. Every f.g. fully residually free group is a subgroup of a group obtained from a free group as a finite series of extensions of centralizers.*

From this we have:

COROLLARY 8. *Every $fg$ fully residually free group is finite represented.*

PROOF. Sketch. If

$$G = \left\langle H, t | t^{-1} a t = b, a \in A, b \in B \right\rangle,$$

and $\underset{A=B}{G * H}$ is $fg$, and $A$ is $fg$ then $G$ and $H$ are $fg$. Then by a normal form argument conclude. □

We know that a $fg$ fully residue free group $G$ is a subgroup of $G_n$. Then

$$F \leq G_{\circ} \leq G_1 \leq \cdots \leq G_n,$$

then

$$G_n = \left\langle G_{n-1}, t | \left[ C_{G_{n-1}} (u), t \right] = 1 \right\rangle.$$

Induction on $n$. For the case $n = 0$ implies that $G \leq F$ obvious since is finitely presented. Assume all $fg$ are fully residually free groups that a subgroups of a series of length $< n$ are finitely presented.

$$G \leq G_n = \left\langle G_{n-1}, t | \left[ C_{G_{n-1}} (u), t \right] = 1 \right\rangle,$$

$G$ is a fundamental group of graphs of groups with abelian $H$ groups. In particular $H$ groups are $fg$ vertices groups, and the vertices groups are given by

$$V = G \cap G_{n-1}^g.$$

Since $G$ is $fg$ and also $G_{n-1}^g$ is $fg$ then the intersection of two $fg$ is also $fg$ group. Therefore by induction they are finitely presented. So $G$ is finitely presented as the fundamental graphs of groups with finitely presented groups and finitely presented edges groups.

EXAMPLE 95. (**Algebraic Sets**) *Let $G[X] = G * F(X)$, where $X = (x_1, \cdots, x_n)$. Then $G = F(a_1, \cdots, a_n)$. Then the system of solutions is given by*

$$S(X, A) = S(x_1, \cdots, x_n, x_1, \cdots, x_m).$$

*Where*

$$S(g_1, , g_n, a_1, , a_m) = 1.$$

*Then* $x_1^2x_2^2x_3^2 = 1$, *where* $g_1 = u^n, g_2^k$, *and* $g_3^2 = u^{-n-k}$, $\ni$ *all solutions live in* $G^n$. *The normal closure is given by*

$$ncl\left(x_1^2x_2^2x_3^2\right),$$

*and the radical is given by*

$$R\left(x_1^2x_2^2x_3^2\right) = \langle [x_1, x_2], [x_2, x_3], [x_1, x_2], x_1x_2x_3 \rangle.$$

*Note that the radical can be larger then the normal closure.*

Radicals and coordinates groups. The maximal subset

$$R(S) \subseteq G * F(X)$$

with

$$V_G(R(S)) = V_G(S)$$

is the radical of $S = 1$ in $G$. The quotient group

$$G_{R(S)} = \frac{G[X]}{R(S)} = \frac{G * F(X)}{R(S)}.$$

is the coordinate group of $S = 1$. Solutions of $S(X) = 1$ in $G \Leftrightarrow G$ homomorphisms $G_{R(S)} \to G$. This is solutions are also homomorphism

$$\begin{array}{rcl} G[X] & \to & G, \\ \varphi & : & G \to G, \text{ identically} \\ \varphi & : & x_i \to g_i. \end{array}$$

Then $\varphi$ kills $s$ and $ncl(S) \leq \ker \varphi$. Then

$$\varphi : \frac{G[X]}{ncl(S)} \to S.$$

Then

$$\varphi : \frac{G[X]}{R(S)} \to G,$$

is the coordinate group for $S$.

EXAMPLE 96. *Is* $S(X) = [x_1, x_2][x_3, x_4]$, *then*

$$R(S) = ncl(S).$$

CONJECTURE 1. *Radical of many interesting equations are not known.*

EXAMPLE 97. *Consider*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{11i} & x_{12i} \\ x_{21i} & x_{22i} \end{pmatrix} = \begin{pmatrix} a_{11}x_{12i} + a_{12}x_{12i} & * \\ * & * \end{pmatrix}$$

*for* $a_{ij} \in \mathbb{Z}$ *then the solution is given by*

$$S(x_1, \cdots, x_n, a_1, \cdots, a_n) = 1 \; iff \; \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix},$$

*where*

$$\left\{ \begin{array}{l} f_{11} = 1 \\ f_{12} = 0 \\ f_{21} = 0 \\ f_{22} = 1 \end{array} \right\}.$$

## 1. Diophantine problem in free groups

THEOREM 58. [ *Makanin, 1982*] *There is an algorithm to verify whether a given system of equations has a solution in a free group ( free semigroup) or not.*

He showed that if there is a solution of an equation $S(X,A)=1$ in $F$ then there is a "short" solution of length $f(|S|)$ where $f$ is some fixed computable function. Extremely hard theorem! Now it is viewed as a major achievement in group theory, as well as in computer science.

## 2. Complexity

The original Makanin's algorithm is very inefficient, not even primitive recursive. Plandowski gave a much improved version ( for free semigroups) : P-space. Gutierrez devised a $P$- space algorithm for free groups.

THEOREM 59. [ *Kharlampovich, Lysenock, Myasnikov, Touikan* (2008)] *The Diophantine problem for quadratic equations in free groups is $NP$ complete.*

THEOREM 60. [ *announced by Lysenok*] *The Diophantine problem in free semigroups ( groups) is $NP$ complete.*

If so many interesting consequences for algorithmic group theory and topology.

## 3. Existential and Universal theories of $F$

THEOREM 61. *The existential $Th_{\exists}(F)$ and the universal $Th_{\forall}(F)$ theories of $F$ are decidable.*

It was known long before that non Abelian free groups have the same existential and universal theories. Main questions was: what are finitely generated groups $G$ with $Th_{\exists}(F)=Th_{\forall}(F)$?

EXAMPLE 98. *For*

$$\exists x_1\cdots x_n u(x_1,\cdots x_n)=1,$$

*no negation. Then*

$$\exists x_1\cdots x_n\left(u(x_1,\cdots x_n)=1\ \&\ v(x_1,\cdots x_n)\neq 1\right).$$

*Then the negation of this is*

$$\forall x_1\cdots x_n(u\rightarrow v).$$

## 4. Groups universally equivalent to $F$

THEOREM 62. *(Unification theorem 1) Let $G$ be a finitely generated group and $F\leq G$. Then the following conditions are equivalent:*

(1) *$G$ is discriminated by $F$ (fully residually free), i.e. for any finite subset $M\subseteq G$ there exists a homomorphism $G\rightarrow F$ injective on $M$.*
(2) *$G$ is universally equivalent to $F$.*
(3) *$G$ is the coordinate group of an irreducible variety over $F$.*
(4) *$G$ is a Sela's limit group.*
(5) *$G$ is a limit of free groups in Gromov- Hausdorff metric.*
(6) *$G$ embeds into an ultrapower of free groups.*

Definition 67. *Ultrafilter* $\mathcal{U}$ *is a mapping*

$$f:\mathcal{P}(\mathbb{N})\rightarrow\{1,0\},$$

*such that if*

$$A\cap B=\varnothing,$$

*then*

$$\begin{aligned} f(A)+f(B) &= f(A\cup B), \\ f(\mathbb{N}) &= 1. \end{aligned}$$

*So is an additive measure.*

Remark 39. *The ultrafilter is noprinciple if* $f\,(\text{finite})=0$

Definition 68. *Ultraproduct of* $G_i$, $i\in I$ *and*

$$\prod G_i/\sim$$

*then*

$$\{g_i\}\sim\{h_i\},$$

*if*

$$g_i=h_i,$$

*for all* $i\in\mathcal{U}$*.*

Remark 40. *The Ultraproduct is non principal ultrafilter.*

Theorem 63. *(Unification theorem 2 with coefficients) Let* $G$ *be a finitely generated group containing a free non abelian group* $F$ *as a subgroup. Then the following conditions are equivalent:*

(1) *$G$ is $F$ discriminant by $F$.*
(2) *$G$ is universally equivalent to $F$ (in the language with constants).*
(3) *$G$ is the coordinate group of an irreducible variety over $F$.*
(4) *$G$ is Sela's limit group.*
(5) *$G$ is a limit of free groups in Gromov- Hausdorff metric.*
(6) *$G$ $F$- invariant into an ultrapower of $F$.*

Proof. Equivalence (1) $\Leftrightarrow$ (3) has been already proved (see theorem 56 chapter 8). We will prove the equivalence

(1) $\Leftrightarrow$ (2) .

Let $L_A$ be the language of group theory with generators $A$ of $F$ as constants. Let $G$ be a $fg$ group which is $F$- discriminated by $F$. Consider a formula

$$\exists X\left(U(X,A)=1\bigwedge W(X,A)\neq 1\right).$$

This formula is true in $F$, then it is also true in $G$, because $F\leq G$. If it is true in $G$, then for some $\overline{X}\in G^m$ holds

$$U(X,A)=1,$$

and

$$W(X,A)\neq 1.$$

Since $G$ is $F$- discriminated by $F$, there is an $F$- homomorphism

$$\phi:G\rightarrow F,$$

∍

$$\phi\left(W\left(X,A\right)\right)\neq 1,$$

this is

$$W\left(\overline{X}^{\phi},A\right)\neq 1.$$

Of course

$$U\left(\overline{X}^{\phi},A\right)=1.$$

Therefore the above formula is true in $F$. Since in $F$- group a conjunction os equations [inequalities] is equivalent to one equation [respectively, inequality], the same existential sentence in the language $L_A$ are true in $G$ and in $F$.

$(2)\Longrightarrow(1)$.

Suppose now that $G$ is $F$- universally equivalent to $F$. Let

$$G=\left\langle X,A|S\left(X,A\right)=1\right\rangle,$$

be a presentation of $G$ and

$$w_1\left(X,A\right),\cdots w_k\left(X,A\right),$$

nontrivial elements in $G$. Let $Y$ be the set of the same cardinality as $X$. Consider a system of equations

$$S\left(Y,A\right)=1,$$

in variables $Y$ in $F$. Since the group $F$ is equationally Noetherian, this system is equivalent over $F$ to a finite subsystem

$$S_1\left(Y,A\right)=1.$$

The formula

$$\psi=\forall Y\left(S_1\left(Y,A\right)=1\rightarrow\left(w_1\left(Y,A\right)=1\vee\cdots\vee w_k\left(Y,A\right)=1\right)\right),$$

is false in $G$, therefore it is false in $F$. This means that there exists a set of elements $B$ in $F$ such that

$$S_1\left(B,A\right)=1,$$

and, therefore,

$$S\left(B,A\right)=1,$$

such that

$$w_1\left(B,A\right)\neq 1\wedge\cdots\wedge w_k\left(B,A\right)\neq 1.$$

The map

$$X\rightarrow B$$

that is identical on $F$ can be extended to the $F$- homomorphism from $G$ to $F$. □

## 5. Limits of free groups

DEFINITION 69. ***Limit group by Sela***. *Let $H$ be a $fg$, consider a sequence $\{\phi_i\}$ in $Hom\left(H,F\right)$ is stable if, for all $h\in H$, $h^{\phi_i}$ is eventually always $1$ or eventually never $1$. Then the stable kernel is*

$$\underrightarrow{\ker\phi_i}=\left\{h\in H|h^{\phi_i}=1\ \mathit{for\ almost\ all}\ i\right\}.$$

*$G$ is a limit group if there is a $fg$ $H$ and a stable sequence $\{\phi_i\}$* ∍

$$G\cong\frac{H}{\underrightarrow{\ker\phi_i}}.$$

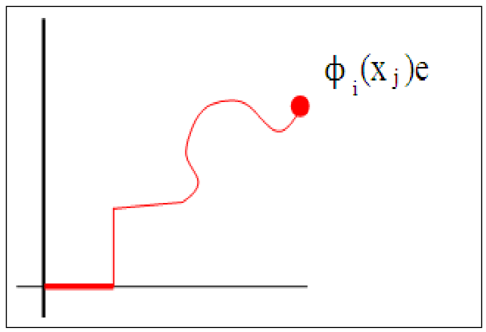
$H$ acts on the $Cayley\,(F)$ through $\varphi_i$ then

$$h\,(v) = \varphi_i\,(h)\,v,$$

fix $i$. Then

$$\begin{aligned} |\varphi_i\,(x_j)\,e| &\rightarrow \infty, \\ \text{as } i &\rightarrow \infty. \end{aligned}$$

We rescale for $i$ the metric in $Cayley\,(F)$ by dividing distances by $\max\limits_{j=1,\cdots,n} |\varphi_i\,(x_j)\,e|$. The sequence of $\mathbb{R}$-trees




In Gromov-Hausdorff the sequence of actions converges to the action of $H$ on $\mathbb{R}$-trees. Then

$$\ker\varphi = \underrightarrow{\ker\phi_i}.$$

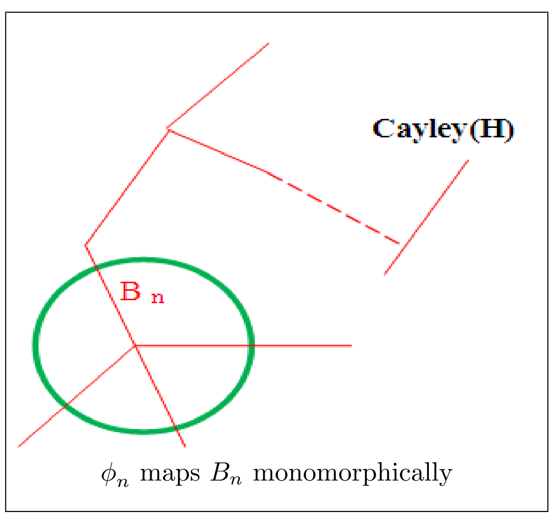
Proof. We will prove now the equivalence $(1) \Longrightarrow (4)$.
Suppose that

$$H = \langle g_1, \cdots, g_k \rangle\,,$$

is $fg$ and discriminated by $F$. There exists a sequence of homomorphism

$$\phi_n : G \rightarrow F,$$

so that $\phi_n$ maps the elements in a ball of radius $n$ in the Cayley graph of $H$ to distinct elements in $F$.



This is a stable sequence of homomorphism $\phi_m$. In general, $G$ is a quotient of $H$, but since the homomorphism were chosen so that $\phi_n$ maps a ball of radius $n$ monomorphically into $F$, $G$ is isomorphic to $H$ and, therefore, $H$ is a limit group.

$$G \cong \frac{H}{\underrightarrow{\ker \phi_i}} \cong H.$$

To prove the converse we may assume that a limit group $G$ is non Abelian because the statement is, obviously, true for Abelian groups. By definition, there exists a $fg$ group $H$, and a sequence of homomorphisms

$$\phi_i : H \to F,$$

so that

$$G \cong \frac{H}{\underrightarrow{\ker \phi_i}}.$$

With out lost of generality we can assume that $H$ is finitely presented. Add a relation one at a time to $H$ to obtain

$$\begin{array}{ccccccccccc} F_1 \to & H_1 \to & H_2 & \to & \cdots & \to & H_i & \to & G \\ \searrow & \searrow & \downarrow & Hom\,(H_2, F) & & \swarrow & Hom\,(H_i, F) & \swarrow & Hom\,(G, F) \\ & & & F & & & & & \end{array}$$

$Hom\,(G, F)$ sequence of $Hom\,(H_i, F)$ stabilizes because Noetherian property at some point

$$Hom\,(G, F) = Hom\,(H_i, F)$$

for some $i$ since $H_i$ is finitely presented. We assume $Hom\,(H, F) = Hom\,(G, F)$ and hence that the $\phi_i$'s are defined on $G$. Each non trivial element of $G$ is mapped to 1 by only finitely many $\phi_i$ so $G$ is $F$ -discriminated. $\square$

We proved the equivalence $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$. The other statements we will prove later.

# Chapter 10

EXAMPLE 99. *The universal sentence,* $\forall x,y\ [x,y]\neq 1\rightarrow w(x,y)\neq 1$, *let* $w$ *be an arbitrary group word in* $x,y$. *Also this is true for a free group. Then this is true for finite residually free group. Therefore they generate a subgroup* $F$. *Thus,* $F$ *is true:*

$$F\models \forall x,y\ \ [x,y]\neq 1\rightarrow w(x,y)\neq 1.$$

*Therefore* $G\models\varphi$ *finite residually free, every generator nonabelian subgroup is free in* $G$.

THEOREM 64. *(Unification theorem 1) Let* $G$ *be a finitely generated group and* $F\leq G$. *Then the following conditions are equivalent:*

(1) *$G$ is discriminated by $F$ (fully residually free), i.e. for any finite subset $M\subseteq G$ there exists a homomorphism $G\rightarrow F$ injective on $M$.*
(2) *$G$ is universally equivalent to $F$.*
(3) *$G$ is the coordinate group of an irreducible variety over $F$.*
(4) *$G$ is a Sela's limit group.*

DEFINITION 70. ***Limit group by Sela***. *Let* $H$ *be a* $fg$, *consider a sequence* $\{\phi_i\}$ *in* $Hom(H,F)$ *is stable if, for all* $h\in H$, $h^{\phi_i}$ *is eventually always* $1$ *or eventually never* $1$. *Then the stable kernel is*

$$\underrightarrow{\ker\phi_i}=\left\{h\in H|h^{\phi_i}=1\ \textit{for almost all}\ i\right\}.$$

*$G$ is a limit group if there is a $fg$ $H$ and a stable sequence $\{\phi_i\}$ $\ni$*

$$G\cong\frac{H}{\underrightarrow{\ker\phi_i}}.$$

[Ch. Champetier and V. Guirardel 2004] A marked group $(G,S)$ is a group $G$ with a prescribed family of generators $S=(s_1,\cdots,s_n)$. Two marked groups $(G,(s_1,\cdots,s_n))$ and $\left(G^{'},\left(s_1^{'},\cdots,s_n^{'}\right)\right)$ are isomorphic as marked groups if the bijection $s_i\leftrightarrow s_i^{'}$ extends to an isomorphic. Equivalently, their Cayley graphs are isomorphic as labelled graphs.

EXAMPLE 100. *If we consider*

$$\begin{array}{ccc} F_n(S) & \overset{\varphi_1}{\rightarrow} & (G_1,S)\\ \varphi_2\searrow & & \downarrow\varphi\\ & & (G_2,S)\end{array}$$

*then we say that* $\varphi_1$ *is equivalent to* $\varphi_2$. *So that*

$$\frac{F_n(S)}{N_1}=G_1,\frac{F_n(S)}{N_2}=G_2$$

EXAMPLE 101. *Consider* $(\langle a\rangle, (1,a))$ *and* $(\langle a\rangle, (a,1))$ *are not isomorphic as marked groups.*

DEFINITION 71. *Denote by* $\mathcal{G}_n$ *the set of groups marked by* $n$ *elements up to isomorphism of marked groups.*

## 1. A topology on spaces of marked groups

DEFINITION 72. *Topology on* $\mathcal{G}_n$. *The topology in terms of normal subgroups it will be constructed in the following way: The generating set* $S$ *of* $(G,S)$ *induces a word metric on* $G$. *We denote by* $B_{(G,S)}(R)$ *its ball of radius* $R$ *center at the identity in the free group. Let* $2^{F_n}$ *be the set of all subsets of the free group* $F_n$. *For any subsets* $A, A'$ *consider*

$$V\left(A, A'\right) = \max\left\{R \in \mathcal{N} \cup \infty | A \cap B_{F_n(s_1,\cdots,s_n)}(R) = A' \cap B_{F_n(s_1,\cdots,s_n)}(R)\right\}.$$

EXAMPLE 102. *If* $A = A'$ *this implies that* $V\left(A, A'\right) = V(A,A) = \infty$, *because the intersect every where.*

The above definition induces a metric $d$ on $2^{F_n}$ defined by $d\left(A, A'\right) = e^{-V\left(A,A'\right)}$. This metric is ultrametric and makes $2^{F_n}$ in a compact metric space.

DEFINITION 73. *An ultrametric space is a set of points* $M$ *with an associated distance function*

$$d : M \times M \to \mathbb{R},$$

$\ni \forall x, y, z$ *in* $M$ *one has*

(1) $d(x,y)$,
(2) $d(x,y) = 0$, *if* $x = y$,
(3) $d(x,y) = d(y,x)$,
(4) $d(x,y) \leq \max(d(x,z), d(y,z))$, *the ultrametric.*

**The topology in terms of epimorphisms**. Two epimorphisms

$$F_n \to G_i,$$

$i = 1, 2$ of $\mathcal{G}_n$ are closed if their kernels are closed.

**The topology in terms of relations.** One can define a metric on $\mathcal{G}_n$ by setting the distance between two marked groups $(G,S)$ and $\left(G', S'\right)$ to be $e^{-N}$ if they have exactly the same relations of length at most $N$ ( under the bijection $S \longleftrightarrow S'$) (Grigorchuk, Gromov's metric). Finally, a limit group is a limit ( with respect to the metric above) of free group marked as elements

EXAMPLE 103. *Limits groups*

(1) *An infinite presented group* $\langle s_1, \cdots, s_n | r_1, \cdots, r_i, \cdots\rangle$ *marked by* $s_1, \cdots, s_n$ *is a limit of finitely presented groups*

$$\langle s_1, \cdots, s_n | r_1, \cdots, r_i\rangle$$

*when* $i \to \infty$.

(2) $\frac{\mathbb{Z}}{i\mathbb{Z}}$ *converges to* $\mathbb{Z}$ *as* $i \to \infty$.

(3) *A free Abelian group of rank* 2 *is a limit of a sequence of cyclic groups with marking*

$$(\langle a\rangle, (a, a^n)), n \to \infty.$$

(4) *A sequence of markings of* $\mathbb{Z}$ *converges to some marking of* $\mathbb{Z}^k$.
(5) *A residually finite groups is a limit of finite groups.*

EXAMPLE 104. *Take* $w_1, \cdots, w_k$ *generators of* $F_2(a,b)$. *Then* $F_k$ *is a limit of markings of* $F_2$. *Let* $F(a,b)$ *and* $w_1(a,b), \cdots, w_k(a,b)$ *satisfy the small cancellation property* $C\left(\frac{1}{100}\right)$. *Then the piece* $|u| < \frac{1}{100}\min\left(|w_i|, |w_j|\right)$. *There is no piece which is longer than* $\frac{1}{100}$. *No relations in* $F_2(a,b)$ *between* $w_1, \cdots, w_k$ *of length* $< 100$. *Iterate then take the limit and in the limit we do not have a relation.*

In the definition of a limit group, $F$ can be replaced by any equationally Noetherian group or algebra. A direct limit of a direct system of $n$ generated finite partial subalgebras of $A$ such that all products of generators eventually appear in these partial subalgebras, is called a limit algebra over $A$. It is intersecting to study limits of free semigroups.

LEMMA 19. *Let* $(G,S)$ *be a marking of a finitely presented groups. There exist a neighborhood of* $(G,S)$ *containing only marked quotients of* $(G,S)$.

REMARK 41. *If there is a finite number of relations on the limit we get finite number of relations. Then all this relations but a bit more still which are still relations will be in the quotient space.*

COROLLARY 9. *A finitely presented group which is a limit of the set of finite groups is residually finite.*

CONJECTURE 2. *Describe the closure of the set of finite groups in* $\mathcal{G}_n$.

DEFINITION 74. *Ultrafilter* $\mathcal{U}$ *is a mapping*

$$f : \mathcal{P}(\mathbb{N}) \to \{1, 0\},$$

*such that if*

$$A \cap B = \varnothing,$$

*then*

$$\begin{aligned} f(A) + f(B) &= f(A \cup B), \\ f(\mathbb{N}) &= 1. \end{aligned}$$

*So is an additive measure.*

DEFINITION 75. *Ultraproduct of* $G_i$, $i \in I$ *and*

$$\prod G_i / \sim$$

*then*

$$\{g_i\} \sim \{h_i\},$$

*if*

$$g_i = h_i,$$

*for all* $i \in \mathcal{U}$.

THEOREM 65. *(Los) Let* $G$ *be a group and* ${}^*G$ *an ultrapower of* $G$ *for nonprincipal ultrafilter. Then* $G$ *and* ${}^*G$ *have the same elementary theory.*

EXAMPLE 105. *Let $G$ be a residually finite group, $G = \langle S, R\rangle$. Let $B_n(S)$ be a ball of radius $n$ in the Cayley graph $Cay(G)$. Let $G_n$ be a finite group where $B_n(S)$ is mapped monomorphically, $G$ is a limit of $\langle G_n, S^{\varphi_n}\rangle$,*

$$\varphi : G \to G_n.$$

*If $g \in G$ and $g \neq 1$, $|g| = n$, then $g^{\varphi_m} \neq 1$ in $G_m$, $m > n$. So $g$ is nontrivial in the limit.*

REMARK 42. *If $H \leq G$, then $Th_\forall(G) \leq Th_\forall(H)$.*

REMARK 43. *If $G =^* \prod F$, where $H \leq G$, then $Th_\forall(H) \leq Th_\forall(F)$.*

THEOREM 66. *Groups universally equivalent to $K$ are subgroups of ${}^*\prod F$.*

THEOREM 67. *(Unification theorem 2 with coefficients) Let $G$ be a finitely generated group containing a free non abelian group $F$ as a subgroup. Then the following conditions are equivalent:*

(1) *$G$ is $F$ discriminant by $F$.*
(2) *$G$ is universally equivalent to $F$ (in the language with constants).*
(3) *$G$ is the coordinate group of an irreducible variety over $F$.*
(4) *$G$ is Sela's limit group.*
(5) *$G$ is a limit of free groups in Gromov- Hausdorff metric.*
(6) *$G$ $F$- invariant into an ultrapower of $F$.*
(7) *$G$ is a limit of free groups in Gromov- Hausdorff metric.*
(8) *$G$ embeds into an ultrapower of free groups.*

REMARK 44. *The equivalence $(2) \Leftrightarrow (6)$ is a particular case of general results in model theory.*

PROOF. $(5) \Leftrightarrow (6)$ It is shown by Christopher Champetier, and Vincent Guirardel (Limit groups as limits of free groups) that a group is a limit group iff it is a finitely generated subgroup of an Ultraproduct of free groups, for a non principal ultrafilter, and any such Ultraproduct of free groups contains all the limit groups.

$(5) \Longrightarrow (6)$.

Let $(G_k, S_k) \to (G, S)$ $(k \to \infty)$, let $\omega$ be any nonprincipal ultrafilter. Let $G_k = F$. Then we have

$$G \leq^* \prod_\omega (G_k).$$

Consider $S = \{s_1, \cdots, s_k\}$,

$$\varphi : s_i \to \left(s_i^{(k)}\right)_{i=1}^{\infty}. \tag{1.1}$$

We need to show (1.1) is an embedding.

$\boxed{\text{Monomorphism}}$ Take a nontrivial element in $G$, say $g \in G$, then $g \neq 1$ in $G_n$ then the corresponding element in non trivial only in finite set of $G_k$'s on the sets of measure zero.

$\boxed{\text{Homomorphism}}$ The neighborhood of $G$ are quotients. Therefore all the components of the direct product are quotients.

$(6) \Longrightarrow (5)$.

Let

$$G \leq^* \prod_\omega (F_k) = H$$

we are going to take the images. Let $G_k \leq P_k(H)$, where $P_k(H)$ is a subgroup into the projection of the Ultraproduct. Take

$$s_i \to \left(s_i^{(k)}\right),$$

and $G_k = \left\langle s_1^{(k)}, \cdots, s_i^{(k)} \right\rangle$, then $(G, S)$ is the limit of $\left(G_k, \left(s_1^{(k)}, \cdots, s_i^{(k)}\right)\right)$. $\square$

Proposition 11. *Let $G$ be a fully residually free group. Then $G$ possesses the following properties*

1. *Each Abelian subgroup of $G$ is contained in a unique maximal finitely generated Abelian subgroup, in particular, each Abelian subgroup of $G$ is finitely generated.*
2. *$G$ is finitely presented, and has only finitely many conjugacy classes of its maximal non cyclic Abelian subgroups.*
3. *$G$ has solvable word problem.*
4. *Every 2 generated subgroup of $G$ is either free or Abelian.*
5. *$G$ is linear.*
6. *If rank $(G) = 3$ then either $G$ is free of rank 3, free Abelian of rank 3, or a free rank one extension of centralizer of a free group of rank 2, this is $G = \langle x, y, t | [u(x, y), t] = 1\rangle$, where the word $u$ is not a proper power.*

Proof. OF $(5)$. Then we have

$$G_{fg} \leq SL_2\left({}^{*}\prod_{\omega}\mathbb{Z}\right)$$

$\square$

Proof. The Ultraproduct of $SL_2(\mathbb{Z})$ is $SL_2({}^{*}\mathbb{Z})$, where ${}^{*}\mathbb{Z}$ is the Ultraproduct of $\mathbb{Z}$. Indeed, the direct product $\prod SL_2(\mathbb{Z})$ is isomorphic to $SL_2\left(\prod \mathbb{Z}\right)$. Therefore, one can define a homomorphism from the Ultraproduct of $SL_2(\mathbb{Z})$ onto $SL_2({}^{*}\mathbb{Z})$. Since the intersection of a finite number of sets from an ultrafilter again belongs to the ultrafilter, this epimorphism is a monomorphism. Being finitely generated $G$ embeds in $SL_2(R)$, where $R$ is a finitely generated subring in ${}^{*}\mathbb{Z}$. Then

$$G_{fg} \leq SL_2(R),$$

and $R$ is $f.g.$ subring in ${}^{*}\mathbb{Z}$.

$$\prod SL_2(\mathbb{Z}) \to \quad SL_2\left(\prod \mathbb{Z}\right) \to SL_2\left({}^{*}\prod_{\omega}\mathbb{Z}\right),$$

then $\overline{\varphi} \circ \overline{\psi}$, defines

$$\prod SL_2(\mathbb{Z}) \to \quad {}^{*}\prod SL_2(\mathbb{Z}) \overset{\overline{\varphi}\circ\overline{\psi}}{\to} SL_2\left({}^{*}\prod_{\omega}\mathbb{Z}\right).$$

$\square$

Theorem 68. [*Announced by Louder*] *There exist a function $g(n)$ such that the length of every proper descending chain of closed sets in $F^n$ is bounded by $g(n)$.*

Equationally Noetherian property: Every proper decreasing sequence of irreducible varieties is finite in $F^n$. Suppose we have $S(x_1, \cdots, x_n) = 1$, irreducible. Then if $s_1, s_2$ are irreducible then you cannot do it more then $g(n)$- times.

## 2. Description of solutions

Theorem 69. [*Razborov*] *Given a finite system of equations $S(X) = 1$ in $F(A)$ there is an algorithm to construct a finite solution diagram that describes all solutions of $S(X) = 1$ in $F$.*

Factor sets, let $G = \langle X|S\rangle$. Let $G$ be a $f.g.$ group we want to study $Hom(G, F)$, we can always think as $Hom(G, F)$ is in $1-1$ correspondence with solutions of systems of equations $S(X) = 1$, for all $Hom\left(F_{R(S)}, F\right)$.

Theorem 70. *$\forall f.g$ group $G$ $\exists$ a finite set of proper limit quotients of $G_i$, $\ni$ $L_1, \cdots, L_k$ $\ni \forall\varphi \in Hom(G, F)$ $\exists$ $\alpha \in Aut(G)$, and*

$$\psi_i : L \to F,$$

*such that $\varphi = \psi \circ \alpha$. This is*

$$\begin{tikzcd}
 & G = F_{R(S)} \arrow[loop above, "\alpha"] \arrow[dl, "\pi_1"'] \arrow[drrr, "\pi_k"] & & & \\
L_1 \arrow[loop above] \arrow[d] \arrow[dl] \arrow[dr] & \cdots & L_i & \cdots & L_k \\
 & & & &
\end{tikzcd}$$

*the diagram is finite.*

For free groups the diagram is algorithmically.

Solution diagram is

$$\begin{tikzcd}
 & & F_{R(S)} \arrow[dl] \arrow[dr] & \\
 & F_{R\left(\Omega_{V_1}\right)} \arrow[loop above, "\sigma_1"] \arrow[dl] \arrow[d] & \cdots & F_{R\left(\Omega_{V_N}\right)} \\
F_{R\left(\Omega_{V_{21}}\right)} & \cdots \quad F_{R\left(\Omega_{V_m}\right)} \arrow[d] \arrow[d, shift left] & & \\
 & \cdots & & \\
 & F(A) * F(T) \arrow[d] & & \\
 & F(A) & &
\end{tikzcd}$$

Conjecture 3. *In the Hyperbolic group it is not known, if we can construct such diagram algorithmically*

# Chapter 11

Example 106. *$B_{2,3} = \left\langle a, t | t^{-1}a^2t = a^3 \right\rangle$ is not equationally Noetherian, consider*

$$S = \left\{ \{ \left[ x^{-y^n}, z \right] = 1 | n = 1, \cdots \right\},$$

*the system of equations. This triples*

$$x = a^{3^n}, y = t, z = a.$$

*Solves*

$$S_l = \left\{ \left[ x^{y^{-i}}, z \right] = 1 | i = 1, \cdots, n \right\},$$

*but not $S$.*

*Why? Take $t^{-1}a^2t = a^3$, then*

$$\begin{aligned} ta^3t^{-1} &= a^2, \\ t\overbrace{a^3 \cdots a^3}^{n \ times} t^{-1} &= \overbrace{a^2 \cdots a^2}^{n \ times} \\ ta^{3^n}t^{-1} &= a^{2^n}. \end{aligned}$$

*By induction if $i = n + 1$, then*

$$\left[ ta^{2^n}t^{-1}, a \right] \neq 1.$$

*Then $ta^{-2^n}t^{-1}a^{-1}ta^{2^n}t^{-1}a \neq 1$, and $t^{-1}a^2t = a^3$, with $ta^3t^{-1} = a^2$. Therefore it does not commutes. Then any subsystem is not equivalent to the whole system.*

Definition 76. *A group is Hopfian if any epimorphism of $\theta : G \to G$ is an isomorphism.*

Example 107. *$B(2,3) = \left\langle a, b | ba^2b^{-1} = a^3 \right\rangle$ is non Hopfian. This is there exist an epimorphism*

$$\begin{aligned} \theta &: a \to a^2, \\ \theta &: b \to b, \end{aligned}$$

*with nontrivial kernel. This is $\left[ bab^{-1}, a^3 \right] \in \ker \theta$. To see this we have*

$$\begin{aligned} \left[ bab^{-1}, a^3 \right] &= bab^{-1}a^3 \left( bab^{-1} \right)^{-1} \left( a^3 \right)^{-1} \\ &= bab^{-1} \left( ba^2b^{-1} \right) \left( bab^{-1} \right)^{-1} \left( ba^2b^{-1} \right)^{-1} \\ &= ba^3a^{-1}a^{-2}b^{-1} = 1. \end{aligned}$$

Theorem 71. *(Kharl, Mias, ra) Suppose $H$ is not free. Then there is a finite set $S = \{ s : H \twoheadrightarrow H_s \}$ of proper epimorphisms such that:*

(1) *for all* $h \in Hom(H, \mathbb{F})$, *there exists* $a \in Aut(H)$ *such that* $h \circ a$ *factors through* $S$.

$$\begin{array}{ccc} H & \to & H \\ \downarrow s & & \downarrow h \\ H_s & \to & \mathbb{F} \end{array}$$

*where* $H_s$ *is the proper qoutient.*

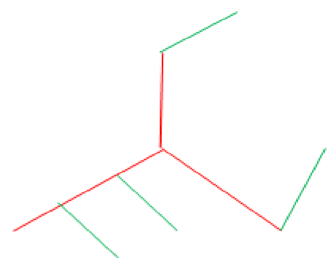
(2) *We will refine this statement and discuss a proof.*

## 1. Real trees

DEFINITION 77. *A real tree* $(T, d_T)$ *is a metric space such that between any two points* $t, t^{'} \in T$, *there is a unique arc: (the image of an embedding* $\sigma : \left[x, x^{'}\right] \to T$, *with* $\sigma(x) = t$ *and* $\sigma\left(x^{'}\right) = t^{'}$*)*

*from* $t$ *to* $t^{'}$ *and this arc is the image of an isometric embedding of an interval.*

EXAMPLE 108. *If we assign length to the interval we have a real tree.*



*Which is not a countable, why? Because* $[0, 1]$ *is not countable. We can get an infinite tree by adding intervals. Recall that if the group is finitely generated then the tree is finitely generated. This is* $G$ *acts on* $T$ *by isometry,*

$$\varphi : G \to Isom(T).$$

EXAMPLE 109. *A finite simplicial real tree is a finite tree with each edge identified with an interval.*

EXAMPLE 110. *A countable increasing union of finite simplicial real trees.*

EXAMPLE 111. $0-$*hyperbolic spaces embed into real trees.*

DEFINITION 78. *If* $g \in G, G \circlearrowleft T$, *then the translation length*

$$\|g\| = \min_{x \in T} \{d(x, gx)\},$$

$g$ *is elliptic if* $\|g\| = 0$. *Otherwise* $g$ *is hyperbolic.*

THEOREM 72. *If* $g$ *is elliptic, then there exists* $A_g$ *a closed subtree of fixed points for* $g$.

DEFINITION 79. *A closed subtree is the intersection of closed intervals in* $T$ *with* $T_\circ$, *i.e.*

$$T_\circ \cap \{ \text{ closed intervals in } T\} = \text{closed interval.}$$

EXAMPLE 112. $(0, 1)$ *is open in* $\mathbb{R}$.

PROOF. of theorem $(72)$. Set $A_g$ of fixed points is non-empty. If $g$ fixes $p_1, p_2$ in $T$, it fixes geodesics between then, this implies that $A_g$ is a subtree, and obviously closed. $\square$

## 2. Isometries of real trees

REMARK 45. *An isometry $\eta$ of a real tree $T$ is either elliptic or hyperbolic.*

REMARK 46. *Elliptic $\eta$ fixes a point of $T$. The axis of $\eta$ is $Fix\,(\eta)$.*

REMARK 47. *Hyperbolic $\eta$ leaves invariant an isometrically embedding $\mathbb{R}$ (its axis $A_\eta$). Points on $A_\eta$ are translation by*

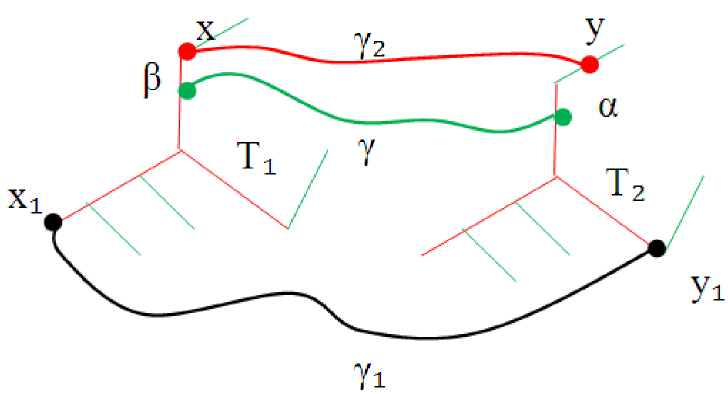
$$l_T\,(\eta) := \min\left\{d_T\,(t, \eta\,(t))\,|t \in T\right\}.$$

LEMMA 20. *If $T_1, T_2$ with $T_1 \cap T_2 = \varnothing$, are subtrees of $T$ then there exists a unique arc $\gamma\ \ni$, where $\gamma = [x, y]$, where $x$ is the initial point and $y$ is the final point. Let $x \in T_1$ and $y \in T_2$, then*

$$\gamma \cap T_1 = x, \text{ and } \gamma \cap T_2 = y.$$

*Then $\gamma$ is called a bridge.*

PROOF.




Take an arbitrary points $x$ in $T_1$ and $y$ in $T_2$. Let $\alpha$ be a geodesic joining them. Let

$$\gamma = \alpha - T_1 - T_2.$$

Take $\bar{\gamma}$ so that

$$\gamma \cap T_1 = x, \text{ and } \gamma \cap T_2 = y.$$

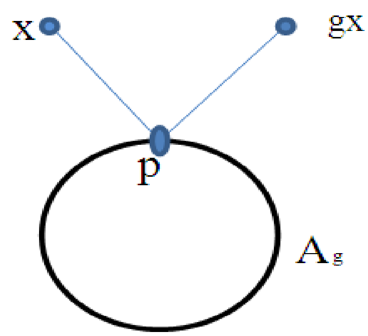
Then $\gamma^{-1}\beta\gamma$ is a geodesic $=[x, x_1]$ and not in $T_2 \Longrightarrow\Longleftarrow$ since $T_2$ is a subtree. $\square$

By this lemma we have

COROLLARY 10. *Let $g \in G$ be elliptic $x \in T$, $[x, p]$ the bridge from $x$ to $A_g$ then*

$$d\,(x, gx) = 2\,|[x, p]|\,.$$

*Consider*




Why? Since $|[x,p]|$, we know where the interval is send, and

$$|[x,p]| = |[gx,p]|,$$

because $g$ is an isometry.

THEOREM 73. *If* $\|g\| > 0$. *Then* $\exists\ A_g$ *a linear subtree and*

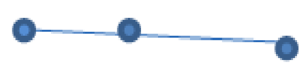
$$A_g = \{p | d(p, gp) = \|g\|\},$$

*axis of* $g$. *Also* $g$ *acts on* $A_g$ *by translation through* $\|g\|$ *.For any* $x \in T$

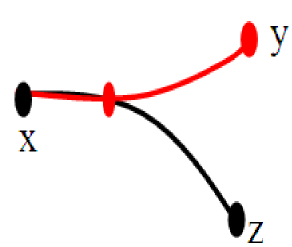
$$d(x, gx) = \|g\| + 2|[x,p]|,$$

*where* $[x,p]$ *is the bridge between* $x$ *and* $A_g$*.Then* $A_g$ *is isometric to* $\mathbb{R}$.

REMARK 48. *Linear means that every three points are collinear. I.e.*

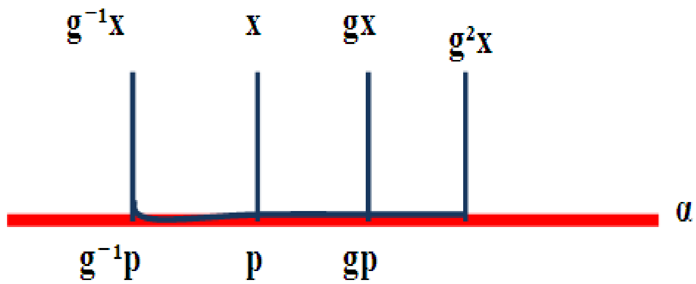


DEFINITION 80. *By a Gromov product we mean*




*we write* $(y \bullet z)_x =$ *common beginning of* $[x,y]\, z\, [x,z]$.

PROOF. of theorem (73). Take $x \in T$. Let $p = \left(g^{-1}x, gx\right)_x$ be a Gromov product,



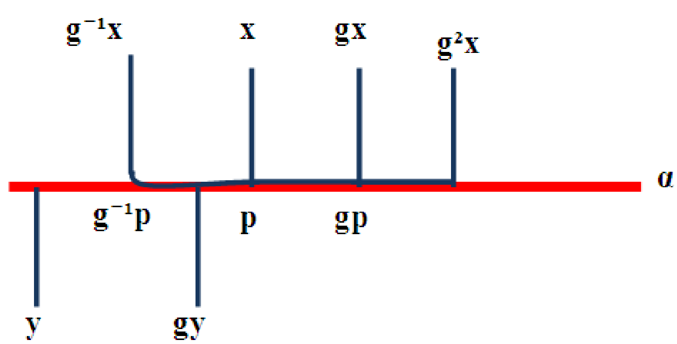

Then

$$A_g = \bigcup_{i=-\infty}^{\infty} \left[p^{g^i}, p^{g^{i+1}}\right],$$

if we take another point $y$ there is a bridge. Where

$$\left[g^{-1}p, p\right] \cap [p, gp] = \{p\}.$$

Then $g$ acts on $A_g$ acts by translations,




For $y \notin A_g$ we have

$$d(y, gy) = 2d(y, p) + d(p, gp),$$

where $d(y,p)$ is the bridge. Therefore $A_g$ is the axis, and hence isometric to the real line □

We will be interested in isometric actions of a finitely generated group $H$ on $T$. The $H$ tree $T$ is minimal if $T$ contains no proper invariant $H$ subtrees.

LEMMA 21. *If $H$ is finitely generated and $T$ is a minimal $H$ tree, then $T$ is either a point or the union of he axes of the hyperbolic elements of $H$.*

## 3. Spaces of real trees.

Let $\mathcal{A}(H)$ be the set of isometry classes of non- trivial minimal $H$- trees.

DEFINITION 81. *Gromov topology:*

$$\lim\{(T_n, d_n)\} = (T, d),$$

*iff for all finite $K \subset T$, $\epsilon > 0$, and finite $P \subset H$ ($P$ is a finite subset) there are, for all large $n$, subsets $K_n$ of $T_n$ and bijections*

$$f_n : K_n \to K,$$

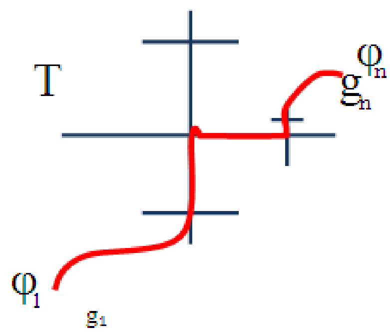
*such that*

$$|d(\eta f_n(s_n), f_n(t_n)) - d_n(\eta s_n, t_n)| < \epsilon,$$

*for all $s_n, t_n \in K_n$ and $\eta \in P$.*

Intuitively, large and larger pieces of the limit tree with their restriction actions (approximately) appear in nearby trees. This is $H$ - f.g. group and $Hom(H \to F)$. If $\varphi_n \in Hom(H, F)$. $H$ acts on $Cay(F)$ through $\varphi_n$. If $\forall h \in H, x \in Cay(F)$, $h(x) = h^{\varphi_n} x = \varphi_n(h(x))$.

EXAMPLE 113. *The Cayley graph of free group is a tree. If $H = \langle g_1, \cdots g_k\rangle$, then*




*where*

$$\|\varphi_n\| = \sum_{i=1}^{\infty} |g_i^{\varphi_i}|,$$

*since we are considering the projectivized space, define by*

$$\|g\|_{\varphi_n} = \frac{\|g\|}{\sum_{i=1}^{\infty} |g_i^{\varphi_i}|},$$

*which is a metric. An take $\left(T, \|g\|_{\varphi_n}\right)$ as representative.*

IF $\eta \in H$ fixes a point in $T_{\varphi_n}$ then

$$\varphi_n(\eta) = 1.$$

Then $\ker(T_n) = \ker(h)$.

We are interested in projectivized spaces of non-trivial $H$- trees, i.e.,

$$(T, d) \sim (T, \lambda d),$$

for $\lambda > 0$.

$$\mathcal{PA}(H) := \frac{\mathcal{A}(H)}{(0, \infty)}.$$

Also nontrivial $h \in Hom(H, \mathbb{F})$ gives an action of $H$ on the Cayley tree for $\mathbb{F}$ and so determines $T_h \in \mathcal{PA}(H)$. Since the Cayley tree is a free $\mathbb{F}$ - tree, if $\eta \in H$ fixes a point in $T_h$, then $h(\eta) = 1$. In particular $\ker(T_n) = \ker(h)$. And $T_h$ and $T_{i_\phi \circ h}$ are isometric where $i_\phi$ denotes conjugation by $\phi \in \mathbb{F}$.

THEOREM 74. *(Paulin, Culler-Morgan). Let $\{h_i\}$ be a sequence in $Hom(H, \mathbb{F})$. Then $\{T_{h_i}\}$ has a convergent subsequence in $\mathcal{PA}(H)$.*

Such limits have nice properties.

DEFINITION 82. *An Htree $T$ is super stable if the following property holds: If $J \subset I$ are non-degenerate arcs with $Fix_T(I)$ non-trivial, then*

$$Fix_T(J) = Fix_T(I).$$

REMARK 49.

$$Stab(J) = Stab(I),$$

*if is nontrivial.*

REMARK 50. *An $H$ -tree $T$ is very small if it is nontrivial (i.e. not a point), minimal, non-degenerated tripod stabilizers are trivial.*

PROPOSITION 12. *Suppose $\{h_i\}$ is a sequence in $Hom(H, \mathbb{F})$, no $h_i$ has cyclic image, and*

$$T = \lim T_{h_i}.$$

*Then:*

1. *$T$ is irreducible (i.e. not linear and no fixed end).*
2. $\ker(T) = \underrightarrow{\ker h_i}$, *and*
3. $\frac{T}{\ker(T)}$ *is very small and super stable.*

In particular, $\frac{H}{\ker(T)}$ is a limit group.

PROOF. We will show tripod stabilizers are trivial. The rest is similar. Assume $\eta$ stabilizes the endpoints of a tripod. Nearby $T_{h_i}$ have tripods with endpoints nearly stabilized by $\eta$. So, $\eta$ fixes the cone point and $h_i(\eta) = 1$. $\square$

## 4. Shortening

DEFINITION 83. *Let $H$ have a fixed finite generating set $S$. For $h \in Hom(H, \mathbb{F})$,*

$$\|h\| := \max_{s \in S} |h(x)|.$$

*then*

1. *$h \sim h'$ if $h' = i_\gamma \circ h \circ a$, where $\gamma \in \mathbb{F}$ and $a \in Aut(H)$.*

$$\begin{array}{ccc} & a & \\ & \curvearrowleft & \\ & H & \\ h & \downarrow\downarrow & h' \\ & F & \\ & \circlearrowright & \\ & i_\gamma & \end{array}$$

2. *$h$ is shortest if $h = \min_{h \sim h'} \left\| h' \right\|$*

THEOREM 75. *(Shortening) Suppose $H$ is f.g., freely indecomposable, and not (Abelian) $\mathbb{Z}$. If $T = \lim T_{h_i}$ where $\{h_i\}$ is a sequence of shortest elements, then $T$ (real tree) is not faithful, (there are elements which are in the kernel of this action).*

THEOREM 76. $(K - M - S)$. *Suppose $H$ is f.g. and not free. Then there is a finite set$\mathcal{S} = \{s : H \twoheadrightarrow H_s\}$ of proper epimorphisms such that:*

(1) *for all $h \in Hom(H, \mathbb{F})$, there exist $a \in Aut(H)$ such that $h \circ a$ factors through $\mathcal{S}$ .*
   *Further, if $H$ is not a limit group then we may always take $a = id|_H$.*

PROOF. If $H$ is not a limit group then there is a finite set $\{\eta_1, \cdots, \eta_n\}$ such that every $h \in Hom(H, \mathbb{F})$ kills some $\eta_i$. We may take

$$\mathcal{S} = \left\{ \frac{H}{\langle\langle \eta_1 \rangle\rangle}, \cdots, \frac{H}{\langle\langle \eta_n \rangle\rangle} \right\}.$$

By Grushko, we may assume that $H$ is freely indecomposiable. We may also assume that $H$ is non-Abelian. It is enough to show that there is $\mathcal{S}$ through which every shortest $h \in Hom(H, \mathbb{F})$ factors. Suppose not. Let $\{\eta_1, \cdots \eta_i\}$ enumerate the non-trivial elements of $H$. If

$$\mathcal{S}_i = \left\{ \frac{H}{\langle\langle \eta_1 \rangle\rangle}, \cdots, \frac{H}{\langle\langle \eta_i \rangle\rangle} \right\},$$

then there is shortest $h_i$ that is injective on $\{\eta_1, \cdots \eta_i\}$. A subsequence of $\{h_i\}$ converges to a faithful tree contradicting theorem ( Shortening). □

COROLLARY 11. *We may assume each $H_s$ is a limit group.*

## 5. Examples of factor sets

EXAMPLE 114. *The sets that appear in the theorem are called factors sets. There are not many explicit examples.*

EXAMPLE 115. *If $H = H_1 * H_2$, then*

$$\begin{aligned} \mathcal{S}(H) &= \mathcal{S}(H_1) * \mathcal{S}(H_2) \\ &= \{s_1 * s_2 | s_i \in \mathcal{S}(H_i)\}. \end{aligned}$$

EXAMPLE 116. $\mathcal{S}(\mathbb{Z}^n) = \{s : \mathbb{Z}^n \to \mathbb{Z}\}$.

EXAMPLE 117. *If $H$ is a closed, orientable, genus $g$ surface group and if*

$$s : H \to F_g$$

*represents the standard retraction of the surface onto a rank $g$ graph, then*

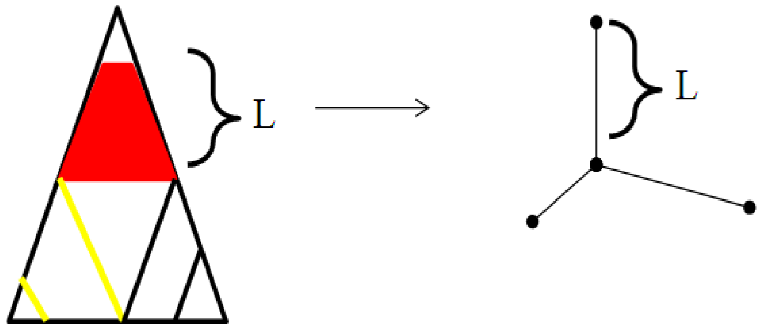
$$\mathcal{S}(H) = \{s\}.$$

(*Zieschang, Stallings*).

EXAMPLE 118. *Then non-orientable version is due to Grigorchuk and Kuchanov and one map does not suffice. Note: for $n = 1, 2,$ or $3, n\mathbb{P}$ is not a limit point group.*

We need a better understanding of real trees. We will use foliated $2-$ complexes to visualize real trees.

EXAMPLE 119. ( *Triangle*)




*Each colored band is decomposed into parallel line segments with a transverse Lebesgue measure.*

# Chapter 12

## 1. JSJ decomposition, presentation by Alexander Taam

DEFINITION 84. *A splitting of a group $G$ is a triple $(G, (G(x), T), \phi)$, where*

$$\phi : G \to \pi(\mathcal{G}, T),$$

*is an isomorphism.*

DEFINITION 85. *An elementary $\mathbb{Z}-$splitting is a one edge splitting if*

$$G = A \underset{C}{*} B, \text{ or } G = A\underset{C}{*},$$

*with $C \simeq \mathbb{Z}$.*

DEFINITION 86. *The splitting is reduced if every vertex group of valery 1 or 2 properly contains images of incident edge groups.*

DEFINITION 87. *Nondegenerate if at least one edge, and reduced.*

DEFINITION 88. *If*

$$G = A \underset{C}{*} B,$$

*and $C \subseteq C_1 \subseteq A$ then*

$$G = A \underset{C_1}{*} B,$$

*where*

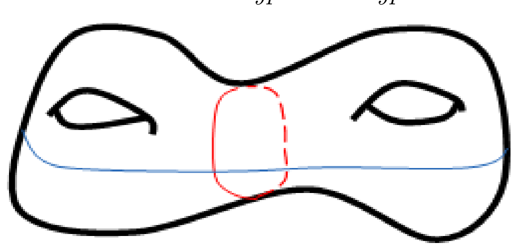
$$B_1 = C_1 \underset{C}{*} B$$

*is called a folding and inverse is unfolding.*

DEFINITION 89. *If $D_i$ for $i = 1, 2$ are elementary splittings with edge group $C_i$, if $C_1$ is contained in a conjugate of either subgroup given by $D_2$ say $C_1$ is elliptic with respect to $D_2$ otherwise say it is hyperbolic.*

THEOREM 77. *(Rips and Sela) Let $G$ be freely indecomposiable. If $D_i$, $i = 1, 2$ is one elementary $\mathbb{Z}-$ splitting of $G$ with edge group $C_1$ then $C_1$ is elliptic with respect to $D_2$ iff $C_2$ is elliptic with respect to $D_1$.*

EXAMPLE 120. *We can consider a hyperbolic-hyperbolic*



## 2. Quadratically Hanging subgroup

Let $G$ be one order f.g. group. Let $Q$ be a subgroup of $G$ an we called $Q$ a quadratically Hanging if $G$ has a $\mathbb{Z}$ splitting with $Q$ is a vertex group for vertex $r$, are incident edge groups or cyclic puncture subgroup of $Q$. Then $Q$ admits one of the following groups

(1) $\left\langle p_1, \cdots, p_m, a_1, \cdots, a_g, b_1, \cdots, b_g | \prod_{i=1}^{g} [a_i, b_i] \prod_{k=1}^{m} p_k \right\rangle$,

(2) $\left\langle p_1, \cdots, p_m, a_1, \cdots, a_g | \prod_{i=1}^{g} a_i^2 \prod_{k=1}^{m} p_k \right\rangle$, since is a quadratic equations we should attain closed curves,

(3) For each edge $e$ outgoing from $v$, $G_e$ is conjugate to one of the $\langle p_k \rangle$ ,

(4) For each $p_k$ $\exists$ $e_k$ outgoing from $v$ $\ni G_{e_K}$ conjugate to $\langle p_k \rangle$ .

Definition 90. *We said maximal quadratic hanging subgroup if for every elementary $\mathbb{Z}$ splitting $D$ of $G$ either $Q$ is elliptic to $D$ or edge group $C$ of $D$, if it can be conjugated into $Q$. Where $D$ is inherited from splitting of $Q$ along $C$. (Rips and Sela).*

Let $H$ be a finitely presented torsion free (Hyperbolic) group there exists a reduced unfold $\mathbb{Z}$ splitting of $H$ (called JSJ decomposition) $\ni$

(1) Every maximal QH subgroup can be conjugated into a vertex group.
(2) Every QH subgroup can be conjugated into a maximal QH subgroup.
(3) Every nonmaximal QH vertex group is elliptic in every $\mathbb{Z}$ splitting of $H$.
(4) If an elementary $\mathbb{Z}$ splitting with edge group $C$ is hyperbolic with respect to other elementary $\mathbb{Z}$ splitting then $C$ can be conjugated into some maximal QH subgroup.

Remark 51. *JSJ "encodes" all splitting of groups: congenerated by splitting along the curve in some QH subgroup and collapsing all other edges.*

Example 121. *$Mod(G) \subset Aut(G)$ degenerated*

Example 122. *Denn twists of edges groups.*

Example 123. *Modular automorphism QH groups that fix incident edge groups to conjugacy.*

Example 124. *Linear automorphism of Abelian vertex group that fix incident edge groups.*

Example 125. *Inner automorphism.*

JSJ is good to solve isomorphism for groups.

Theorem 78. *Let $G$ be a freely indecomposable limit group $\exists$ finite collection of limit group quotients*

$$\begin{aligned} \eta &: L \to M_j, \ni \\ h &: L \to F, \end{aligned}$$

*for every homomorphism $\exists$ some $\phi \in Mod(G)$, and some $j$ and also*

$$\begin{aligned} V_j &: M_j \to F \ni \\ h &= v_j \circ \eta_j \circ \phi. \end{aligned}$$

REMARK 52. *In general there is no canonical JSJ decomposition.*

## 3. Open problems

(1) The Poincaré Conjecture holds iff every $g \geq 2$ epimorphism

$$\varphi : S_g \to F_g \times F_g,$$

factors through the free product

$$S_1 * S_{g-1}.$$

Where $S_g$ is the fundamental group of an oriented closed surface.

$$\begin{array}{ccc} & & S_g \\ & \swarrow & \downarrow \\ S_1 * S_{g-1} & \to & F_g \times F_g \end{array}$$

As reference see

- Stallings. How not to prove the Poincaré Conjecture
- Gregorchach- Kurchamoc. Some questions in group theory related to geometry.

(2) If the equation

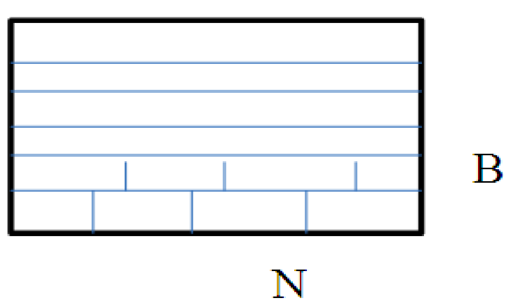
$$[[x, y], y, y, y, y, y] = \prod_{i=1}^{n} z_i^5,$$

does not have a solution in a free group for any $n$ then $B(2,5)$ is infinite. This is

$$B(2,5) = \langle a, b | x^5 = 1, \forall x \in B(2,5) \rangle.$$

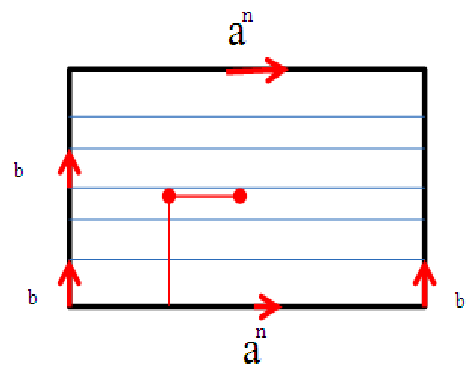
(3) Bin packing is $NP$ complete problem, let $R = \{r_1, \cdots, r_k\} \subseteq \mathbb{N}$, $\ni$

$$\sum_{i=1}^{k} r_i = NB.$$




If there exist a partition of $R$ into $B$ parts such that the sum of the numbers is equal to $N$.

The quadratic equation is equivalent to the bin packing problem




$$[a^n, b^B] = \prod_{i=1}^{k} x_i^{-1} [a^r, b] x_i, \qquad (\heartsuit\heartsuit)$$

this is a quadratic equation since $x$ appears twice. The bin can be packed #($\heartsuit\heartsuit$) has a solution $F(a, b)$ .Do this for hyperbolic groups.

# Chapter 13

## 1. The Coarse Geometric of Groups by Tim Susse.

### Outline

(1) Groups as Geometric Objects.
(2) Quasi- isometries.
(3) Hyperbolicity and Hyperbolic Groups.

## 2. Groups as Geometric Objects.

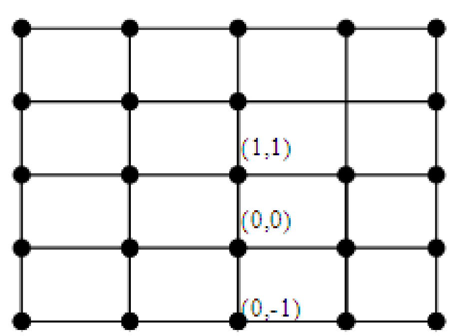
Given a presentation of a group $G = \langle S|R\rangle$ we associate a geometric object called the Cayley graph, denoted Cayley $(G, S)$.

Vertex Set $= G$

$$g_1 \sim g_2$$

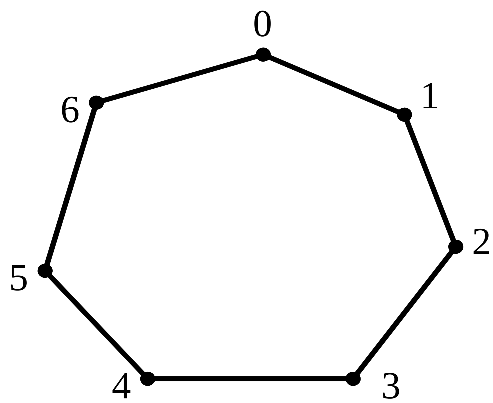
iff there exists $s \in S \cup S^{-1}$ with $g_1 = g_2 s$. Thus the Cayley graph of a group depends on the choice of generating set.

EXAMPLE 126. *for* $Cayley\big(\mathbb{Z}^2, \{(1,0), (0,1)\}\big)$.



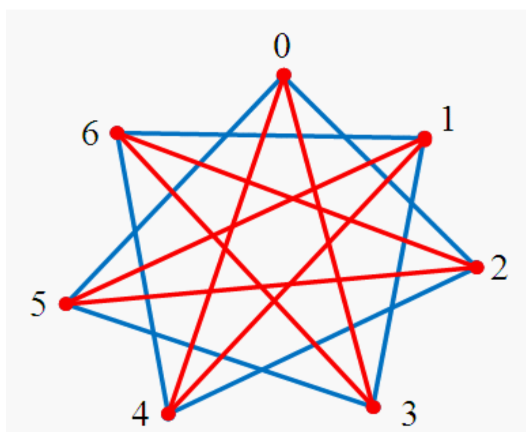

$Cayley\big(\mathbb{Z}^2, \{(1,0), (0,1)\}\big)$.

EXAMPLE 127. *for Cayley*$\left(\frac{\mathbb{Z}}{7\mathbb{Z}}, \{1\}\right)$.



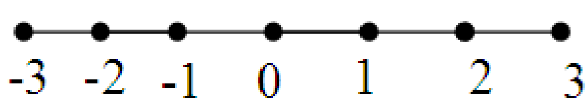

*Cayley*$\left(\frac{\mathbb{Z}}{7\mathbb{Z}}, \{1\}\right)$.

EXAMPLE 128. *of Cayley* $\left(\frac{\mathbb{Z}}{7\mathbb{Z}}, \{2,3\}\right)$.



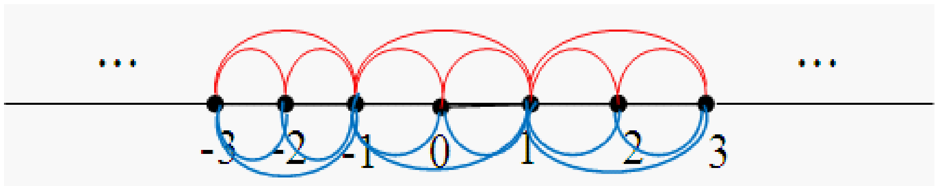

*Cayley* $\left(\frac{\mathbb{Z}}{7\mathbb{Z}}, \{2,3\}\right)$.

EXAMPLE 129. *of Cayley*$(\mathbb{Z}, \{1\})$.




*Cayley*$(\mathbb{Z}, \{1\})$.

EXAMPLE 130. *of Cayley*$(\mathbb{Z}, \{2, 3\})$.




*Cayley*$(\mathbb{Z}, \{2, 3\})$.

REMARK 53. *You might notice that if you step really far back, or zoom out on the graphic, the two Cayley graphs look very similar. We want to formalize and take advantage of this.*

## 3. Words metric on a group

Given a generating set $S$ for $G$, we define the distance between two points in $G$ to be their distance in the Cayley graph Cayley$(G, S)$. Again, this depends on the choice of generating set. This metric $d_S$ is called the word metric on $G$.

Facts:

(1) For any $g \in G$, $d_S(g, e) = l_S(g)$, the word length of $G$.
(2) For any $g, h \in G$, $d_S(g, h) = l_S(g^{-1}h)$.
(3) $G$ acts on $(G, d_S)$ by left multiplication, an isometry.
(4) Is $S$ is a finite generating set, closed balls are finite, so $(G, d_S)$ is a proper metric space. The converse is also true.

## 4. Quasi-isometries

DEFINITION 91. *(bi-Lipschitz equivalence) Given two metric spaces* $(X, d_X)$ *and* $(Y, d_Y)$ *we say that a map*

$$f : X \to Y,$$

*is a* $k-$*bi-Lipschitz map if for any pair of points* $x_1, x_2 \in X$ *we have*

$$\frac{1}{k} d_X(x_1, x_2) \le d_Y(f(x_1), fx(_2)) \le kd(x_1, x_2).$$

REMARK 54. *A bi-Lipschitz map is like a stretching of the metric at every point (by bounded amounts). It is worth noting that a bi-Lipschitz map is always a topological embedding.*

A weaker notion of equivalence is a large scale or coarse bi-Lipschitz condition. We call this a quasi- isometry.

DEFINITION 92. *We say that a map*

$$f : X \to Y$$

*is a* $(k, c)-$ *quasi- isometric embedding if for every pair* $x_1, x_2$ *we have that*

$$\frac{1}{k} d_X(x_1, x_2) - c \le d_Y(f(x_1), fx(_2)) \le kd(x_1, x_2) + c.$$

Further, if the map is coarsely onto (i.e. if every point in $Y$ is distance at most $c$ from $f(X)$, we call it a quasi-isometry). Quasi-isometry defines an equivalence relation on metric spaces:

(1) Clearly the identity map is an isometry, so any space is quasi-isometric to itself.
(2) A quick computation shows that if $f$ and $g$ are quasi-isometrics, so is $f \circ g$.
(3) Symmetric is a little tricky.

To show symmetry we need to construct a coarse inverse of a quasi-isometry $f$. By this we mean a quasi-isometry $f^{-1}$ so that there exists a constant $r$ with

$$d_X\left(x, f^{-1} \circ f(x)\right) \leq r.$$

To construct the inverse, we first need some quick facts. If

$$f : X \to Y,$$

is a $(k, c)-$ quasi- isometry, then it's failure to be injective is bounded. In particular, if $f(x_1) = f(x_2)$, then

$$d(x_1, x_2) \leq kc.$$

To see this, look at the left side of the definition of quasi- isometry, i.e.:

$$\frac{1}{k} d_X(x_1, x_2) - c \leq d_Y(f(x_1), fx(_2)) = 0.$$

So up to a bounded diameter "error", for each $y \in f(X)$, we can choose $g(y) = x$, where we choose some element $x \in f^{-1}(y)$. For each element $y \notin f(X)$, we note that there exists some $y^{'} \in f(X)$ with

$$d_Y\left(y, y^{'}\right) \leq c.$$

Choose some $y^{'}$ with this property and let $g(y) = x$ where $x \in f^{-1}(y)$.

REMARK 55. *This function is well defined if we ignore these finite diameter "error". We call this sort of thing coarsely well defined.*

**Note:** Why is the Quasi-isometry right?

Given a finitely generated group $G$, there exist many finite generating sets.

PROPOSITION 13. *Let $S$ and $T$ be two finite generating sets of a group $G$, then the identity map on $G$ is a quasi- isometry*

$$id_G : Cayley(G, S) \to Cayley(G, T).$$

PROOF. Let $S = \{s_1, \cdots, s_n\}$ and $T = \{t_1, \cdots, t_m\}$. Since $S$ generates $G$ there exists a shortest (geodesic) spelling of each $t_i$ in the language of $S$. In particular, $t_i = s_{1_i}^{\epsilon_1}, \cdots, s_{n_i}^{\epsilon_k}$, where $\epsilon_i = \pm 1$. Let

$$K = \max\{k : l_S(t_i) = k\},$$

i.e. the longest word length of a $t_i$ in Cayley$(G, S)$. Similarly, let $L$ be the maximum word length of the $s_i$ in Cayley$(G, T)$. Take $g \in G$. Then say $l_S(g) = r$, so

$$g = s_1(g) \cdots s_n(g),$$

where each $s_i(g) \in S \cup S^{-1}$. Replace each of the $s_i$ with their spellings in $t$, and we see that

$$l_T(g) \leq Kr.$$

Similarly, if $l_T(g) = r$, then

$$l_S(g) \leq Lr.$$

So, for any $g \in G$ we get

$$\frac{1}{K} \bullet l_T(g) \leq l_S(g) \leq L \bullet l_T(g).$$

However, for any $g, h \in G$,

$$d_S(g,h) = l_S\left(gh^{-1}\right),$$

and similarly for $T$. So, this turns into a quasi-isometry. In fact, this is a bi-Lipschitz equivalence. □

REMARK 56. *In our previous examples of Cayley graphs for* $\mathbb{Z}$ *we had* $K = 3, L = 2$ *for* $S = \{1\}$ *and* $T = \{2,3\}$.

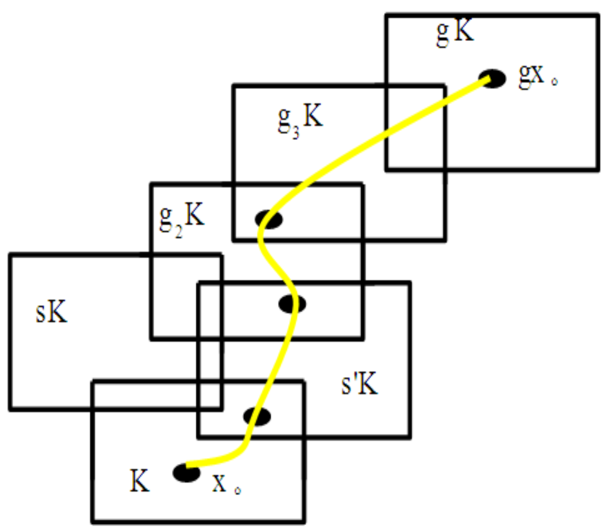
One of the fundamental tools in studying the coarse geometry of groups is the following fact.

THEOREM 79. *(Milnor-Svarc Lemma) Let* $X$ *be a proper metric space and let* $G$ *act on* $X$ *geometrically (properly discontinuously and cocompactly by isometries). Then* $G$ *is finitely generated and for a fixed* $x \in X$ *the orbit map* $g \mapsto gx$ *is a quasi-isometry.*

The proof involves taking a closed ball $K$ (which is compact, since $X$ is proper) that contains a fundamental domain for the action on $X$. Let

$$S = \{s \in G : K \cap sK \neq \varnothing\}.$$

Since the action is properly discontinuous, this set is finite. We would then show that $G = \langle S \rangle$ by making a path from $x \in K$ to $gx \in gK$ for $g \in G$ by making steps that are small enough that the corresponding translates of $K$ intersect. These correspond to elements of $S$. In doing this, the quasi-isometry bound fall out.




Consequences of the Milnor-Svarc Lemma:

(1) If $G$ is a finitely generated group and $H \leq G$ and $[G : H] < \infty$, then $H \curvearrowright$ (acts on) $G$ by multiplication on the right. This action is cocompact, so $H$ is quasi-isometric to $G$. This is an example of what is called commensurability (which is stronger the quasi-isometry).

(2) If $G = \pi_1(M)$ where $M$ is a compact Riemannian manifold, then $G$ is finitely generated and is quasi-isometric to $\widetilde{M}$. For instance $\pi_1(S)$, a closed hyperbolic surface group ($\chi(S) < 0$), is quasi-isometric to the hyperbolic plane.
(3) In a similar vein, $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$.

There are many algebraic properties that have geometric content (i.e. are quasi-isometry invariant). In particular:

(1) Having a finite presentation is quais-isometry invariant.
(2) Having a finite index free subgroup is a quasi-isometry invariant.
(3) Having two topological ends is equivalent to being virtually $\mathbb{Z}$, so it is a quasi-isometry invariant.
(4) Having one end is also a quasi-isometry invariant
(5) Having a finite index nilpotent subgroup (virtual nilpotent) is equivalent to having polynomial growth (this is Gromov's Polynomial Growth Theorem). The latter is a quasi-isometry invariant for groups.

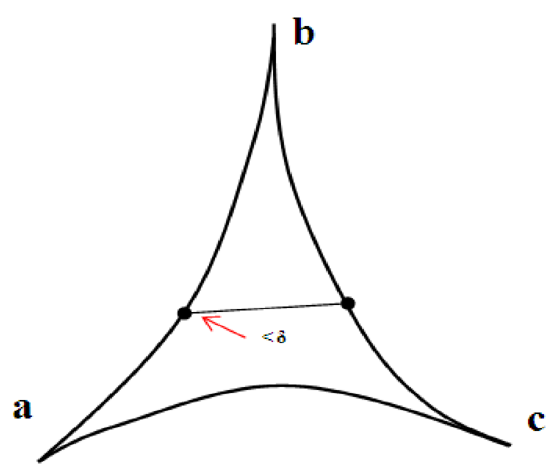
## 5. Hyperbolicity and Hyperbolic Groups

### Slim triangle property

Definition 93. *We say that a metric space $(X, d)$ is $\delta-$ hyperbolic if for any geodesic triangle $[a, b, c]$ we have that*

$$d([a, b], [a, c] \cup [b, c]) \leq c$$

*and similarly for all other permutations of the letters. We call such a triangle $\delta-$ thin or slim.*




Hyperbolicity is innately a statement about "large" triangles in a metric space. In fact, any compact metric space is automatically hyperbolic with $\delta$ equal to the (finite) diameter.

There are other, equivalent, notions of Hyperbolicity.

(1) Rips proved that a space is $\delta-$ hyperbolic if and only if every quadruple of points $a, b, p \in X$ satisfy
$$\langle b|c\rangle_p \geq \min\left\{\langle a|b\rangle_p, \langle a|c\rangle_p\right\} - \delta.$$
Where
$$\langle x|y\rangle_p = \frac{1}{2}\left(d(x,p) + d(y,p) - d(x,y)\right),$$
is called the Gromov product.

(2) Bowditch proved in his paper on the curve complex that Hyperbolicity can be formulated as a statement about coarse centers of triangles $[Bo]$.

(3) As the name suggests, the hyperbolic plane $\mathbb{H}$, which we identify with the complex upper half plane with the Riemannian metric
$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$
is
$$\log\left(1+\sqrt{2}\right) - \text{hyperbolic}.$$
Further, hyperbolic space in any dimension is hyperbolic.

(4) A tree (of any kind, including an $\mathbb{R}$- tree) is trivially $0-$ hyperbolic, since any triangle is actually a tripod.

Proposition 14. *If $f : X \to Y$ is a quasi-isometry and $X$ is $\delta-$ hyperbolic, then $Y$ is also hyperbolic.*

To prove this proposition, we need to study the properties of quasi-isometric images of geodesics.

(1) A geodesic is an isometric embedding of an interval.
(2) A $(k,c)$ $-$quasigeodesic is a $(k,c)-$ quasi-isometric embedding of an interval.

Remark 57. *If $\gamma$ is a geodesic in $X$ and*
$$f : X \to Y$$
*is a $(k,c)-$ quasi-isometric embedding, then $f(\gamma)$ is a $(k,c)-$ quasigeodesic.*

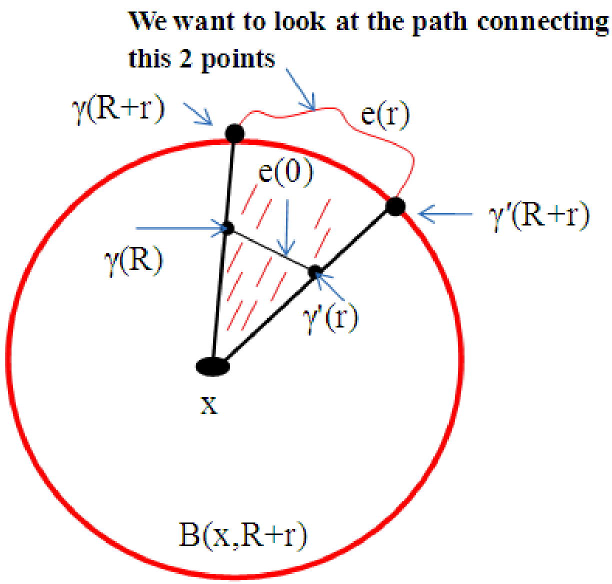
## Quasi-isometry invariance

Definition 94. *A function*
$$e : \mathbb{N} \to \mathbb{R},$$
*is called divergence function for a metric (length) space $X$ if for every $R, r \in \mathbb{N}$ and any pair of geodesics*
$$\gamma : [0, a] \to X,$$
*and*
$$\gamma' \left[0, a'\right] \to X,$$
*with*
$$\gamma(0) = \gamma'(0) = x,$$
*we want to know if the geodesics are long enough. If*
$$R + r \leq \min\left\{a, a'\right\},$$

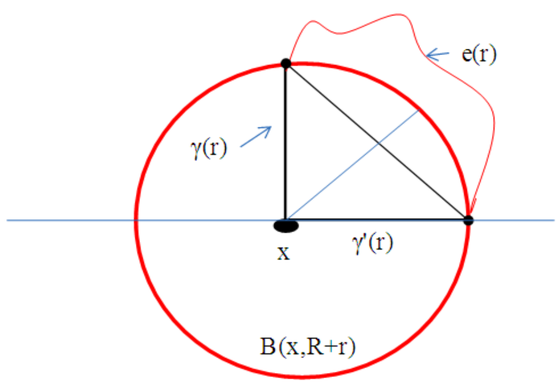
*and*

$$d\left(\gamma\left(R\right),\gamma\left(R^{'}\right)\right)\geq e\left(0\right),$$

*any path connecting* $\gamma\left(R+r\right)$ *to* $\gamma^{'}\left(R+r\right)$ *outside* $B\left(x,R+r\right)$ *must have length at least* $e\left(r\right).$




Example 131. *The Euclidean plane has divergence* $e\left(r\right)=\pi r$. *As you might think, divergence has to do with the size of the sphere of radius* $r$.




*You get a linear function.*

Example 132. *In an infinite tree, the divergence is infinite, since there is only one path between two points. This is called a cut point.*

Example 133. $\mathbb{H}^2$ *has an exponential divergence function, this can be figured out by computing the circumference of a circle of Euclidean radius* $r$ *centered at* $0$ *in the disc model.*

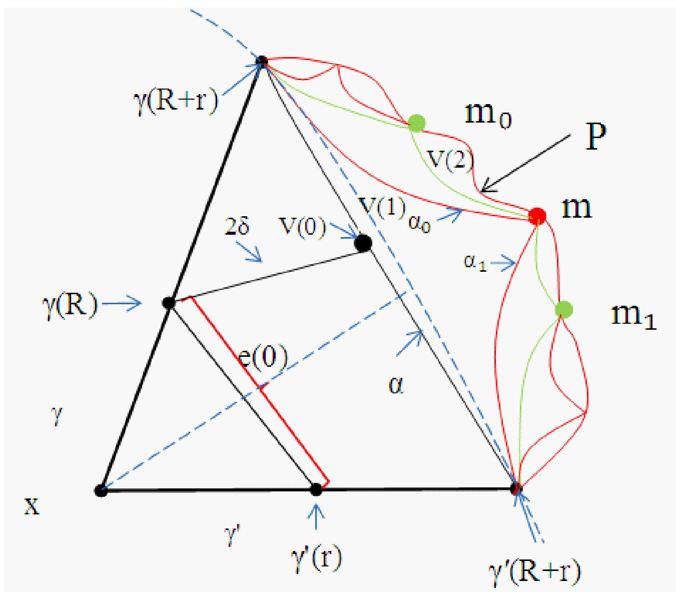
Example 134. *Is this also true in other hyperbolic spaces?*

THEOREM 80. *If $X$ is a $\delta-$ hyperbolic metric space, then it has an exponential divergence function.*

PROOF. Fix $R, r \in \mathbb{N}$. Let $\gamma$ and $\gamma^{'}$ be two geodesics based at some point $x \in X$ with

$$d\left(\gamma(R), \gamma\left(R^{'}\right)\right) > \delta,$$

and set $e(0) = \delta$. Let $p$ be a path in $X \backslash B(x, R+r)$ from $\gamma(R+r)$ to $\gamma^{'}(R+r)$ and let $\alpha$ be the geodesic from $\gamma(R+r)$ to $\gamma^{'}(R+r)$. Now let $m$ be the middle point on the path $p$ and let $\alpha_{\circ}$ be the geodesic from $\gamma(R+r)$ to $m_{\circ}$ and $\alpha_1$ the geodesic from $m$ to $\gamma^{'}(R+r)$.




Now, for any binary string $b$, let $m_b$ be the midpoint of the segment of $p$ between the endpoints of $\alpha_{b_{\circ}}$ be the geodesic between the beginning of $\alpha_b$ and $m_b$ and $\alpha_{b_1}$ the geodesic between $m_b$ and the end of $\alpha_b$. Keep subdividing $p$ in this way until each segment in the division has length between $\frac{1}{2}$ and 1. If $n$ is the number of pieces, then

$$\log_2 l(p) \leq n \leq \log_2 (l(p)+1).$$

For each $b$, the segments $\alpha_b, \alpha_{b_{\circ}}, \alpha_{b_1}$ form a geodesic triangle, and so are $\delta-$ slim. Since

$$d\left(\gamma(R), \gamma^{'}(R)\right) > \gamma,$$

there exists a point $v(0)$ on $\alpha$ with

$$d(v(0), \gamma(R)) < \delta.$$

Continuing inductively, we can find $v(1)$ on $\alpha \cup \alpha_1$ with

$$d(v(0), v(1)) \leq \delta.$$

And so if $v(i)$ is on $\alpha_b$ we find $v(i+1)$ on either $\alpha_{b_{\circ}}$ or $\alpha_{b_1}$ with

$$d(v(i), v(i+1)) < \delta.$$

Let $v(m)$ be the point obtained at the level of iteration. There is a point $y \in P$ whose distance from $v(m)$ is at most 1 and so its distance from $x$ is at most

$$R + \delta \log_2 (l(p)) + 2.$$

But

$$d(x, P) \geq R + r,$$

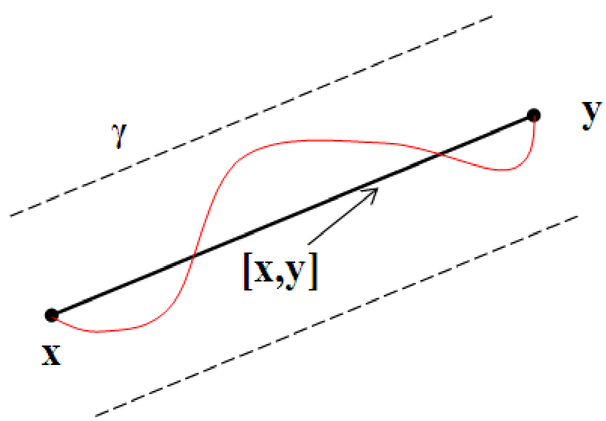
so

$$R + r \leq R + \delta \log_2 (l(p)) + 2,$$

this is $l(p)$ is at least exponential $r$. Where $l(p) > 2^{\frac{r-2}{\delta}}$ this is the exponential divergence function. $\square$

Proposition 15. *Let $\gamma$ be a $(k,c)-$ quasigeodesic with end points $x$ and $y$ and let $[x,y]$ denote a geodesic (not necessary unique) connecting $x$ to $y$. Then, there exists $M = M(k,c,\delta)$ so that the Hausdorff distance between $\gamma$ and $[x,y]$ is less than $M$. In particular, $\gamma$ is in the $M-$ neighborhood of the geodesic between its endpoints.*

Proof. The proof of this is an application of the theorem from the previous theorem.



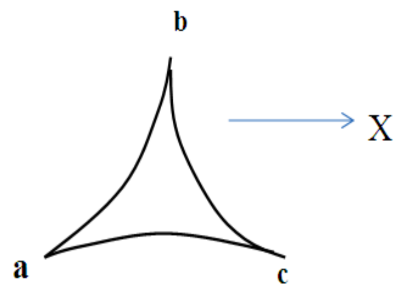

$\square$

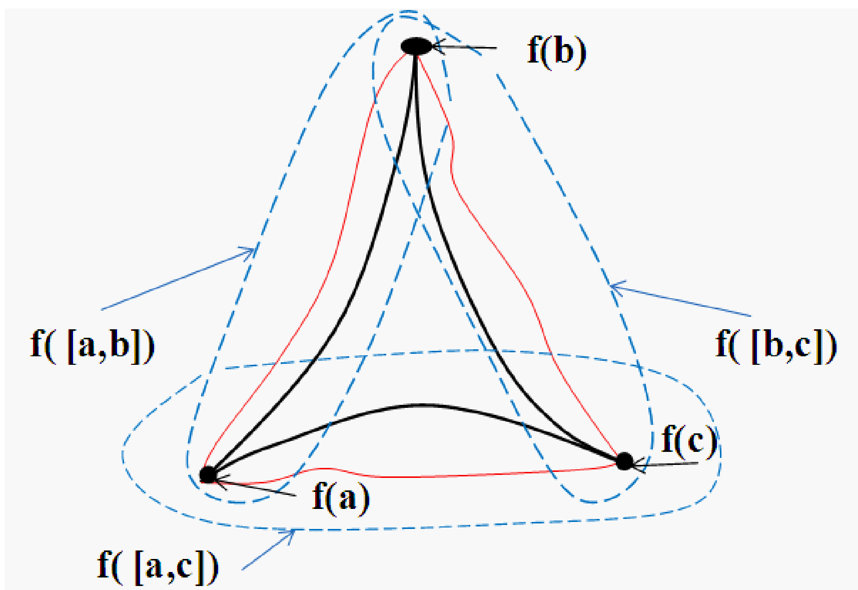
Corollary 12. *If $X$ is a $\delta-$ hyperbolic space and*

$$f : Y \to X$$

*be a $(k,c)-$ quasi-isometry, then $Y$ is $k(2M+\delta)+c-$ hyperbolic.*

Proof. Let $a, b, c \in Y$ and let $[a, b, c]$ be some geodesic triangle in $Y$ and consider its image $f([a, b, c])$ in $X$.




Well $f([a, b])$ is a quasi-geodesic, so it is in the $M-$ neighborhood of a geodesic $[f(a), f(b)]$ and similarly for $[b, c]$ and $[a, c]$. Take a point $x \in f([a, b])$. Let $y \in f([a, b])$ be such that

$$d(x, y) < M$$

and let $z \in [f(b), f(c)] \cup [f(a), f(c)]$ be such that

$$d(x, y) < \delta.$$

Further more, there exist $w \in f([b, c]) \cup f([a, c])$ such that

$$d(z, w) < M.$$

So,

$$d(x, w) < 2M + \delta.$$

Since $f$ was a quasi-isometry, and $x$ and $w$ are in the image of $f$, their preimages on the triangle $[a, b, c]$ are most $K(2M + \delta) + C$ apart. Thus the triangle in $Y$ is slim. $\square$

REMARK 58. *Corollary 12 shows us that it makes sense to talk about a finite group being hyperbolic. If we transition between finite generating sets, the Cayley graphs are quasii-isometric, so if one is hyperbolic, so is any other.*

EXAMPLE 135. *If $M$ is a closed hyperbolic manifold (i.e. $\widetilde{M}$ is isometric to $\mathbb{H}^n$), then $\pi_1(M)$ is $\delta-$hyperbolic. (This is a consequence of the Milnor-Svarc Lemma).*

EXAMPLE 136. *Given a hyperbolic group $G$, a subgroup $H$ is quaisconvex if and only if the inclusion map*

$$H \hookrightarrow G,$$

*is a quasi-isometric embedding.*

EXAMPLE 137. *In a hyperbolic group, if $g \in G$ is infinite order, then the centralizer of $g$ is quasiconvex.*

EXAMPLE 138. *No hyperbolic group contains as $\mathbb{Z}^2$ subgroup. In fact, it cannot contain a Baumslag-Solitar subgroup,*

$$\left\langle a, b | b^{-1} a^m b = a^n \right\rangle.$$

EXAMPLE 139. *(Gromov) If $G$ satisfies the small cancellation condition $C' \left(\frac{1}{6}\right)$, then $G$ is hyperbolic.*

DEFINITION 95. *Let $G = \langle S | R \rangle$. We say that the presentation is a Dehn presentation if for any reduced word $w$ with $w = 1$ in $G$, there exists a relator $r \in R$ so that $r = r_1 r_2$,*

$$l(r_1) > l(r_2),$$

*and*

$$w = w_1 r_1 w_2.$$

In other words, any word that represents the identity in $G$ contains more than one half of a relator, and so it can be shortened. A word which cannot be further reduced or shortened by this method (replacing $r_1$ by $r_2^{-1}$, a shorter word) is called Dehn reduced.

If $G$ has a finite Dehn presentation ($G$ is finitely generated, $R$ is finite) then you can check all subwords of length at most

$$N = \max \{ l(r) | r \in R \},$$

to see if a reduction can be made. This procedure for solving the word problem is called Dehn's Algorithm, originally created by Max Dehn in 1910 to solve the word problem in surface groups. Its run time is $O\left(|w|^2\right)$ in its simplest iteration. There are atmost $|w| - N$ subwords in each step, and at most $|w|$ steps in the reduction.

EXAMPLE 140. *Let $G = \langle a, b, c, d | [a, b] [c, d] \rangle$. Let $R$ be the symmetrized set of generators, so that $R$ contains all cyclic conjugates of $[a, b] [c, d]$ and its inverse.*

What about other hyperbolic groups?

DEFINITION 96. *A path $\gamma$ in a metric space $X$ is called a $k-$ local geodesic if every subpath of length $k$ is a geodesic.*

EXAMPLE 141. *On a sphere of radius 1, a great circle is a $\pi-$local geodesic.*

EXAMPLE 142. *If $M$ is a Riemannian manifold, let $r(M)$ be the injectivity radius of $M$. Then any image of a ray in $T_X M$ under the exponential map is an* $r(M) - loca\lg eodesic$.

Local geodesics have to do with loops in a metric space, or relations in a Cayley graph. That makes them natural to study when considering presentations.

LEMMA 22. *Let $G$ be $\delta-$hyperbolic group and let $\delta$ be a $4\delta-$local geodesic. Let $g$ be the geodesic between the endpoints of $\gamma$ (called $\gamma_+$ and $\gamma_-$). Assume*

$$l(g) > 2\delta,$$

*and let $r$ and $s$ be points on $\gamma$ and $g$ respectively, both distance $2\delta$ from $\gamma_+$. Then*

$$d(r,s) \leq \delta.$$

The proof here is by induction on the length of $\gamma$ and uses the thin triangle property multiple times.

THEOREM 81. *If $\gamma$ is a $4\delta-$local geodesic in a $\delta-$hyperbolic group $G$, then $\gamma$ is contained in the $3\delta$ neighborhood of the geodesic between its endpoints.*

We will use this theorem to create a shortening algorithm in our hyperbolic group $G$.

THEOREM 82. *Let $G$ be a $\delta-$ hyperbolic group with generators $S$. Let $R$ be equal to the set of words*

$$R = \{w : w = 1 \in G, |w| < 8\delta \text{ and } w \text{ represents the identity element in } G\}.$$

*We aim to show that $\langle S|R\rangle$ is a Dehn presentation for $G$. In particular $G$ is finitely presented and has a solvable word problem.*

PROOF. Take a word $w$ in the generators so that $w = 1$ in $G$. If the loop $w$ in the Cayley graph is already a $4\delta$ geodesic, then it is in the $3\delta$ neighborhood of the origin, so it is already an element of $R$ and we can see that it represents the identity. Now say that $w$ is not a $4\delta$ which is not a geodesic between its endpoints. Replace it in $w$ by the geodesic between its endpoints, call it $w_2$. Then the path $w_1 w_2^{-1}$ has length less than $8\delta$, so that word is in $R$. Further, we note that $w$ contained $w_1$, the longer part of the relator.

Now, we can continue this process until we reduced $w$ to a $4\delta-$local geodesic, a case we have already covered. Thus

$$G = \langle S|R\rangle,$$

is a Dehn presentation. □

## 6. Open questions

(1) [Gromov] Given a hyperbolic group $G$ with one topological end (i.e. a freely indecomposable hyperbolic group), does it contain a surface subgroup?
(2) Are hyperbolic groups residually finite?

(3) [Bestvina] Say that $G$ admits a finite dimensional $K(G,1)$ and does not contain any Baumslag- Solitar groups. Is $G$ necessarily hyperbolic? If $G$ embeds in a hyperbolic group is this true? Note: Gromov proved that every hyperbolic group admits a finite dimensional $K(G,1)$, making this question more natural than it seems.

(4) [Canary] Let $H \leq G$, $G$ a hyperbolic group. If there exists some $n$ so that $g^n \in H$ for every $g \in G$, is $H$ necessarily finite index in $G$? The answer is yes if $H$ is quasiconvex.

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