Lectures on Geometric Group Theory

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Preface

The main goal of this book is to describe several tools of the *quasi-isometric rigidity* and to illustrate them by presenting (essentially self-contained) proofs of several fundamental theorems in this area: Gromov's theorem on groups of polynomial growth, Mostow Rigidity Theorem and Schwartz's quasi-isometric rigidity theorem for nonuniform lattices in the real-hyperbolic spaces. We conclude with a survey of the quasi-isometric rigidity theory.

The main idea of the geometric group theory is to treat finitely-generated groups as *geometric objects*: With each finitely-generated group G one associates a metric space, the *Cayley graph* of G. One of the main issues of the geometric group theory is to recover as much as possible algebraic information about G from the geometry of the Cayley graph. (A somewhat broader viewpoint is to say that one studies a finitely generated group G by analyzing geometric properties of spaces X on which G acts *geometrically*, i.e., properly discontinuously, cocompactly and isometrically. The Cayley graph is just one of such spaces.) A primary obstacle for this is the fact that the Cayley graph depends not only on G but on a particular choice of a generating set of G. Cayley graphs associated with different generating sets are not isometric but *quasi-isometric*. The fundamental question which we will try to address in this book is:

> *If G, G' are quasi-isometric groups, to which extent G and G' share the same algebraic properties?*

The best one can hope here is to recover the group G *up to virtual isomorphism* from its geometry. Groups G_1, G_2 are said to be virtually isomorphic if there exist subgroups

$$F_i \subset H_i \subset G_i, i = 1, 2,$$

so that H_i has finite index in G_i, F_i is a finite normal subgroup in H_i, $i = 1, 2$, and H_1/F_1 is isomorphic to H_2/F_2.

Virtual isomorphism implies quasi-isometry but, in general, the converse is false, see Example 1.49.

An example when quasi-isometry implies weak commensurability is given by the following theorem due to R. Schwartz [**93**]:

THEOREM. *Suppose that G is a nonuniform lattice acting on the hyperbolic space $\mathbb{H}^n, n \geq 3$. Then for each group Γ quasi-isometric to G, the group Γ is virtually isomorphic to G.*

We will present a proof of this theorem in chapter 7. Another example of *quasi-isometric rigidity* is a corollary of M. Gromov's theorem on groups of polynomial growth:

THEOREM. *(M. Gromov) Let G be a finitely generated group of polynomial growth. Then G is virtually nilpotent.*

The converse to this theorem:

Every virtually nilpotent group has polynomial growth (see Theorem 5.13)

is much easier and is due to J. Wolf [**112**].

COROLLARY. *Suppose that G is a group quasi-isometric to a nilpotent group. Then G itself is* virtually nilpotent*, i.e. contains a nilpotent subgroup of finite index.*

Gromov's theorem and its corollary will be proven in chapter 5.

Proving these theorems is the main objectives of this book. Among other results, we will prove Mostow Rigidity Theorem for hyperbolic manifolds and will outline a proof of the Tits alternative. Along the way, we will introduce several tools of the geometric group theory: coarse topology, ultralimits and quasiconformal mappings.

Other sources. Our choice of topics in geometric group theory was rather narrow. We refer the reader to [**1**], [**3**], [**13**], [**15**], [**23**], [**24**], [**41**], [**44**], [**25**], [**86**], [**90**], [**104**], for the discussion of other parts of the theory.

Since these notes were first written, Bruce Kleiner [**66**] gave a completely different (and much shorter) proof of Gromov's polynomial growth theorem. We decided to retain, however, original Gromov's proof since it contains wealth of other ideas.

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Contents

CHAPTER 1

Preliminaries

1. Groups, actions and generating sets

In this and the following two sections we review basics of the (non-geometric) group theory. We refer to [**52, 70, 107**] for detailed treatment of this material.

A *topological group* is a group G equipped with the structure of a topological space, so that the group operations (multiplication and inversion) are continuous maps. If G is a group without specified topology, we will always assume that G is *discrete*, i.e., is given the discrete topology.

Let G be a group or a semigroup and E be a set. An *action* of G on E is a map

$$\mu : G \times E \to E, \quad \mu(g, a) = g(a),$$

so that

1. $\mu(1, x) = x$, $\mu(g_1 g_2, x) = (g_1, \mu(g_2)x)$ for all $g_1, g_2 \in G$ and $x \in E$ (in case G is a semigroup).

In addition, if G is a group, we require

2. $\mu(g, \mu(g^{-1}, x)) = x$ for all $g \in G$ and $x \in E$.

An action of G on E is often denoted $G \curvearrowright E$. Given two actions $\mu : G \curvearrowright X$ and $\nu : G \curvearrowright Y$, a map $f : X \to Y$ is called *G–equivariant* if

$$f\mu(g, x) = \nu(g, f(x)), \quad \forall g \in G, x \in X.$$

In other words, for each $g \in G$ we have a commutative diagram,

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

If G is a topological group and E is a topological space, a *continuous action* of G on E is a continuous map μ satisfying the above *action* axioms.

If E is a metric space, an *isometric action* is an action so that $\mu(g, \cdot)$ is an isometry of E for each $g \in G$.

A group action $G \curvearrowright X$ is called *free* if for every $x \in X$, the *stabilizer of x in G*,

$$G_x = \{g \in G : g(x) = x\}$$

is $\{1\}$.

A topological group action $\mu : G \curvearrowright X$ is called *proper* if for every compact subsets $K_1, K_2 \subset X$, the set

$$G_{K_1,K_2} = \{g \in G : g(K_1) \cap K_2 \neq \emptyset\} \subset G$$

is compact. If G has discrete topology, a proper action is called *properly discontinuous* action, as G_{K_1,K_2} s finite.

Exercise 1. Suppose that X is locally compact and $G \curvearrowright X$ is proper. Show that the quotient X/G is Hausdorff.

A topological action $G \curvearrowright X$ is called *cocompact* if X/G (with the quotient topology) is compact.

Suppose now that X is a metric space. The group of isometries $Isom(X)$ of the space X has the compact-open topology, equivalent to the topology of uniform convergence on compacts. A subgroup $G \subset Isom(X)$ is called *discrete* if it is discrete with respect to the subset topology.

EXERCISE 2. Suppose that X is proper. Show that $G \subset Isom(X)$ is discrete iff the action $G \curvearrowright X$ is properly discontinuous.

In the case when the set E is a group H, we require an action $G \curvearrowright H$ to determine a homomorphism $\phi : G \to Aut(H)$, $\phi(g)(h) = \mu(g, h)$. Given such action, one forms the *semidirect product*

$$H \rtimes_\phi G$$

which consists of the pairs (h, g) with the product rule given by

$$(h_1, g_1)(h_2, g_2) = (\phi(g_2)(h_1)h_2, g_1g_2).$$

In case when G is isomorphic to $\mathbb{Z}$ and the generator of $\mathbb{Z}$ acts on H via an automorphism ψ we will use the notation

$$H \rtimes_\psi \mathbb{Z}$$

for the above semidirect product. It is easy to see that $F := H \rtimes_\phi G$ is a group and H embeds in F as a normal subgroup via the map

$$h \mapsto (h, 1).$$

Accordingly, the elements of F will be denoted gh, where $g \in G, h \in H$.

Suppose that $H \subset G$ is a subgroup. A subset $S \subset H$ is called a *generating set* of H if H is the smallest subgroup of G containing S. In other words, every element of H is represented as the product of the elements of S and of their inverses. The subgroup H (denoted $H = \langle S \rangle$) is said to be *generated by* S.

We say that a normal subgroup $K \lhd G$ is *normally generated* by a set $R \subset K$ if K is the smallest normal subgroup of G which contains R. We will use the notation

$$K = \langle\langle R \rangle\rangle$$

for this subgroup.

The *commutator* of elements x, y in a group G is

$$[x, y] := xyx^{-1}y^{-1}.$$

The n-fold *iterated commutator* of the elements $x_0, x_1, ..., x_n$ is

$$[x_0, x_1, ..., x_n] := [...[[x_0, x_1], x_2], ..., x_n].$$

The *commutator subgroup* $G' = [G, G]$ of G is the subgroup of G generated by the set of all commutators

$$\{[x, y] : x, y \in G\}.$$

More generally, given a subgroup $H \subset G$, define $[G, H]$ to be the subgroup of G generated by the set

$$\{[x, y] : x \in G, y \in H\}.$$

Note that it is not necessarily true that the commutator subgroup G' of G consists entirely of commutators $[x, y] : x, y \in G$ (see **[107]** for some finite group examples). This leads to an interesting invariants (of geometric flavor) called the *commutator norm* (or *commutator length*) $\ell_c(g)$ of $g \in G'$, which is the least number k so that g can be expressed as a product

$$g = [x_1, y_1]...[x_k, y_k],$$

as well as the *stable commutator norm* of g:

$$\limsup_{n\to\infty} \frac{\ell_c(g^n)}{n}.$$

See **[6, 19]** for further details. For instance, let G be the free group on two generators. Then, every nontrivial element of G' has stable commutator norm is greater than 1.

The *center* $Z(G)$ of a group G is defined as the subgroup consisting of elements $h \in G$ so that $[h, g] = 1$ for each $g \in G$. It is easy to see that the center is a normal subgroup of G.

A group G is called *abelian* if every two elements of G *commute*, i.e., $ab = ba$ for all $a, b \in G$. In other words, G is abelian iff $G' = \{1\}$. The quotient $G^{ab} := G/G'$ is called the *abelianization* of G.

EXAMPLE 3. Suppose that $\phi : G \to A$ is a homomorphism to an abelian group A. Then ϕ *factors through* the abelianization, i.e., there exists a homomorphism

$$\bar{\phi} : G^{ab} \to A$$

so that $\phi = \bar{\phi} \circ p$, where $p : G \to G^{ab}$ is the canonical projection.

2. Nilpotent and solvable groups

A group G is called *polycyclic* if there is a finite series

$$G = H_0 \rhd H_1 \rhd ... \rhd H_n = \{1\}$$

where H_i is a normal subgroup in H_{i-1} and H_{i-1}/H_i is cyclic for each $i = 1, ..., n$.

The *derived series* of G is defined as

$$G = G^0 \supset [G, G] = G^1 \supset [G^1, G^1] = G^2 \supset [G^2, G^2] = G^3 ...$$

Thus, G^i/G^{i-1} is abelian for each i.

A group G is called *solvable* if the derived series terminates at the identity subgroup of G after finitely many steps. Equivalently, G is solvable if it admits a finite series

$$G = H_0 \rhd H_1 \rhd ... \rhd H_n = \{1\}$$

where H_i is a normal subgroup in H_{i-1} and H_{i-1}/H_i is abelian for each $i = 1, ..., n$.

In particular, every polycyclic group is solvable. The converse is, in general, false. For instance, the Baumslag–Solitar group

$$BS(2, 1) = \langle a, b | aba^{-1} = b^2 \rangle$$

is solvable but not polycyclic: The difference between the two concepts is that for solvable groups we do not require the quotients H_{i-1}/H_i to be finitely generated. To see why $G = BS(2, 1)$ fails to be polycyclic one observes that subgroup $G' = \langle\langle b \rangle\rangle$ is not finitely generated: It is isomorphic to the group of diadic rational numbers.

A group G is called *nilpotent* if it admits a (finite) central series, which is a sequence of normal subgroups of G

$$\{1\} = \Gamma_0 \subset ...\Gamma_i \lhd \Gamma_{i+1} \lhd ...\Gamma_{n-1} \lhd \Gamma_n = G,$$

so that $\Gamma_{i+1}/\Gamma_i \subset Z(G/\Gamma_i)$, or, equivalently, $[G, \Gamma_{i+1}] \subset \Gamma_i$. Thus, the quotients G_{i+1}/G_i are abelian for each i.

The *lower central series* of a group G is:

$$G = G_0 \supset [G, G] = G_1 \supset [G, G_1] = G_2 \supset [G, G_2] = G_3 ... \tag{4}$$

and the upper central series is

$$1 \subset Z_1(G) \subset Z_2(G) \subset Z_3(G) \subset ... \tag{5}$$

where $Z_1(G)$ is the center of G and $Z_{i+1}(G)$ is the largest subgroup of G so that

$$[G, Z_{i+1}(G)] \subset Z_i(G).$$

If G is nilpotent then lower central series terminates at the identity subgroup and its upper central series terminates at the group $Z_n(G) = G$. A group G is called s-step nilpotent if its lower central series is

$$G_0 \supset G_1 \supset ...G_{s-1} \supset \{1\},$$

where $G_{s-1} \neq \{1\}$. The number s is called the *nilpotency class* of G. Equivalently, the nilpotency class is the number of nontrivial (i.e. different from $\{1\}$) terms in the upper central series; it also equals the least number of nontrivial terms in a central series of G.

LEMMA 1.1. *(R. Baer,* **[2]**, *see also* **[52**, Theorem 10.2.4]*.) If G is a finitely generated nilpotent group then each group G_i in the (lower) central series (4) of G is finitely generated.*

Proof. Suppose that G is 2-step nilpotent, $S = \{x_1, ..., x_N\}$ is a generating set of G. The reader will verify that $[x_i x_j, x_k] = [x_i, x_k][x_j x_k]$ for any $1 \le i, j, k \le N$. Thus the commutators $[x_i, x_j]$, $1 \le i, j \le N$, generate the subgroup $G_2 = [G, G]$ of G. Similarly, if G is n-step nilpotent, for each k the subgroup G_{k+1}/G_{k+2} is generated by the k-fold commutators:

$$[....[[x_{i_1}, x_{i_2}], x_{i_3}], ..., x_{k+1}], \text{ where } x_j \in S.$$

This implies that G_n is finitely generated, hence by the reverse induction each G_i is finitely generated as well, $1 \le i \le n$. □

Corollary 1.2. *If G is a finitely-generated nilpotent group then each quotient in the lower central series G_i/G_{i+1} is a finitely–generated abelian group. In particular, G is polycyclic.*

Lemma 1.3. *If G is s-step nilpotent then every subgroup $H \subset G$ is also s-step nilpotent.*

Proof. Consider the lower central series

$$G_0 \supset G_1 \supset ... \supset G_{s-1} \supset \{1\}$$

$$H_0 \supset H_1 \supset ... \supset H_{s-1} \supset H_s...$$

of the groups G and H. Then for each i,

$$H_i \subset G_i.$$

It follows that $H_s = \{1\}$. □

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Lemma 1.4. *(A. Malcev, [**71**].) If G is a nilpotent group with the lower central series (4) such that G has torsion-free center, then:*

(a) Each quotient $Z_{i+1}(G)/Z_i(G)$ is torsion-free.

(b) G is torsion-free.

Proof. REDO (a) We argue by induction. It clearly suffices to prove the assertion for the quotient $Z_{n-1}(G)/Z_n(G)$. We will show that for each nontrivial element $\bar{x} \in Z_{n-1}(G)/Z_n(G)$ there exists a homomorphism $\varphi \in Hom(Z_{n-1}(G)/Z_n(G), Z_n(G))$ such that $\varphi(\bar{x}) \neq 1$. Since $Z_n(G)$ is torsion-free this would imply that $Z_{n-1}(G)/Z_n(G)$ is torsion-free. Let $x \in Z_{n-1}(G)$ be the element which projects to $x \in Z_{n-1}(G)/Z_n(G)$. Thus $x \notin Z_n(G)$, therefore there exists an element $g \in G$ such that $[g, x] \in Z_n(G) - \{1\}$. Define the map $\tilde{\varphi} : Z_{n-1}(G) \to Z_n(G)$ by

$$\tilde{\varphi}(y) := [y, g].$$

Obviously $\tilde{\varphi}(x) \neq 1$ and, since $Z_n(G)$ is the center of G, the map $\tilde{\varphi}$ descends to a map $Z_{n-1}(G)/Z_n(G) \to Z_n(G)$. We leave it to the reader to verify that $\tilde{\varphi}$ is a homomorphism.

(b) In view of (a), for each i, $m \ge 0$ and each $x \in Z_i(G) - Z_{i+1}(G)$ we have: $x^m \notin Z_{i+1}(G)$. Thus $x^m \neq 1$. By induction it follows that G is torsion-free. □

3. Automorphisms of nilpotent groups

The material of this section will be needed only in Section 1 and the reader can skip it until getting to that section.

Lemma 1.5. *Let $M \in GL(n, \mathbb{Z})$ be a matrix, such that each eigenvalue of M has the absolute value 1. Then all eigenvalues of M are roots of unity.*

Proof. One can derive this lemma from [**11**, Theorem 2, p. 105] as follows. Take the number field $K \subset \mathbb{C}$ defined by the characteristic polynomial p_M of M. Then each root of p_M belongs to the ring of integers $\mathfrak{O} \subset K$. Since $det(M) = \pm 1$, it follows that each root of p_M is a unit in $\mathfrak{O}$. Then the assertion immediately follows from [**11**, Theorem 2, p. 105]. Another proof along the same lines will be given in Section 2 using discreteness of the embedding of F into its ring of adeles.

Below we will give an elementary proof, taken from [**91**].

Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of M listed with multiplicity. Then

$$tr(M^k) = \sum_{i=1}^{n} \lambda_i^k.$$

Since $M \in GL(n, \mathbb{Z})$, the above sums are integers for each $k \in \mathbb{Z}$. Consider the following elements

$$v_k := (\lambda_1^k, ..., \lambda_k^n) \in (S^1)^n$$

of the compact group $G = (S^1)^n$. Since the sequence (v_k) contains a convergent subsequence v_{k_l}, we obtain

$$\lim_{l \to \infty} v_{k_{l+1}} v_{k_l}^{-1} = 1 \in G.$$

Setting $m_l := k_{l+1} - k_l > 0$, we get

$$\lim_{l \to \infty} \lambda_i^{m_l} = 1, i = 1, ..., n.$$

Thus the sequence

$$s_l = \sum_{i=1}^{n} \lambda_i^{m_l}$$

converges to n; since $tr(M^{m_l})$ is an integer, it follows s_l is constant (and, hence, equals n) for all sufficiently large l. Therefore, for sufficiently large l,

$$\sum_{i=1}^{n} Re(\lambda_i^{m_l}) = n. \tag{6}$$

Since $|\lambda_i| = 1$,

$$Re(\lambda_i^{m_l}) \leq 1, \quad \forall l, i.$$

The equality (6) implies that $Re(\lambda_i^{m_l}) = 1$ for all i. Thus $\lambda_i^{m_l} = 1$ for all i and all sufficiently large l; hence all eigenvalues of M are roots of unity. $\square$

For the rest of this section, let ψ be an automorphism of a finitely-generated nilpotent group G. Then ϕ preserves the lower central series and hence induces automorphisms ψ_i of the abelian groups $A_i = G_p/G_{i+1}$. Clearly, each ψ_i preserves the maximal finite subgroup $A_i^f \subset A_i$ for each i. Hence ψ_i induces an automorphism $\bar{\psi}_i$ of the finitely generated free abelian group

$$\bar{A}_i := A_i/A_i^f.$$

Thus $\bar{\psi}_i$ is given by a matrix $M_i = M_i(\psi) \in GL(m_i, \mathbb{Z})$, where m_i is the rank of $\bar{A}_i$. After taking a sufficiently high power of ψ we can assume that none of these matrices has a root of unity (different from 1) as an eigenvalue and that $\psi_i|A_i^f$ is the identity.

Proposition 1.6. *Suppose in addition that each M_i has all its eigenvalues equal to* 1. *Then the semidirect product*

$$\Gamma := G \ltimes_\psi \mathbb{Z}$$

is nilpotent.

Proof. Consider a matrix $M \in GL(m, \mathbb{Z})$ which has only 1 as its eigenvalue. This matrix induces an automorphism $\phi \in Aut(\mathbb{Z}^m)$. Then M has a primitive integer vector v as its eigenvector. The automorphism ϕ induces an automorphism of the free abelian group $\mathbb{Z}^m/\mathbb{Z}v$. It is clear, by looking at the Jordan normal form of M, that the matrix of this automorphism again has only 1 as its eigenvalue. Continuing inductively, we obtain a filtration of free abelian groups

$$\mathbb{Z}^m \supset \mathbb{Z}^{m-1} \supset ... \supset \{1\}$$

which is invariant under ϕ and so that the action of ϕ on each cyclic group $\mathbb{Z}^i/\mathbb{Z}^{i-1}$ is trivial. Using this observation we *refine* the lower central series of G to a central series

$$G = H_0 \supset H_1 \supset ... \supset H_n = \{1\}$$

so that each quotient H_i/H_{i+1} is cyclic (possibly finite), ψ preserves the above series and induces the trivial automorphism of H_i/H_{i+1} for each i. Then H_i is a normal subgroup in Γ for each i. Let t

denote the generator of $\mathbb{Z}$ (it acts on G via the automorphism ψ). Then, for each $h \in H_i$ and $k \in \mathbb{Z}$ we have $[t^k, h] \in H_{i+1}$. Therefore for each $xt^m \in \Gamma$ and $h_i \in H_i$ we have

$$[h_i, xt^m] = hxh^{-1}h_{i+1}x^{-1} = hxh^{-1}x^{-1}g_{i+1} = [h, x]g_{i+1} \in H_{i+1}$$

for some $h_{i+1}, g_{i+1} \in H_{i+1}$. Thus

$$[\Gamma, H_i] \subset H_{i+1}.$$

It follows that Γ is nilpotent. $\square$

REMARK 1.7. Mark Sapir has pointed out that the above proposition was first proven by B. Plotkin.

LEMMA 1.8. *Let A be a finitely generated free abelian group and $\alpha \in Aut(A)$. Then:*
If α has an eigenvalue ρ such that $|\rho| \geq 2$ then there exists $a \in A$ such that

$$\epsilon_0 a + \epsilon_1 \alpha(a) + ... + \epsilon_m \alpha^m(a) + ... \in A$$

(where $\epsilon_i \in \{0, 1\}$ and $\epsilon_i = 0$ for all but finitely many i's) are distinct for different choices of the sequences (ϵ_i).

PROOF. The transpose matrix α^T also has ρ as its eigenvalue. Hence, there exists a nonzero linear function $\beta : A \to \mathbb{C}$ such that $\beta \circ \alpha = \rho\beta$. Pick any $a \in A \setminus Ker(\beta)$. Then,

$$\beta(\sum_{i=0}^{\infty} \epsilon_i \alpha^i(a)) = (\sum_{i=0}^{\infty} \epsilon_i \rho^i)\beta(a).$$

Suppose that

$$\sum_{i=0}^{\infty} \epsilon_i \alpha^i(a) = \sum_{i=0}^{\infty} \delta_i \alpha^i(a).$$

Then, for some $\eta_i \in \{0, \pm 1\}$,

$$\sum_{i=0}^{\infty} \eta_i \alpha^i(a) = 0,$$

and, hence,

$$\sum_{i=0}^{\infty} \eta_i \rho^i \beta(a) = 0.$$

Let N be the maximal value of i for which $\eta_i \neq 0$. Then, since $\beta(a) \neq 0$,

$$\rho^N = \pm \sum_{i=0}^{N-1} \eta_i \rho^i.$$

Since $|\eta_i| \leq 1$ for each i,

$$|\rho|^N \leq \sum_{i=0}^{N-1} |\rho|^i = \frac{|\rho|^N - 1}{|\rho| - 1} \leq |\rho|^N - 1.$$

Contradiction. $\square$

COROLLARY 1.9. *Suppose that G is a finitely-generated nilpotent group and $\psi \in Aut(G)$. Then one of the following holds:*

(a) Either

$$\Gamma := G \rtimes_\psi \mathbb{Z}$$

is virtually nilpotent,

(b) Or, for some i there exists $N \in \mathbb{Z} \setminus \{0\}$ so that for the automorphism $\phi := \psi^N$ of G induces an automorphism of $\bar{A}_i$ satisfying the conclusion of Lemma 1.8.

Proof. Case (a). Suppose that for each i the matrix $M_i = M_i(\psi)$ has only roots of unity as its eigenvalues. Then there exists $N \in \mathbb{N}$ for which all eigenvalues of the matrices $M_i(\psi^N)$ are equal to 1. Thus, by Proposition 1.6, the semidirect product

$$\Gamma' := G \rtimes_{\psi^N} \mathbb{Z}$$

is nilpotent. It is clear that Γ' is a finite index subgroup in Γ. Thus Γ is virtually nilpotent.

Case (b). Suppose that for some i the matrix $M_i = M_i(\psi)$ has an eigenvalue which is not a root of unity. Therefore, according to Lemma 1.5, there exists an eigenvalue λ of M_i whose absolute value is different from 1. Since $\det(M_i) = \pm 1$, it follows that we can choose λ so that $|\lambda| > 1$. By taking N so that

$$\rho := N|\lambda| \geq 2,$$

and applying Lemma 1.8, we conclude that the assertion (b) holds. □

4. Jordan's theorem

In this section we sketch a proof of the following theorem due to C. Jordan, for the details we refer the reader to [**88**, Theorem 8.29].

Theorem 1.10. *(C. Jordan) Let L be a Lie group with finitely many connected components. Then there exists a number $q = q(L)$ such that each finite subgroup F in L contains an abelian subgroup of index $\leq q$.*

Proof. Recall that each connected Lie group H acts on itself smoothly via conjugation

$$g : h \mapsto ghg^{-1}.$$

This action fixes $e \in H$, therefore we consider the derivatives $d_e(g) : T_eH \to T_eH$. We obtain a linear action of H on the vector space T_eH (the Lie algebra of H) called *adjoint representation Ad*. The kernel of this representation is contained in the center $Z(H)$ of H. EXPLAIN. Therefore, each connected Lie group embeds, modulo its center, to the group of real matrices. Therefore, without loss of generality, we will be assuming that $L = GL_n(\mathbb{R})$.

Given a subset $\Omega \subset L$ define inductively subsets $\Omega^{(i)}$ as $\Omega^{(i+1)} = [\Omega, \Omega^{(i)}]$, $\Omega^{(0)} := \Omega$.

Lemma 1.11. *There is a neighborhood Ω of $1 \in L$ such that*

$$\lim_{i\to\infty} \Omega^{(i)} = \{1\}.$$

Proof. Let $A, B \in L$ be near the identity; then $A = \exp(\alpha), B = \exp(\beta)$ for some α, β in the Lie algebra of L. Therefore

$$[A,B] = [1 + \alpha + \frac{1}{2}\alpha^2 + ..., 1 + \beta + \frac{1}{2}\beta^2 + ...] =$$

$$(1 + \alpha + \frac{1}{2}\alpha^2 + ...)(1 + \beta + \frac{1}{2}\beta^2 + ...)(1 - \alpha + \frac{1}{2}\alpha^2 - ...)(1 - \beta + \frac{1}{2}\beta^2 - ...)$$

By opening the brackets we see that the linear term in the commutator $[A, B]$ is zero and each term in the resulting infinite series involves both nonzero powers of α and of β. Therefore

$$\|1 - [A, B]\| \leq C\|1 - A\| \cdot \|1 - B\|.$$

Therefore, by induction, if $B_{i+1} := [A, B_i]$, $B_1 = B$, then

$$\|1 - [A, B_i]\| \leq C^i\|1 - A\|^i \cdot \|1 - B\|.$$

By taking Ω be such that $\|1 - A\| < C$ for all $A \in \Omega$, we conclude that

$$\lim_{i\to\infty} \|1 - B_i\| = 0. \quad \square$$

Lemma 1.12 (Zassenhaus lemma). *Let $\Gamma \subset L$ be a discrete subgroup. Then the set $\Gamma \cap \Omega$ generates a nilpotent subgroup.*

Proof. There exists a neighborhood V of 1 in L such that $V \cap \Gamma = \{1\}$; it follows from the above lemma that all the iterated commutators of the elements of $\Gamma \cap \Omega$ converge to 1. It thus follows that the iterated m-fold commutators of the elements in $\Gamma \cap \Omega$ are trivial for all sufficiently large m. Therefore the set $\Gamma \cap \Omega$ generates a nilpotent subgroup in Γ. □

The finite subgroup $F \subset L$ is clearly discrete, therefore the subgroup $\langle F \cap \Omega \rangle$ is nilpotent. Then $\log(F \cap \Omega)$ generates a nilpotent subalgebra in the Lie algebra of L. Since F is finite, it is also compact, hence, up to conjugation, it is contained in the maximal compact subgroup $K = O(n) \subset GL(n, \mathbb{R}) = L$. The only nilpotent Lie subalgebras of K are abelian subalgebras, therefore the subgroup F' generated by $F \cap \Omega$ is abelian. It remains to estimate the index. Let $U \subset \Omega$ be a neighborhood of 1 in K such that $U \cdot U^{-1} \subset \Omega$ (i.e. products of pairs of elements xy^{-1}, $x, y \in U$, belong to Ω). Let q denote $Vol(K)/Vol(U)$, where Vol is induced by the biinvariant Riemannian metric on K.

Lemma 1.13. *$|F : F'| \le q$.*

Proof. Let $x_1, ..., x_{q+1} \in F$. Then

$$\sum_{i=1}^{q+1} Vol(x_i U) = (q+1) Vol(U) > Vol(K).$$

Hence there are $i \neq j$ such that $x_i U \cap x_j U \neq \emptyset$. Thus $x_j^{-1} x_i \in UU^{-1} \subset \Omega$. Hence $x_j^{-1} x_i \in F'$. □

This also proves Jordan's theorem. □

5. Limits of virtually solvable subgroups of $GL(n, \mathbb{C})$

Throughout this section we set $V := \mathbb{C}^n$ and $G := GL(n, \mathbb{C}) = GL(V)$.

Lemma 1.14. *If $n \geq 2$ then each abelian subgroup $A \subset G$ has a proper invariant subspace in V. Moreover, A has a fixed point in $P(V)$.*

Proof. Let $a \in A$ be an infinite order element which is not of the form $\lambda \cdot I$, $\lambda \in \mathbb{C}^*$. Consider the Jordan normal form of a: It is a direct sum decomposition

$$V = \oplus_j V_j$$

into a-invariant subspaces so that for each j either:

1) $a\big|_{V_j} = \lambda_j \cdot I$, $\lambda_j \in \mathbb{C}^*$, or

2) $dim V_j \geq 2$ and $a\big|_{V_j} = \mu_j \cdot b$, where $\mu_j \in \mathbb{C}^*$ and b has a unique fixed vector in V_j.

Here we are assuming that $\lambda_j \neq \lambda_k$, $j \neq k$.

Each $a' \in A$ commutes with a; hence, it has to preserve the above decomposition of V; the transformation a' could permute the factors V_j, V_k of V which have the same non-diagonal normal form and $\mu_j = \mu_k$.

Therefore, if one of the V_j has the type given in (1), then we are done since it has to be invariant under the entire A. Otherwise, we have only factors of the non-diagonal type. Let $a = dim(V_1)$; by renumbering the factors we can assume that $V_1, ..., V_s$ are all the d-dimensional factors in V so that $\mu_1 = ... = \mu_s$. Then, clearly, the diagonal subspace in $\oplus_{j=1}^s V_s$ is invariant under all the elements of A.

Finally, if each element of A is a multiple of the identity operator, then each 1-dimensional subspace in V is A-invariant.

The proof of the existence of a fixed point in $P(V)$ follows immediately from Part 1 and the dimension induction argument. □

Lemma 1.15. *Let $S < G$ be a solvable subgroup. Then, if $n \geq 2$, the group S has a proper invariant subspace and, hence, a fixed point in $P(V)$.*

Proof. The proof is by induction on dimension and the length of S.

1. If S is abelian, both assertions follow from the previous lemma.

2. Suppose the assertion holds for all m-step solvable groups and all solvable groups acting on vector spaces of dimension $< n$. Then $S' = [S, S]$ has a fixed point in $P(V)$. Since S' is normal in S, its fixed-point set is S-invariant. If the fixed-point set of S' is a proper subspace in $P(V)$, then we are done by the dimension induction. Suppose that S' acts on V by dilations, i.e., it fixes $P(V)$ pointwise. Although the sequence

$$1 \to S' \to S \to A \to 1$$

need not split, nevertheless, A lifts to an abelian subgroup A' of $GL(V)$. By Lemma 1.14, the group A' has a proper invariant subspace W in V. Clearly, W is also S'-invariant, so it is invariant under S as well. The assertion about the fixed point follows from the dimension induction as in the proof of Lemma 1.14. □

Lemma 1.16. *There exists a number $N = N(n)$ so that the following holds. Suppose that $H < G$ is a virtually solvable group. Then H contains a normal subgroup of index $\leq N$ which has a fixed point in $P(V)$.*

Proof. We prove lemma by the induction on dimension. Let $S \triangleleft H$ be a normal subgroup of finite index. Then the fixed-point set F of $S \curvearrowright P(V)$ is nonempty by Lemma 1.15. Clearly, F is H-invariant. If $F \neq P(V)$, then the entire group H has a proper invariant subspace in V and we are done by the dimension induction. Otherwise, S acts on $P(V)$ trivially and we have the induced action $Q = H/S \curvearrowright P(V)$. The image of Q in $PGL(V)$ is finite and, hence, by Jordan's theorem, it contains a normal abelian subgroup A of index $\leq M = M(n)$. As in the proof of Lemma 1.15, we conclude that A has a fixed point in $P(V)$. The preimage of A under the quotient map $H \to Q$ has index $\leq M(n)$ and a fixed point in $P(V)$. □

We can now prove

Proposition 1.17. *Let $\Gamma \subset G$ be a finitely-generated irreducible subgroup which is not virtually abelian. Then there exists a neighborhood Ξ of id in $Hom(\Gamma, G)$ so that every $\rho \in \Xi$ has image which is not virtually solvable.*

Proof. Suppose to the contrary that there exists a sequence $\rho_j \in Hom(\Gamma, G)$ converging to id, so that each $\rho_j(\Gamma)$ is virtually solvable. Let $\Gamma_0 \triangleleft \Gamma$ be a normal subgroup of index $\leq N(n)$ so that each $\rho_j(\Gamma_0)$ has a fixed point in $P(V)$. By passing to a subsequence, we can assume that these fixed points converge to a fixed point of Γ_0. The fixed-point set of Γ_0 in $P(V)$ is Γ-invariant. If this is a proper subspace of $P(V)$, we obtain a contradiction with irreducibility of the action $\Gamma \curvearrowright V$. Otherwise, Γ_0 is abelian and we obtain a contradiction with the assumption that Γ is not virtually abelian. □

Although the above proposition will suffice for the proof of the Tits' alternative, we will prove a slightly stronger assertion:

Theorem 1.18. *Let $\Gamma \subset G$ be a finitely-generated subgroup which is not virtually solvable. Then there exists a neighborhood Σ of id in $Hom(\Gamma, G)$ so that every $\rho \in \Sigma$ has image which is not virtually solvable.*

Proof. First, we generalize lemma 1.16:

Lemma 1.19. *There exists a number $s = s(n)$ so that the following holds. Suppose that $H < G$ is a virtually solvable group. Then H contains a solvable subgroup of index $\leq s$.*

Proof. By Lemma 1.16, there exists a subgroup $H_0 \subset G$ of index $\leq N(n)$, which has an invariant line $L \subset V$. Thus, the group H_1 acts on the quotient space $W := V/L$. The group H_0 is, of course, still virtually solvable. Hence, by the dimension induction, it contains a solvable subgroup S of index $\leq N(n-1)$. The kernel A of the homomorphism $\phi : H_0 \to GL(W)$ is abelian (as a subgroup of $\mathbb{C}^*$). Hence, the group $H_1 := \phi^{-1}(S)$ is solvable and has index $\leq N(n-1)$ in H_0. Hence, we can take

$$s := N(2) \ldots N(n). \quad \square$$

We next bound the lengths of solvable subgroups of $GL(V)$:

LEMMA 1.20. *Each solvable subgroup $H < G$ has length $\leq n$.*

PROOF. By Lemma 1.15, H has an invariant line $L \subset V$. The image of H in $GL(W)$, $W = V/L$ is again solvable. Applying this argument inductively, we conclude that H has a complete invariant flag in V:

$$0 = V_0 \subset V_1 \subset ... \subset V_{n-1} \subset V,$$

where $dim(V_k) = k$, $k = 1, ..., n-1$. Clearly, $H' = [H, H]$ acts trivially on the 1-dimensional space V/V_{n-1}. Hence, H' embeds in $GL(V_{n-1})$. Now, the assertion follows from the dimension induction. □

We are now ready to prove Theorem 1.18. Suppose to the contrary that there exists a sequence $\rho_j \in Hom(\Gamma, G)$ converging to id, so that each $\rho_j(\Gamma)$ is virtually solvable. Let $\Gamma_0 \lhd \Gamma$ be a normal subgroup of index $\leq s(n)$ so that each $\Gamma_j := \rho_j(\Gamma_0)$ is solvable. Thus, each Γ_j satisfies the law:

$$[g_1, ..., g_n] := [...[[g_1, g_2], g_3]...] = 1$$

for the n-fold iterated commutator of every n elements $g_1, ..., g_n \in \Gamma_j$. Therefore, Γ_0 satisfies the same law, since

$$[\gamma_1, ..., \gamma_n] = \lim_{j\to\infty} [\rho_j(\gamma_1), ..., \rho_j(\gamma_n)] = 1, \quad \forall \gamma_1, ..., \gamma_n \in \Gamma.$$

However, it is immediate that the above law is equivalent to the property that Γ_0 is solvable of length $\leq n$. Contradiction. □

6. Virtual isomorphisms

We say that groups G_1 and G_2 are *virtually isomorphic* (abbreviated as VI) if there exist finite index subgroups $H_i \subset G_i$ and finite normal subgroups $F_i \lhd H_i$, $i = 1, 2$, so that the quotients H_1/F_1 and H_2/F_2 are isomorphic.

PROPOSITION 1.21. *Virtual isomorphism (VI) is an equivalence relation between groups.*

PROOF. The only non-obvious property is transitivity. We need

LEMMA 1.22. *Let F_1, F_2 be normal finite subgroups of a group G. Then their normal closure $F = \langle\langle F_1, F_2 \rangle\rangle$ (i.e., the smallest normal subgroup of G containing F_1 and F_2) is again finite.*

PROOF. Let $f_1 : G \to G_1 = G/F_1$, $f_2 : G_1 \to G_1/f_1(F_2)$ be the quotient maps. Since the kernel of each f_1, f_2 is finite, it follows that the kernel of $f = f_2 \circ f_1$ is finite as well. On the other hand, the kernel of f is clearly the subgroup F. □

Suppose now that G_1 is VI to G_2 and G_2 is VI to G_3. Then we have

$$F_1 \lhd H_1 < G_1, |G_1 : H_1| < \infty, |F_1| < \infty,$$

$$F_2 \lhd H_2 < G_2, |G_2 : H_2| < \infty, |F_2| < \infty,$$

$$F_2' \lhd H_2' < G_2, |G_2 : H_2'| < \infty, |F_2'| < \infty,$$

$$F_3 \lhd H_3 < G_3, |G_3 : H_3| < \infty, |F_3| < \infty,$$

so that

$$H_1/F_1 \cong H_2/F_2, \quad H_2'/F_2' \cong H_3/F_3.$$

The subgroup $H_2'' := H_2 \cap H_2'$ has finite index in G_2. By the above lemma, the normal closure in H_2''

$$K_2 := \langle\langle F_2 \cap H_2'', F_2' \cap H_2'' \rangle\rangle$$

is finite. We have quotient maps

$$f_i : H_2'' \to C_i = f_i(H_2'') \subset H_i/F_i, i = 1, 3,$$

with finite kernels and cokernels. The subgroups $E_i := f_i(K_2)$, are finite and normal in C_i, $i = 1, 3$. We let $H'_i, F'_i \subset H_i$ denote the preimages of C_i and E_i under the quotient maps $H_i \to H_i/F_i$, $i = 1, 3$. Then $|F'_i| < \infty, |G_i : H'_i| < \infty, i = 1, 3$. Lastly,

$$H'_i/F'_i \cong C_i/E_i \cong H''_2/K_2, i = 1, 3.$$

Therefore, G_1, G_3 are virtually isomorphic. $\square$

7. Free groups and group presentations

For a set X the *free group* F_X with the generating set X, is defined as follows. Define X^{-1} to be the set consisting of the symbols $a^{-1}, a \in X$. We will think of $X \sqcup X^{-1}$ as an *alphabet*. Consider the set $W = W_X$ of *words* in the alphabet $X \cup X^{-1}$, i.e., the elements of W are expressions of the form

$$a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} ... a_{i_k}^{\epsilon_k}$$

where $a_i \in X, \epsilon_i = \pm 1$. We will use the notation 1 for the *empty word* (the one which has no letters). For instance

$$a_1 a_2 a_1^{-1} a_2 a_2 a_1 \in W.$$

The *length* of a word w is the number of letters in this word.

We define the equivalence relation $\sim$ on W generated by

$$u a_i a_i^{-1} v \sim uv, \quad u a_i^{-1} a_i v \sim uv$$

where $u, v \in W$. We then form the quotient $F = F_X = W/\sim$. We can identify elements of F with the *reduced words*, i.e., the words which contain no occurrences of consecutive letters a_i, a_i^{-1}. For instance

$$1, a_2 a_1, a_1 a_2 a_1^{-1}$$

are reduced, while

$$a_2 a_1 a_1^{-1} a_3$$

is not reduced. Define the product operation on F by the *concatenation* of words:

$$u \cdot v = uv.$$

Define the inverse of a reduced word

$$w = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} ... a_{i_k}^{\epsilon_k}$$

by

$$w^{-1} = a_{i_k}^{-\epsilon_k} a_{i_{k-1}}^{-\epsilon_{k-1}} ... a_{i_1}^{-\epsilon_1}.$$

It is clear that ww^{-1} project to the empty word 1 in F. Therefore F is a group.

DEFINITION 1.23. The group F_X is called the *free group with the generating set* X.

The *rank* of a free group is the cardinality of its generating set.

EXERCISE 7. Show that isomorphic free groups have the same rank. Hint: Use the abelianization of F to reduce the problem to the invariance of dimension for vector spaces.

Most of the time we will be dealing with free groups of finite rank.

The free semigroup F_X^s with the generating set X is defined in the fashion similar to F_X, except that we only allow the words in the alphabet X (and not in X^{-1}).

Presentation of groups. Suppose that G is a group with a generating set X. Define the free group F_X with the generating set X. Then we have a canonical epimorphism

$$\phi : F_X \to G$$

which sends each reduced word

$$w = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} ... a_{i_k}^{\epsilon_k}$$

to the product of the corresponding generators (and their inverses) in G. The kernel of ϕ is a subgroup $K \lhd F_X$. Suppose that K is *normally generated* by a subset $R \subset K$. Then

$$\langle X|R\rangle$$

denotes the *presentation* for the group G. Sometimes we will write presentations in the form

$$\langle x_i, i \in I | r_j = 1, j \in J\rangle$$

where

$$X = \{x_i\}_{i\in I}, \quad R = \{r_j\}_{j\in J}.$$

Conversely, given an alphabet X and a set R of (reduced) words in the alphabet $X \cup X^{-1}$ we can form the quotient

$$G := F_X/\langle\langle R\rangle\rangle.$$

Then $\langle X|R\rangle$ is a presentation of G. By abusing notation, we will often write

$$G = \langle X|R\rangle$$

if G is a group with the presentation $\langle X|R\rangle$. If w is a word in the generating set X, we will use $[w]$ to denote its projection to the group G. An alternative notation for the equality

$$[v] = [w]$$

is

$$v \equiv_G w.$$

A group is called *finitely generated* if it admits a finite generating set. A group is called *finitely presented* if it admits a finite presentation $\langle X|R\rangle$, i.e. both X and R are finite.

Here are some useful examples of finite presentations:

1. *Surface groups*:

$$G = \langle a_1, b_1, ..., a_n, b_n | [a_1, b_1]...[a_n, b_n]\rangle,$$

is the fundamental group of the closed connected oriented surface of genus n, see e.g., **[72]**.

2. *Right–angled Artin groups.* Let $\mathcal{G}$ be a finite graph with the vertex set $V = \{x_1, ..., x_n\}$ and the edge set E consisting of the edges $\{[x_i, x_j]\}_{i,j}$. Define the *right–angled Artin group* by

$$A_{\mathcal{G}} := \langle V | [x_i, x_j], \text{whenever } [x_i, x_j] \in E\}.$$

Here we commit a useful abuse of notation: In the first instance $[x_i, x_j]$ denotes the commutator and in the second instance it denotes the edge of $\mathcal{G}$ connecting x_i to x_j.

Example 1.24. a. If $\mathcal{G}$ contains no edges then $A_{\mathcal{G}}$ is a free group on n generators.

b. If $\mathcal{G}$ is the complete graph on n vertices then

$$A_{\mathcal{G}} \cong \mathbb{Z}^n.$$

3. *Coxeter groups.* Let $\mathcal{G}$ be a finite graph without loops (i.e. edges connecting a vertex to itself) and bigons (two distinct edges connecting a given pair of vertices). Let V and E denote be the vertex and the edge set of $\mathcal{G}$ respectively. Put a label $m(e) \in \mathbb{N} \setminus \{1\}$ on each edge $e = [x_i, x_j]$ of $\mathcal{G}$. Call the pair

$$\Gamma := (\mathcal{G}, m : E \to \mathbb{N} \setminus \{1\})$$

a *Coxeter graph.* Then Γ defines the *Coxeter group* C_Γ:

$$C_\Gamma := \{x_i \in V | (x_i x_j)^{m(e)}, \text{ whenever there exists an edge } e = [x_i, x_j]\rangle.$$

See **[24]** for the detailed discussion of Coxeter groups.

4. *Artin groups.* Let Γ be a Coxeter graph. Define

$$A_\Gamma := \{x_i \in V | \underbrace{x_i x_j \cdots}_{m(e) \text{ terms}} = \underbrace{x_j x_i \cdots}_{m(e) \text{ terms}}, \text{ whenever } e = [x_i, x_j] \in E\rangle.$$

Then A_Γ is a right-angled Artin group iff $m(e) = 2$ for every $e \in E$. In general, C_Γ is the quotient of A_Γ by the subgroup normally generated by the set

$$\{x_i \in V\}.$$

5. *Discrete Heisenberg group:*

$$H := \langle x_1, x_2, t | [x_i, t] = 1, i = 1, 2, [x_1, x_2] = t\rangle$$

EXERCISE 8. Show that H is 2-step nilpotent.

6. *Baumslag–Solitar groups:*

$$BS(p, q) = \langle a, b | ab^p a^{-1} = b^q\rangle.$$

EXERCISE 9. Show that $BS(1, q)$ is solvable but not polycyclic.

Presentations $G = \langle X|R\rangle$ provide a compact form for defining the group G. They were introduced by Max Dehn in the early 20-th century. The main problem of the combinatorial group theory is to get algebraic information about G out of its presentation.

Dehn problems in the combinatorial group theory.

Word Problem. Given a word w in the generating set X and a (finite) presentation $G = \langle X|R\rangle$, determine if w represents the trivial element of G, i.e., if

$$w \in \langle\langle R\rangle\rangle.$$

Conjugacy Problem. Given a pair of words v, w in the generating set X and a (finite) presentation $G = \langle X|R\rangle$, determine if v and w represent conjugate elements of G, i.e., if there exists $g \in G$ so that

$$[w] = g^{-1}[v]g.$$

Triviality Problem. Given a (finite) presentation $G = \langle X|R\rangle$ determine if G is trivial, i.e., equals $\{1\}$.

Isomorphism Problem. Given two (finite) presentations $G_i = \langle X_i|R_i\rangle, i = 1, 2$, determine if G_1 is isomorphic to G_2.

It was discovered in the 1950-s in the work of Novikov, Boone and Rabin **[82, 10, 87]** that all four of Dehn's problems are *algorithmically unsolvable*. For instance, in the case of the word problem, given a finite presentation $G = \langle X|R\rangle$, there is no algorithm which whose input would be a (reduced) word w and the output YES is $w \equiv_G 1$ and NO if not. Fridman **[39]** proved that certain groups have solvable word problem and unsolvable conjugacy problem.

8. Free constructions: Amalgams of groups

Amalgams (amalgamated free products and HNN extensions) allow one to build more complicated groups starting with a given pair of groups or a group and a pair of its subgroups which are isomorphic to each other.

Amalgamated free products. As a warm-up we define the *free product* of groups $G_1 = \langle X_1|R_1\rangle, G_2 = \langle X_2|R_2\rangle$ by the presentation:

$$G_1 * G_2 = \langle G_1, G_2| \quad \rangle$$

which is a shorthand for the presentation:

$$\langle X_1 \sqcup X_2|R_1 \sqcup R_2\rangle.$$

For instance, the free group of rank 2 is isomorphic to $\mathbb{Z} * \mathbb{Z}$.

More generally, suppose that we are given subgroups $H_i \subset G_i$ $(i = 1, 2)$ and an isomorphism

$$\phi : H_1 \to H_2$$

Define the *amalgamated free product*

$$G_1 *_{H_1 \cong H_2} G_2 = \langle G_1, G_2 | \phi(h)h^{-1}, h \in H_1 \rangle.$$

In other words, in addition to the relators in G_1, G_2 we identify $\phi(h)$ with h for each $h \in H_1$. A common shorthand for the amalgamated free product is

$$G_1 *_H G_2$$

where $H \cong H_1 \cong H_2$ (the embeddings of H into G_1 and G_2 are suppressed in this notation).

HNN extensions. This construction is named after Higman, Neumann and Neumann who first introduced it in [**55**]. It is a variation on the amalgamated free product where $G_1 = G_2$. Namely, suppose that we are given a group G, its subgroups H_1, H_2 and an isomorphism $\phi : H_1 \to H_2$. Then the HNN extension of G via ϕ is defined as

$$G\star_{H_1 \cong H_2} = \langle G, t | tht^{-1} = \phi(h), \forall h \in H_1 \rangle.$$

A common shorthand for the HNN extension is

$$G\star_H$$

where $H \cong H_1 \cong H_2$ (the two embeddings of H into G are suppressed in this notation).

Exercise 10. Suppose that H_1 and H_2 are both trivial subgroups. Then

$$G\star_{H_1 \cong H_2} \cong G * \mathbb{Z}.$$

Graphs of groups. This definition is a very useful generalization of both the amalgamated free product and the HNN extension.

Suppose that Γ is a graph. Assign to each vertex v of Γ a *vertex group* G_v; assign to each edge e of Γ an *edge group* G_e. We orient each edge e so it has the *initial and the terminal* (possibly equal) vertices e_- and e_+. Suppose that for each edge e we are given *monomorphisms*

$$\phi_{e_+} : G_e \to G_{e_+}, \phi_{e_-} : G_e \to G_{e_-}.$$

Remark 1.25. More generally, one can allow noninjective homomorphisms

$$G_e \to G_{e_+}, G_e \to G_{e_-},$$

but we will not consider them here.

Call the graph Γ together with the collection of vertex and edge groups and the monomorphisms $\phi_{e_\pm}$ a *graph of groups* $\mathcal{G}$.

Definition 1.26. The *fundamental group* $\pi(\mathcal{G})$ of the above graph of groups is a group G satisfying the following:

1. There is a collection of *compatible homomorphisms* $G_v \to G, G_e \to G, v \in V(\Gamma), e \in E(\Gamma)$, so that whenever $v = e_\pm$, we have the commutative diagram

$$\begin{tikzcd} & G_v \arrow[dr] & \\ G_e \arrow[ur] \arrow[rr] & & G \end{tikzcd}$$

2. The group G is *universal* with respect to the above property, i.e., given any group H and a collection of compatible homomorphisms $G_v \to H, G_e \to H$, there exists a unique homomorphism $G \to H$ so that we have commutative diagrams

$$\begin{tikzcd} & G \arrow[dr] & \\ G_v \arrow[ur] \arrow[rr] & & H \end{tikzcd}$$

for all $v \in V(\Gamma)$.

Note that the above definition easily implies that $\pi(\mathcal{G})$ is unique (up to an isomorphism). However, the existence is less obvious. Whenever $G \cong \pi(\mathcal{G})$, we will say that $\mathcal{G}$ determines a *graph of groups decomposition* of G. The decomposition of G is called *trivial* if there is a vertex v so that the natural homomorphism $G_v \to G$ is onto.

EXAMPLE 1.27. 1. Suppose that the graph Γ is a single edge $e = [1, 2]$, $\phi_{e_-}(G_e) = H_1 \subset G_1$, $\phi_{e_+}(G_e) = H_2 \subset G_2$. Then

$$\pi(\mathcal{G}) \cong G_1 *_{H_1 \cong H_2} G_2.$$

2. Suppose that the graph Γ is a single loop $e = [1, 1]$, $\phi_{e_-}(G_e) = H_1 \subset G_1$, $\phi_{e_+}(G_e) = H_2 \subset G_1$. Then

$$\pi(\mathcal{G}) \cong G_1 *_{H_1 \cong H_2} .$$

Once this example is understood, one can show that $\pi(\mathcal{G})$ exists by describing this group in terms of generators and relators in the manner similar to the definition of the amalgamated free product and HNN extension.

Relation to topology. Suppose that for all vertices and edges $v \in V(\Gamma)$ and $e \in E(\Gamma)$ we are given connected cell complexes M_v, M_e with the fundamental groups G_v, G_e respectively. For each edge $e = [v, w]$ assume that we are given a continuous map $f_{e_\pm} : M_e \to M_{e_\pm}$ which induces the monomorphism $\phi_{e_\pm}$. This collection of spaces and maps is called the *graph of spaces*

$$\mathcal{G}_M := \{M_v, M_e, f_{e_\pm} : M_e \to M_{e_\pm} : v \in V(\Gamma), e \in E(\Gamma)\}.$$

In order to construct $\mathcal{G}_M$ starting from $\mathcal{G}$, recall that each group G admits a cell complex $K(G, 1)$ whose fundamental group is G and whose universal cover is contractible. Given a group homomorphism $\phi : H \to G$, there exists (unique up to homotopy) continuous map

$$f : K(H, 1) \to K(G, 1)$$

which induces the homomorphism ϕ. Then one can take $M_v := K(G_v, 1)$, $M_e := K(G_e, 1)$, etc.

To simplify the picture, the reader can think of each M_v as a manifold with several boundary components which are homeomorphic to $M_{e_1}, M_{e_2}, ...$, where e_j are the edges having v as their *initial* or *final vertex*. Then assume that the maps $f_{e_\pm}$ are homeomorphisms onto the respective boundary components.

For each edge e form the product $M_e \times [0, 1]$ and then, the double mapping cylinders for the maps $f_{e_\pm}$, i.e., identify points of $M_e \times \{0\}$ and $M_e \times \{1\}$ with their images under f_{e_-} and f_{e_+} respectively.

Let M denote the resulting cell complex. It then follows from the Seifert- Van Kampen theorem **[72]** that

THEOREM 1.28. *$\pi_1(M) \cong \pi(\mathcal{G})$.*

This allows one to think of the graphs of groups and their fundamental groups *topologically* rather than *algebraically*. Given the above interpretation, one can easily see that for each vertex $v \in V(\Gamma)$ the canonical homomorphism $G_v \to \pi(\mathcal{G})$ is injective.

Graphs of groups and group actions on trees. A *simplicial tree* T is a simply-connected simplicial complex of dimension 0 or 1. In other words, T is a (nonempty) connected graph (of possibly infinite valency) which contains no embedded circles. An *automorphism* of a tree is an automorphism of T as a simplicial complex, i.e., it is a homeomorphism $f : T \to T$ so that the images of the edges of T are edges of T.

An *action* of a group G on a tree T is an action $G \curvearrowright T$ so that each element of G acts as an automorphism of T, i.e., such action is a homomorphism $G \to Aut(T)$. A tree T with the prescribed action $G \curvearrowright T$ is called a G–tree. An action $G \curvearrowright T$ is said to be *without inversions* if whenever $g \in G$ preserves an edge e of T, it fixes e pointwise. The action is called *trivial* if there is a vertex $v \in T$ fixed by the entire group G.

REMARK 1.29. Later on we will encounter more complicated (non-simplicial) trees and actions.

Our next goal is to explain the relation between the graph of groups decompositions of G and actions of G on simplicial trees without inversions.

Suppose that $G \cong \pi(\mathcal{G})$ is a graph of groups decomposition of G. We associate with $\mathcal{G}$ a graph of spaces $M = M_{\mathcal{G}}$ as above. Let X denote the universal cover of the corresponding cell complex M. Then X is the disjoint union of the copies of the universal covers $\tilde{M}_v, \tilde{M}_e \times (0,1)$ of the complexes M_v and $M_e \times (0,1)$. We will refer to this partitioning of X as the *tiling* of X. In other words, X has the structure of a graph of spaces, where each vertex/edge space is homeomorphic to $\tilde{M}_v, v \in V(\Gamma)$, $\tilde{M}_e \times [0,1], e \in E(\Gamma)$. Let T denote the graph corresponding to X: Each copy of $\tilde{M}_v$ determines a vertex in T and each copy of $\tilde{M}_e \times [0,1]$ determines an edge in T.

EXAMPLE 1.30. Suppose that Γ is a single segment $[1,2]$, M_1 and M_2 are surfaces of genus 1 with a single boundary component each. Let M_e be the circle. We assume that the maps $f_{e_\pm}$ are homeomorphisms of this circle to the boundary circles of M_1, M_2. Then, M is a surface of genus 2. The graph T is sketched in Figure 1.

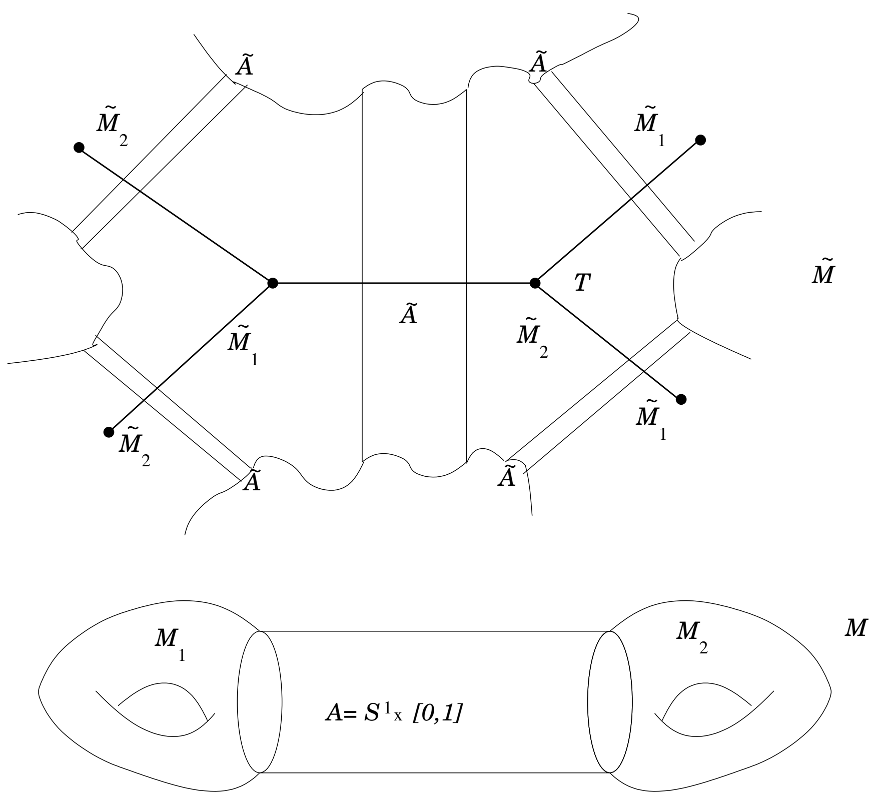

FIGURE 1. *Universal cover of the genus 2 surface.*

The Meyer-Vietoris theorem, applied to the above tiling of X, implies that $0 = H_1(X,\mathbb{Z}) \cong H_1(T,\mathbb{Z})$. Therefore, $T = T(\mathcal{G})$ is a tree. The group $G = \pi_1(M)$ acts on X by deck-transformations, preserving the tiling. Therefore we get the induced action $G \curvearrowright T$. If $g \in G$ preserves some $\tilde{M}_e \times (0,1)$, then it comes from the fundamental group of M_e. Therefore such g also preserves the orientation on the segment $[0,1]$. Hence the action $G \curvearrowright T$ is without inversions. Observe that the stabilizer of each $\tilde{M}_v$ in G is conjugate in G to $\pi_1(M_v) = G_v$. Moreover, $T/G = \Gamma$.

EXAMPLE 1.31. Let $G = BS(p,q)$. Then G clearly, has the structure of a graph of groups since it is isomorphic to the HNN extension of $\mathbb{Z}$,

$$\mathbb{Z}\star_{H_1 \cong H_2}$$

where the subgroups $H_1, H_2 \subset \mathbb{Z}$ have the indices p and q respectively. In order to construct the cell complex $K(G,1)$ take the circle $S^1 = M_v$, the cylinder $S^1 \times [0,1]$ and attach the ends to this cylinder to M_v by the maps of the degree p and q respectively. Now, consider the associated G–tree T. Its vertices have valence $p+q$: Each vertex v has q incoming and p outgoing edges so that for each outgoing edge e we have $v = e_-$ and for each incoming edge we have $v = e_+$. The vertex group $G \cong \mathbb{Z}$ permutes (transitively) incoming and outgoing edges among each other. The stabilizer of each outgoing edge is the subgroup H_1 and the stabilizer of each incoming edge is the subgroup H_2. Thus, the action of $\mathbb{Z}$ on the incoming vertices is via the group $\mathbb{Z}/q$ and on the outgoing vertices via the group $\mathbb{Z}/p$.

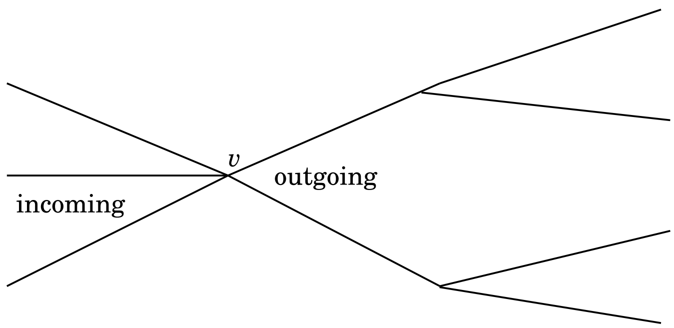

FIGURE 2. *Tree for the group $BS(2,3)$.*

LEMMA 1.32. *$G \curvearrowright T$ is trivial if and only if the graph of groups decomposition of G is trivial.*

PROOF. Suppose that G fixes a vertex $\tilde{v} \in T$. Then $\pi_1(M_v) = G_v = G$, where $v \in \Gamma$ is the projection of $\tilde{v}$. Hence the decomposition of G is trivial. Conversely, suppose that G_v maps onto G. Let $\tilde{v} \in T$ be the vertex which projects to v. Then $\pi_1(M_v)$ is the entire $\pi_1(M)$ and, hence, G preserves $\tilde{M}_{\tilde{v}}$. Therefore, the group G fixes $\tilde{v}$. $\square$

Conversely, each action of G on a simplicial tree T yields a decomposition of G as a graph of groups $\mathcal{G}$, so that $T = T(\mathcal{G})$. We refer the reader to **[94]** and **[95]** for further details.

9. Cayley graphs of finitely generated groups

Let G be a finitely generated group with the finite generating set $S = \{s_1, ..., s_n\}$. We will mostly consider the case when the identity does not belong to S. Define the *Cayley graph* $\Gamma = \Gamma_{G,S}$ as follows: The vertices of Γ are the elements of G. Whenever we have vertices $g, h \in \Gamma$ and a generator $s_i \in S$ such that $h = gs_i$, we insert an edge connecting these vertices and label this edge by s_i. (Thus, there could be several edges connecting g to h, they have different labels.) By abusing notation we will denote this edge $[g,h] = \overline{gh}$. Since S is a generating set of G, it follows that Γ is connected. There are exactly $2n$ edges incident to each vertex of Γ. Define the *word metric* d on Γ by requiring each edge to have unit length. This defines the length for finite piecewise-linear paths in Γ: Concatenation of m edges has length m. Finally, the distance between points $p, q \in \Gamma$ is the infimum (same as minimum) of the lengths of piecewise-linear paths in Γ connecting p to q. For $g \in G$ the *word length* $|g|$ is the distance $d(1,g)$ in Γ. The group G acts on itself (regarded as a set) via the left multiplication

$$g : x \mapsto gx.$$

If $[x, xs_i]$ is an edge of Γ with the vertices x, xs_i, we extend g to the isometry

$$g : [x, xs_i] \to [gx, gxs_i]$$

between the unit intervals. Thus G acts on (Γ, d) isometrically. It is also clear that this action is free, properly discontinuous and cocompact: The quotient Γ/G is homeomorphic to the bouquet of n circles.

Below are two simple examples of Cayley graphs.

EXAMPLE 11. Let G be the rank 2 free Abelian group with the generating set $S = \{s_1, s_2\}$. The Cayley graph $\Gamma = \Gamma_{G,S}$ is the square grid in the Euclidean plane: The vertices are points with integer coordinates, two vertices are connected by an edge if and only if exactly only two of their coordinates are distinct and differ by 1.

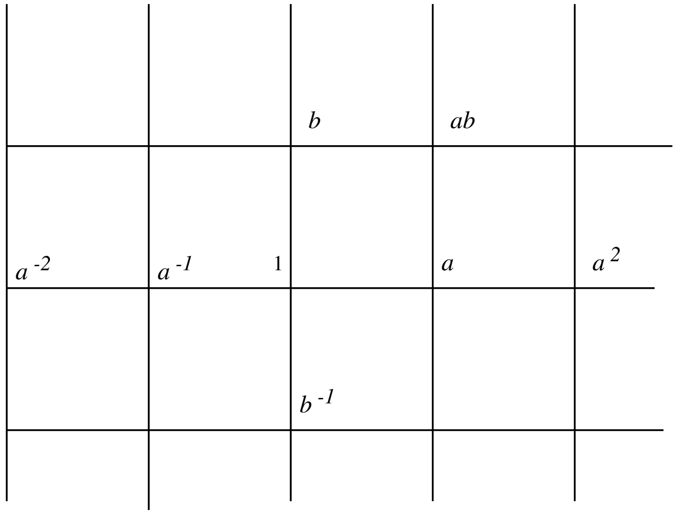

FIGURE 3. *Free abelian group.*

EXAMPLE 12. Let G be the free group on two generators s_1, s_2. Take $S = \{s_i, i = 1, 2\}$. The Cayley graph $\Gamma = \Gamma_{G,S}$ is the 4-valent tree (there are four edges incident to each vertex).

See Figures 3, 4.

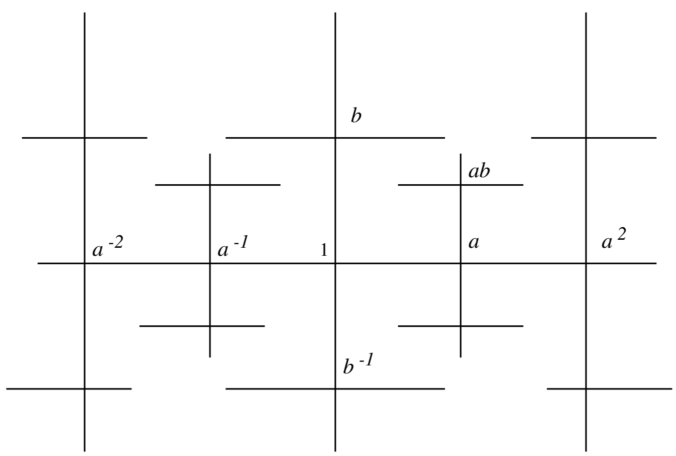

FIGURE 4. *Free group.*

Thus, we succeeded in assigning to every finitely-generated group G as metric space $\Gamma = \Gamma_{G,S}$. The problem, however, is that this assignment $G \to \Gamma$ is far from canonical: Different generating sets could yield completely different Cayley graphs. For instance, the trivial group has the presentations

$$\langle \quad | \quad \rangle, \quad \langle a|a \rangle, \quad \langle a, b|ab, ab^2 \rangle, \ldots,$$

which give rise to the non-isometric Cayley graphs:

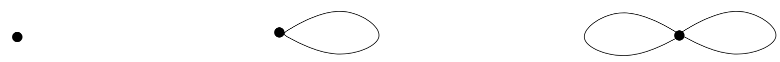

FIGURE 5. *Cayley graphs of the trivial group.*

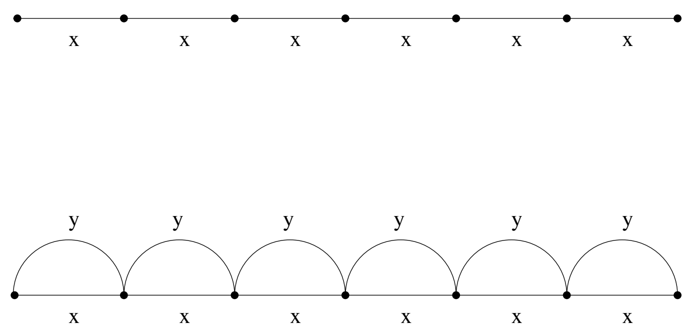

FIGURE 6. *Cayley graphs of $\mathbb{Z} = \langle x | \rangle$ and $\mathbb{Z} = \langle x, y | xy^{-1} \rangle$.*

The same applies to the infinite cyclic group:

Note, however, that all Cayley graphs of the trivial group are finite; the same, of course, applies to all finite groups. The Cayley graphs of $\mathbb{Z}$ as above, although they are clearly non-isometric, are within finite distance from each other (when placed in the same Euclidean plane). Therefore, when seen from a (very) large distance (or by a person with a very poor vision), every Cayley graph of a finite group looks like a "fuzzy dot"; every Cayley graph of $\mathbb{Z}$ looks like a "fuzzy line," etc. Therefore, although non-isometric, they "look alike". On the other hand, it is clear that no matter how poor your vision is, the Cayley graphs of, say, $\{1\}$, $\mathbb{Z}$ and $\mathbb{Z}^2$ all look different: They appear to have different "dimension" (0, 1 and 2 respectively).

Telling apart the Cayley graph Γ_1 of $\mathbb{Z}^2$ from the Cayley graph Γ_2 of the Coxeter group

$$\Delta := \Delta(4,4,4) := \langle a, b, c | a^2, b^2, c^2, (ab)^4, (bc)^4, (ca)^4 \rangle$$

seems more difficult: They both "appear" 2-dimensional. However, by looking at the larger pieces of Γ_1 and Γ_2, the difference becomes more apparent: Within a given ball of radius R in Γ_1, there seems to be less vertices than in Γ_2. The former grows quadratically, the latter grows exponentially fast as R goes to infinity.

The goal of the rest of the book is to make sense of this "fuzzy math".

In the next section we replace the notion of an *isometry* with the notion of a *quasi-isometry*, in order to capture what different Cayley graphs of the same group have in common.

10. Coarse geometry

Let (X, d) be a metric space. We will use the notation $B_R(A)$ to denote the *open R-neighborhood* of a subset $A \subset X$, i.e. $B_R(A) = \{x \in X : d(x, A) < R\}$. In particular, if $A = \{a\}$ then $B_R(A) = B_R(a)$ is the open R-ball centered at a. We will use the notation $\bar{B}_R(A)$, $\bar{B}_R(a)$ to denote the corresponding *closed neighborhoods* and *closed balls* defined by non-strict inequalities.

The *Hausdorff distance* between subsets $A_1, A_2 \subset X$ is defined as

$$d_{Haus}(A_1, A_2) := \inf\{R : A_1 \subset B_R(A_2), A_2 \subset B_R(A_1)\}.$$

Two subsets of X are called *Hausdorff-close* if they are within finite Hausdorff distance from each other.

Given subsets $A_1, A_2 \subset X$, define the *minimal distance* between these sets as

$$d(A_1, A_2) = \inf\{d(a_1, a_2) : a_i \in A_i, i = 1, 2\}.$$

Let $(X, d_X), (Y, d_Y)$ be metric spaces. A map $f : X \to Y$ is called an *isometric embedding* if for all $x, x' \in X$

$$d_Y(f(x), f(x')) = d_X(x, x').$$

A map f is called *an isometry* if its is an isometric embedding and admits an isometric inverse.

Similarly, a map $f : X \to Y$ is called *L-Lipschitz* if

$$d_Y(f(x), f(x')) \le L d_X(x, x'), \quad \forall x, x' \in X. \tag{13}$$

Here L is a certain positive real number. Accordingly, a map is called *L-biLipschitz* if it is L-Lipschitz and admits an L-Lipschitz inverse.

EXAMPLE 1.33. Suppose that X, Y are Riemannian manifolds $(M, g), (N, h)$. Then a smooth map $f : M \to N$ is L-biLipschitz if and only if

$$L^{-1} \le \sqrt{\frac{f^*h}{g}} \le L.$$

In other words, for every tangent vector $v \in TM$,

$$L^{-1} \le \sqrt{\frac{h(df(v))}{g(v)}} \le L,$$

where we think of h and g as quadratic forms on the tangent spaces.

For a Lipschitz function $f : X \to \mathbb{R}$ let $Lip(f)$ denote

$$\inf\{L : f \text{ is } L\text{–Lipschitz}\}.$$

EXAMPLE 14. Suppose that f, g are Lipschitz functions on X. Let $\|f\|, \|g\|$ denote the sup-norms of f and g on X. Show that

1.$Lip(f + g) \le Lip(f) + Lip(g)$.

2. $Lip(fg) \le Lip(f)\|g\| + Lip(g)\|f\|$.

3.

$$Lip(\frac{f}{g}) \le \frac{Lip(f)\|g\| + Lip(g)\|f\|}{\inf_{x \in X} g^2(x)}.$$

Note that in case when f is a smooth function on a Riemannian manifold, these formulae follow from the formulae for the derivatives of the sum, product and ratio of two functions.

A *geodesic* in a metric space X is an isometric embedding γ of an interval in $\mathbb{R}$ into X. Note that this notion is different from the one in Riemannian geometry, where geodesics are isometric embeddings only *locally*.

A metric space X is called *geodesic* if for any pair of points $x, x' \in X$ there exists a geodesic $\gamma : [a, b] \to X$ so that $\gamma(a) = x, \gamma(b) = x'$. For instance, every complete Riemannian manifold (with the distance function determined by the Riemannian metric) is a geodesic metric space. Every Cayley graph is also a geodesic metric space. Of course, if a metric space is not connected then it is not a geodesic metric space. For a more interesting example, consider a circle in $\mathbb{R}^2$ with the induced metric.

The point of the next definition is to loosen up the Lipschitz concept.

DEFINITION 1.34. Let X, Y be metric spaces. A map $f : X \to Y$ is called (L, A)-**coarse Lipschitz** if

$$d_Y(f(x), f(x')) \le L d_X(x, x') + A \tag{15}$$

for all $x, x' \in X$. A map $f : X \to Y$ is called an (L, A)-**quasi-isometric embedding** if

$$L^{-1} d_X(x, x') - A \le d_Y(f(x), f(x')) \le L d_X(x, x') + A \tag{16}$$

for all $x, x' \in X$. Note that a quasi–isometric embedding does not have to be an embedding in the usual sense, however distant points have distinct images.

An (L, A)-quasi-isometric embedding is called an (L, A)-**quasi–isometry** if it admits a **quasi–inverse** map $\bar{f} : Y \to X$ which is also a (L, A)-quasi–isometric embedding so that:

$$d_X(\bar{f}f(x), x) \le A, \quad d_Y(f\bar{f}(y), y) \le A \tag{17}$$

for all $x \in X, y \in Y$.

An (L, A)-quasi-geodesic in X is an (L, A)-quasi–isometric embedding of an interval in $\mathbb{R}$ into X.

We will abbreviate *quasi-isometry*, *quasi–isometric* and *quasi-isometrically* to QI.

In the most cases the *quasi–isometry constants* L, A do not matter, so we shall use the words *quasi–isometries* and *quasi-isometric embeddings* without specifying constants. If X, Y are spaces such that there exists a quasi-isometry $f : X \to Y$ then X and Y are called *quasi-isometric.*

EXERCISE 18. A subset S of a metric space X is said to be r-**dense** in X if the Hausdorff distance between S and X is at most r. Show that if $f : X \to Y$ is a quasi-isometric embedding such that $f(X)$ is r-dense in X for some $r < \infty$ then f is a quasi-isometry. Hint: Construct a quasi-inverse $\bar{f}$ to the map f by mapping a point $y \in Y$ to $x \in X$ such that

$$d_Y(f(x), y) \le r.$$

For instance, the cylinder $X = \mathbb{S}^n \times \mathbb{R}$ is quasi-isometric to $Y = \mathbb{R}$; the quasi-isometry is the projection to the second factor.

EXAMPLE 1.35. Let $h : \mathbb{R} \to \mathbb{R}$ be an L–Lipschitz function. Then the map

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = (x, h(x))$$

is a QI embedding.

Indeed, f is $\sqrt{1 + L^2}$–Lipschitz. On the other hand, clearly,

$$d(x, y) \le d(f(x), f(y))$$

for all $x, y \in \mathbb{R}$.

EXAMPLE 1.36. Let $\varphi : [1, \infty) \to \mathbb{R}_+$ be a differentiable function so that

$$\lim_{r\to\infty} \varphi(r) = \infty,$$

and there exists $C \in \mathbb{R}$ for which $|r\varphi'(r)| \le C$ for all r. For instance, take $\varphi(r) = \log(r)$. Define the function $F : \mathbb{R}^2 \setminus B_1(0) \to \mathbb{R}^2 \setminus B_1(0)$ which in the polar coordinates takes the form

$$(r, \theta) \mapsto (r, \theta + \varphi(r)).$$

Hence F maps radial straight lines to spirals. Let us check that F is L-biLipschitz for $L = \sqrt{1 + C^2}$. Indeed, the Euclidean metric in the polar coordinates takes the form

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Then

$$F^*(ds^2) = ((r\varphi'(r))^2 + 1)dr^2 + r^2 d\theta^2$$

and the assertion follows. Extend F to the unit disk by the zero map. Therefore, $F : \mathbb{R}^2 \to \mathbb{R}^2$, is a QI embedding. Since F is onto, it is a quasi-isometry $\mathbb{R}^2 \to \mathbb{R}^2$.

EXERCISE 19. If $f, g : X \to Y$ are within finite distance from each other, i.e.

$$\sup d(f(x), g(x)) < \infty$$

and f is a quasi-isometry, then g is also a quasi-isometry.

EXERCISE 20. Show that quasi-isometry is an equivalence relation between metric spaces.

A separated net in a metric space X is a subset $Z \subset X$ which is r-dense for some $r < \infty$ and such that there exists $\epsilon > 0$ for which $d(z, z') \ge \epsilon$, $\forall z \ne z' \in Z$.

Alternatively, one can describe quasi-isometric spaces as follows.

Lemma 1.37. *Metric spaces X and Y are quasi-isometric iff there are separated nets $Z \subset X, W \subset Y$, constants L and C, and L-Lipschitz maps*

$$f : Z \to Y, \bar{f} : W \to X,$$

so that $d(\bar{f} \circ f, id) \le C, d(f \circ \bar{f}, id) \le C$.

Proof. Observe that if a map $f : X \to Y$ is coarse Lipschitz then its restriction to each separated net in X is Lipschitz. Conversely, if $f : Z \to Y$ is a Lipschitz map from a separated net in X then f admits a coarse Lipschitz extension to X. □

Definition 1.38. A subset $R \subset X \times Y$ is called an (L, A)-quasi-isometric relation if the following holds:

For $x \in X$ let $R(x)$ denote $\{(x, y) \in X \times Y : (x, y) \in R\}$. Similarly, define $R(y)$ for $y \in Y$. Let π_X, π_Y denote the projections of $X \times Y$ to X and Y respectively.

1. We require each $x \in X$ and each $y \in Y$ be contained within distance $\le A$ from the projection of R to X and Y respectively.

2. We require that for each $x, x' \in \pi_X(R)$

$$d_{Haus}(\pi_Y(R(x)), \pi_Y(R(x'))) \le Ld(x, x') + A.$$

3. Similarly, we require that for each $y, y' \in \pi_Y(R)$

$$d_{Haus}(\pi_X(R(y)), \pi_X(R(y'))) \le Ld(y, y') + A.$$

It then follows that for each pair of points $x, x' \in X$ and $y \in R(x), y' \in R(x')$ we have

$$\frac{1}{L} d(x, x') - \frac{A}{L} \le d(y, y') \le Ld(x, x') + A.$$

The same inequality holds for points $y, y' \in Y$ and $x \in R(y), x' \in R(y')$.

In particular, if R is an (L, A)-quasi-isometric relation between nonempty metric spaces, then it induces an (L_1, A_1)-quasi-isometry $X \to Y$. Conversely, every (L, A)-quasi-isometry is an (L_2, A_2) quasi-isometric relation.

In some cases it suffices to check a weaker version of (17) in order to show that f is a quasi-isometry. We discuss this weaker version below.

Let X, Y be topological spaces. Recall that a (continuous) map $f : X \to Y$ is called *proper* if the inverse image $f^{-1}(K)$ of each compact in Y is a compact in X. A metric space X is called *proper* if each closed and bounded subset of X is compact. Equivalently, the distance function $f : X \to \mathbb{R}_+$, $f(x) = d(x, o)$ is a proper function. (Here $o \in X$ is a base-point.)

Definition 1.39. A map $f : X \to Y$ between proper metric spaces is called *uniformly proper* if f is coarse Lipschitz and there exists a *distortion function* $\psi(R)$ such that $\mathrm{diam}(f^{-1}(B(y, R))) \le \psi(R)$ for each $y \in Y, R \in \mathbb{R}_+$. In other words, there exists a proper function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that whenever $d(x, x') \ge r$, we have $d(f(x), f(x')) \ge \eta(r)$.

To see an example of a map which is proper but not uniformly proper consider the embedding of the bi-infinite curve Γ in $\mathbb{R}^2$ (Figure 7):

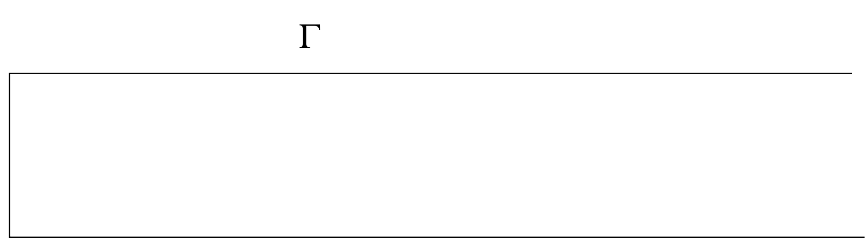

Figure 7.

LEMMA 1.40. *Suppose that Y is a geodesic metric space, $f : X \to Y$ is a uniformly proper map whose image is r-dense in Y for some $r < \infty$. Then f is a quasi-isometry.*

PROOF. Let us construct a quasi-inverse to the map f. Given a point $y \in Y$ pick a point $\bar{f}(y) := x \in X$ such that $d(f(x), y) \le r$. Let's check that $\bar{f}$ is coarse Lipschitz. Since Y is a geodesic metric space it suffices to verify that there is a constant A such that for all $y, y' \in Y$ with $d(y, y') \le 1$, one has:

$$d(\bar{f}(y), \bar{f}(y')) \le A.$$

Pick $t > 1$ which is in the image of the distortion function η. Then take $A \in \eta^{-1}(t)$.

It is also clear that $f, \bar{f}$ are quasi-inverse to each other. □

DEFINITION 1.41. A *geometric action* of a group G on a metric space X is an isometric properly discontinuous cocompact action $G \curvearrowright X$.

The following lemma was proven in the context of Riemannian manifolds first by A. Schwarz [**98**] and, 13 years later, by J. Milnor [**73**]. Both were motivated by relating volumes growth of metric balls in universal covers of compact Riemannian manifolds and growth of their fundamental groups.

REMARK 1.42 (What is in the name?). Schwarz is a German-Jewish name which was translated to Russian (presumably, at some point in the 19-th century) as Шварц. In the 1950-s, the AMS, in its infinite wisdom, decided to translate this name to English as Švarc. A. Schwarz himself eventually moved to the United States and is currently a colleague of the second author at University of California, Davis. See http://www.math.ucdavis.edu/∼schwarz/bion.pdf for his mathematical autobiography. The transformation

Schwarz → Шварц → Švarc

is a good example of a composition of a quasi-isometry and its quasi-inverse.

LEMMA 1.43. *(Milnor–Schwarz lemma.) Let X be a proper geodesic metric space. Let G be a group acting geometrically on X. Pick a point $x_0 \in X$. Then the group G is finitely generated. Moreover, for some choice of a finite generating set S of G, the map $f : G \to X$, given by $f(g) = g(x_0)$, is a quasi-isometry. Here G is given the word metric corresponding to the generating set S.*

PROOF. Our proof follows [**45**, Proposition 10.9]. Let $B = \bar{B}_R(x_0)$ be the closed R-ball of radius in X centered at x_0, so that $B_{R-1}(x_0)$ projects onto X/G. Since the action of G is properly discontinuous, there are only finitely many elements $s_i \in G$ such that $B \cap s_i B \ne \emptyset$. Let S be the subset of G which consists of the above elements s_i (it is clear that s_i^{-1} belongs to S iff s_i does). Let

$$r := \inf\{d(B, g(B)), g \in G \setminus S\}.$$

Since B is compact and $B \cap g(B) = \emptyset$ for $g \notin S$, $r > 0$. We claim that S is a generating set of G and that for each $g \in G$

$$|g| \le d(x_0, g(x_0))/r + 1 \tag{21}$$

where $|\cdot|$ is the word length on G (with respect to the generating set S). Let $g \in G$, connect x_0 to $g(x_0)$ by the shortest geodesic γ. Let m be the smallest integer so that $d(x_0, g(x_0)) \le mr + R$. Choose points $x_1, ..., x_{m+1} = g(x_0) \in \gamma$, so that $x_1 \in B$, $d(x_j, x_{j+1}) < r$, $1 \le j \le m$. Then each x_j belongs to $g_j(B)$ for some $g_j \in G$. Let $1 \le j \le m$, then $g_j^{-1}(x_j) \in B$ and $d(g_j^{-1}(g_{j+1}(B)), B) \le d(g_j^{-1}(x_j), g_j^{-1}(x_{j+1})) < r$. Thus the balls $B, g_j^{-1}(g_{j+1}(B))$ intersect, which means that $g_{j+1} = g_j s_{i(j)}$ for some $s_{i(j)} \in S$. Therefore

$$g = s_{i(1)} s_{i(2)} .... s_{i(m)}.$$

We conclude that S is indeed a generating set for the group G. Moreover,

$$|g| \le m \le (d(x_0, g(x_0)) - R)/r + 1 \le d(x_0, g(x_0))/r + 1.$$

The word metric on the Cayley graph $\Gamma_{G,S}$ of the group G is left-invariant, thus for each $h \in G$ we have:

$$d(h, hg) = d(1, g) \le d(x_0, g(x_0))/r + 1 = d(h(x_0), hg(x_0))/r + 1.$$

Hence for any $g_1, g_2 \in G$

$$d(g_1, g_2) \le d(f(g_1), f(g_2))/r + 1.$$

On the other hand, the triangle inequality implies that

$$d(x_0, g(x_0)) \le t|g|$$

where $d(x_0, s(x_0)) \le t \le 2R$ for all $s \in S$. Thus

$$d(f(g_1), f(g_2))/t \le d(g_1, g_2).$$

We conclude that the map $f : G \to X$ is a quasi-isometric embedding. Since $f(G)$ is R-dense in X, it follows that f is a quasi-isometry. $\square$

COROLLARY 1.44. *Let S_1, S_2 be finite generating sets for a finitely generated group G and d_1, d_2 be the word metrics on G corresponding to S_1, S_2. Then the identity map $(G, d_1) \to (G, d_2)$ is a quasi-isometry.*

PROOF. The group G acts geometrically on the proper metric space

$$(\Gamma_{G,S_2}, d_2).$$

Therefore, the map $id : G \to \Gamma_{G,S_2}$ is a quasi-isometry. $\square$

LEMMA 1.45. *Let (X, d_i), $i = 1, 2$, be proper geodesic metric spaces. Suppose that the action $G \curvearrowright X$ is geometric with respect to both metrics d_1, d_2. Then the identity map $id : (X, d_1) \to (X, d_2)$ is a quasi-isometry.*

PROOF. The group G is finitely generated by Lemma 1.43, choose a word metric d on G corresponding to any finite generating set (according to the previous corollary it does not matter which one). Pick a point $x_0 \in X$; then the maps

$$f_i : (G, d) \to (X, d_i), \quad f_i(g) = g(x_0)$$

are quasi-isometries, let $\bar{f}_i$ denote their quasi-inverses. Then the map $id : (X, d_1) \to (X, d_2)$ is within finite distance from the quasi-isometry $f_2 \circ \bar{f}_1$. $\square$

COROLLARY 1.46. *Let d_1, d_2 be as in Lemma 1.45. Then any geodesic γ with respect to the metric d_1 is a quasigeodesic with respect to the metric d_2.*

LEMMA 1.47. *Suppose that X, X' are proper geodesic metric spaces, G, G' are groups acting geometrically on X and X' respectively and $\rho : G \to G'$ be an isomorphism. Then there exists a ρ-equivariant quasi-isometry $f : X \to X'$.*

PROOF. Pick points $x \in X, x' \in X'$. According to Lemma 1.43 the maps

$$G \to G \cdot x \hookrightarrow X, \quad G' \to G' \cdot x' \hookrightarrow X'$$

are quasi-isometries; therefore the map

$$f : G \cdot x \to G' \cdot x, \quad f(gx) := \rho(g)x$$

is also a quasi-isometry. Thus, f determines a ρ-equivariant quasi-isometry

$$\tilde{f} : X \to X', \tilde{f} = f \circ \pi,$$

where $\pi : X \to G \cdot x$ is the nearest-point projection. $\square$

COROLLARY 1.48. *VI $\Rightarrow$ QI.*

PROOF. Let G be a finitely generated group, $H < G$ a finite index subgroup and $F \triangleleft H$ a finite normal subgroup. Let Γ_G be a Cayley graph of G. The group H acts on Γ_G properly discontinuously and cocompactly. Therefore, H is finitely generated and is QI to G. Let $\Gamma_{H/F}$ be a Cayley graph of H/G. The group H again acts isometrically and cocompactly on $\Gamma_{H/F}$ and, since the kernel F of this action is finite, this action is properly discontinuous. Therefore, H is QI to H/F. Recall now that groups G_1, G_2 are VI (virtially isomorphic) if there exist finite index subgroups $H_i < G_i$ and finite normal subgroups $F_i \triangleleft H_i$, $i = 1, 2$, so that $H_1/F_1 \cong G_2/F_2$. Since G_i is QI to H_i which, in turn, is QI to H_i/F_i, we conclude that G_1 is QI to G_2. $\square$

The next example shows that VI is not equivalent to QI.

EXAMPLE 1.49. Let A be a diagonalizable over $\mathbb{R}$ matrix in $SL(2,\mathbb{Z})$ so that $A^2 \neq I$. Thus the eigenvalues λ, λ^{-1} of A have absolute value $\neq 1$. We will use the notation $Hyp(2,\mathbb{Z})$ for the set of such matrices. Define the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ so that the generator $1 \in \mathbb{Z}$ acts by the automorphism given by A. Let G_A denote the associated semidirect product $G_A := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$. We leave it to the reader to verify that $\mathbb{Z}^2$ is a unique maximal normal abelian subgroup in G_A. By diagonalizing the matrix A, we see that the group G_A embeds as a discrete cocompact subgroup in the Lie group

$$Sol_3 = \mathbb{R}^2 \rtimes_D \mathbb{R}$$

where

$$D(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, t \in \mathbb{R}.$$

In particular, G_A is torsion-free. The group Sol_3 has its left-invariant Riemannian metric, so G_A acts isometrically on Sol_3 regarded as a metric space. Hence, every group G_A as above is QI to Sol_3. We now construct two groups G_A, G_B of the above type which are not VI to each other. Pick two matrices $A, B \in Hyp(2,\mathbb{Z})$ so that A^n is not conjugate to B^m for all $n, m \in \mathbb{Z} \setminus \{0\}$. For instance, take

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(The above property of the powers of A and B follows by considering the eigenvalues of A and B and observing that the fields they generate are different quadratic extensions of $\mathbb{Q}$.) The group G_A is QI to G_B since they are both QI to Sol_3. Let us check that G_A is not VI to G_B. First, since both G_A, G_B are torsion-free, it suffices to show that they are not commensurable, i.e., do not contain isomorphic finite index subgroups. Let $H = H_A$ be a finite-index subgroup in G_A. Then H intersects the normal rank 2 abelian subgroup of G_A along a rank 2 abelian subgroup L_A. The image of H under the quotient homomorphism $G_A \to G_A/\mathbb{Z}^2 = \mathbb{Z}$ has to be an infinite cyclic subgroup, generated by some $n \in \mathbb{N}$. Therefore, H_A is isomorphic to $\mathbb{Z}^2 \rtimes_{A^n} \mathbb{Z}$. For the same reason, $H_B \cong \mathbb{Z}^2 \rtimes_{B^m} \mathbb{Z}$. It is easy to see that an isomorphism $H_A \to H_B$ would have to carry L_A isomorphically to L_B. However, this would imply that A^n is conjugate to B^m. Contradiction.

There are many other examples when QI does not imply VI. For instance, let S be a closed oriented surface of genus $n \geq 2$. Let $G_1 = \pi_1(S) \times \mathbb{Z}$. Let M be the total space of the unit tangent bundle $UT(S)$ of S. Then the fundamental group $G_2 = \pi_1(M)$ is a nontrivial central extension of $\pi_1(S)$:

$$1 \to \mathbb{Z} \to G_2 \to \pi_1(S) \to 1,$$

$$G_2 = \langle a_1, b_1, ..., a_n, b_n, t | [a_1, b_1]...[a_n, b_n]t^{2n-2}, [a_i, t], [b_i, t], i = 1, ..., n\rangle.$$

We leave it to the reader to check that passing to any finite index subgroup in G_2 does not make it a trivial central extension of the fundamental group of a hyperbolic surface. On the other hand, since $\pi_1(S)$ is hyperbolic, the groups G_1 and G_2 are quasi-isometric, see section 14.

11. Gromov-hyperbolic spaces

Roughly speaking, Gromov-hyperbolic spaces are the ones which exhibit "tree-like behavior", at least if we restrict to finite subsets.

Let Z be a geodesic metric space. A geodesic triangle $\Delta \subset Z$ is called *R-thin* if every side of Δ is contained in the R-neighborhood of the union of two other sides. An *R-fat triangle* is a geodesic triangle which is not R-thin. A geodesic metric space Z is called *δ-hyperbolic in the sense of Rips* (who was the first to introduce this definition) if each geodesic triangle in Z is δ-thin. A finitely generated group is said to be Gromov-hyperbolic if its Cayley graph is Gromov-hyperbolic.

Below is an alternative definition of δ-hyperbolicity due to Gromov.

Let X be a metric space (which is no longer required to be geodesic). Pick a base-point $p \in X$. For each $x \in X$ set $|x|_p := d(x,p)$ and define the *Gromov product*

$$(x,y)_p := \frac{1}{2}(|x|_p + |y|_p - d(x,y)).$$

Note that the triangle inequality implies that $(x, y)_p \geq 0$ for all x, y, p; the Gromov product measures how far the triangle inequality if from being an equality.

Exercise 22. Suppose that X is a metric tree, i.e., a geodesic metric space where every path connecting points x, y contains $\overline{xy}$. Then $(x, y)_p$ is the distance $d(p, \gamma)$ from p to the segment $\gamma = \overline{xy}$.

In general we observe that for each point $z \in \gamma = \overline{xy}$

$$(p, x)_z + (p, y)_z = |z|_p - (x, y)_p. \tag{23}$$

In particular, $d(p, \gamma) \geq (x, y)_p$.

Suppose now that X is δ-hyperbolic in the sense of Rips. Then the Gromov product is "comparable" to $d(p, \gamma)$:

Lemma 1.50.

$$(x, y)_p \leq d(p, \gamma) \leq (x, y)_p + 2\delta.$$

Proof. The inequality $(x, y)_p \leq d(p, \gamma)$ was proven above; so we have to establish the other inequality. Note that since the triangle $\Delta(pxy)$ is δ-thin, for each point $z \in \gamma = \overline{xy}$ we have

$$\min\{(x, p)_z, (y, p)_z\} \leq \min\{d(z, \overline{px}), d(z, \overline{py})\} \leq \delta.$$

By continuity, there exists a point $z \in \gamma$ such that $(x, p)_z, (y, p)_z \leq \delta$. By applying the equality (23) we get:

$$|z|_p - (x, y)_p = (p, x)_z + (p, y)_z \leq 2\delta.$$

Since $|z|_p \leq d(p, \gamma)$, we conclude that $d(p, \gamma) \leq (x, y)_p + 2\delta$. □

Now define a number $\delta_p \in [0, \infty]$ as follows:

$$\delta_p := \sup\{\min((x, z)_p, (y, z)_p) - (x, y)_p\}.$$

Exercise 24. If $\delta_p \leq \delta$ then $\delta_q \leq 2\delta$ for all $q \in X$.

Definition 1.51. X is said to be *δ-hyperbolic in the sense of Gromov*, if $\infty > \delta \geq \delta_p$ for all $p \in X$.

In other words, for every quadruple $x, y, z, p \in X$, we have

$$(x, y)_p \geq \min((x, z)_p, (y, z)_p) - \delta.$$

Lemma 1.52. *Suppose that X is δ-hyperbolic in the sense of Rips. Then it is 3δ-hyperbolic in the sense of Gromov.*

Proof. Consider points $x, y, z, p \in X$ and the geodesic triangle $\Delta(xyz) \subset X$. Let $m \in \overline{xy}$ be the point nearest to p. Then, since the triangle $\Delta(xyz)$ is δ-thin, there exists a point $n \in \overline{xz} \cup \overline{yz}$ so that $d(n, m) \leq \delta$. Assume that $n \in \overline{yz}$. Then, by Lemma 1.50,

$$(y, z)_p \leq d(p, \overline{yz}) \leq d(p, \overline{xy}) + \delta.$$

On the other hand, by Lemma 1.50,

$$d(p, \overline{xy}) \leq (x, y)_p - 2\delta.$$

By combining these two inequalities, we obtain

$$(y, z)_p \leq (x, y)_p - 3\delta.$$

Therefore, $(x, y)_p \geq \min((x, z)_p, (y, z)_p) - 3\delta)$. □

We now prove the "converse":

Lemma 1.53. *Suppose that X is a geodesic metric space which is δ-hyperbolic (in the sense Gromov) then X is 2δ-hyperbolic in the sense of Rips.*

Proof. 1. We first show that in such space geodesics connecting any pair of points are "almost" unique, i.e., if α is a geodesic connecting x to y and p is a point in X such that

$$d(x,p) + d(p,y) \le d(x,y) + 2\delta$$

then $d(p,\alpha) \le 2\delta$. We suppose that $d(p,x) \le d(p,y)$. If $d(p,x) \ge d(x,y)$ then $d(x,y) \le 2\delta$ and thus $\min(d(p,x), p(y)) \le 2\delta$ and we are done.

Therefore, assume that $d(p,x) < d(x,y)$ and let $z \in \alpha$ be such that $d(z,y) = d(p,y)$. Since X is δ-hyperbolic in the sense Gromov,

$$(x,y)_p \ge \min((x,z)_p, (y,z)_p) - \delta.$$

Thus we can assume that $(x,y)_p \ge (x,z)_p$. Then

$$d(y,p) - d(x,y) \ge d(z,p) - d(x,z) - 2\delta \iff$$
$$d(z,p) \le 2\delta.$$

Thus $d(p,\alpha) \le 2\delta$.

2. Consider now a geodesic triangle $[x,y,p] \subset X$ and let $z \in \overline{xy}$. Our goal is to show that z belongs to $B_{4\delta}(\overline{px} \cup \overline{py})$. We have:

$$(x,y)_p \ge \min((x,z)_p, (y,z)_p) - \delta.$$

Assume that $(x,y)_p \ge (x,z)_p - \delta$. Set $\alpha := \overline{py}$. We will show that $z \in B_{2\delta}(\alpha)$.

By combining $d(x,z) + d(y,z) = d(x,y)$ and $(x,y)_p \ge (x,z)_p - \delta$, we obtain

$$d(y,p) \ge d(y,z) + d(z,p) - 2\delta.$$

Therefore, by Part 1, $z \in B_{2\delta}(\alpha)$ and, hence, the triangle $\Delta(xyz)$ is 2δ-thin. $\square$

Corollary 1.54. *[See* [**49**]*, section 6.3C.] For geodesic metric spaces, Gromov-hyperbolicity is equivalent to Rips-hyperbolicity.*

The advantage of Gromov's definition of hyperbolicity is that it does not require X to be geodesic and this notion is manifestly QI-invariant:

If X, X' are quasi-isometric and X is δ-hyperbolic in the sense of Gromov then X' is δ'–hyperbolic in the sense of Gromov (for some δ'). In contrast, QI invariance of Rips-hyperbolicity is not a priori obvious (without Corollary 1.54). We will give another proof of QI invariance of Rips-hyperbolicity in Corollary 3.4 as a corollary of Morse lemma.

In what follows, we will refer to δ-hyperbolic spaces in the sense of Rips as being δ-hyperbolic.

Definition 1.55. A finitely-generated group G is called *Gromov-hyperbolic* or *word-hyperbolic* if one of its Cayley graphs is hyperbolic.

Note that, since hyperbolicity is a QI invariant, one Cayley graph of G is hyperbolic iff all its Cayley graphs are hyperbolic.

Here are some examples of Gromov-hyperbolic spaces.

1. Every metric tree is 0-hyperbolic. In particular, free groups are hyperbolic.

2. Let $X = \mathbb{H}^n$ be the hyperbolic n-space. Then X is δ-hyperbolic for appropriate δ. The reason for this is that the "largest" triangle in X is an ideal triangle, i.e. a triangle all whose three vertices are on the boundary sphere of $\mathbb{H}^n$. All such triangles are congruent to each other since $Isom(\mathbb{H}^n)$ acts transitively on triples of distinct points in S^{n-1}. Thus it suffices to verify thinness of a single ideal triangle in $\mathbb{H}^2$, the triangle with the ideal vertices $0, 2, \infty$. We claim that for each point x on the arc between 0 and m the distance to the side γ is < 1. Indeed, since dilations with center at zero are hyperbolic isometries, the maximal distance from x to γ is realized at the point $m = 1 + i$. Computing the hyperbolic length of the horizontal segment between m and $i \in \gamma$ we conclude that it equals 1. Hence $d(x,\gamma) \le d(m,\gamma) < 1$. See Figure 8.

Remark 1.56. By making a more careful computation with the hyperbolic distances one can conclude that $\sinh(d(m,\gamma)) = 1$.

Since Gromov-hyperbolicity of a QI invariant, it follows that every cocompact discrete isometry group of $\mathbb{H}^n$ is Gromov-hyperbolic. For instance, surface groups are hyperbolic.

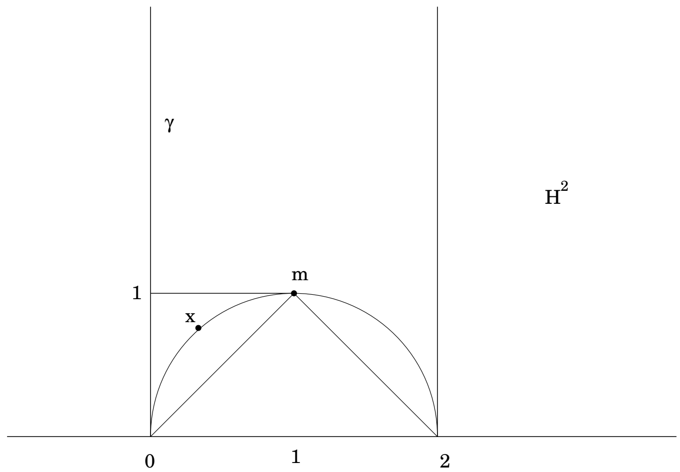

FIGURE 8. *Ideal triangle* $\Delta(0, 2, \infty)$ *in the hyperbolic plane:* $d(x, \gamma) \le d(m, \gamma) < 1$.

3. Suppose that X is a complete Riemannian manifold of sectional curvature $\le \kappa < 0$. Then X is Gromov-hyperbolic. This follows from Rauch-Toponogov comparison theorem. Namely, let $Y = M_\kappa$ be the hyperbolic plane with the curvature normalized to be $= \kappa < 0$. Then Y is δ-hyperbolic. Let $\Delta = \Delta(xyz)$ be a geodesic triangle in X. Construct the *comparison triangle* $\Delta' := \Delta(x'y'z') \subset Y$ whose sides have the same length as for the triangle Δ. Then the triangle Δ' is δ-thin. Pick a pair of points $p \in \overline{xy}, q \in \overline{yz}$ and the corresponding points $p' \in \overline{x'y'}, q' \in \overline{y'z'}$ so that $d(x, p) = d(x', p'), d(y, q) = d(y', q')$. Then Rauch-Toponogov comparison theorem implies that $d(p, q) \le d(p', q')$. It immediately follows that the triangle Δ is δ-thin.

4. More generally, a geodesic metric space X is said to be $CAT(\kappa)$ (for $\kappa \le 0$) if it satisfies the triangle comparison property: Given a triangle $\Delta = \Delta(xyz)$ be a geodesic triangle in X, its comparison triangle $\Delta' = \Delta(x'y'z') \subset M_\kappa$, we have

$$d(p, q) \le d(p', q')$$

for the appropriate points p, q, p', q' on Δ and Δ'. Then the same arguments as above imply that every $CAT(\kappa)$ metric space, for $\kappa < 0$, is Gromov-hyperbolic.

DEFINITION 1.57. A group G is called $CAT(\kappa)$ if it admits a geometric action on a $CAT(\kappa)$ space.

Then, every $CAT(-1)$ group is hyperbolic.

PROBLEM 1.58. Construct a hyperbolic group G which is not a $CAT(-1)$ group.

12. Ideal boundaries

Suppose that X is a proper geodesic metric space. Introduce an equivalence relation on the set of geodesic rays in X by declaring $\rho \sim \rho'$ iff they are *asymptotic*, i.e., are within finite distance from each other. Given a geodesic ray ρ we will denote by $\rho(\infty)$ its equivalence class. Define the *ideal boundary* of X as the collection $\partial_\infty X$ of equivalence classes of geodesic rays in X. The space of geodesic rays (parameterized by arc-length) in X has a natural compact-open topology (we regard geodesic rays as maps from $[0, \infty)$ into X). Thus we topologize $\partial_\infty X$ by giving it the quotient topology τ.

We now restrict our attention to the case when X is δ-hyperbolic. Then for each geodesic ray ρ and a point $p \in X$ there exists a geodesic ray ρ' with the initial point p such that $\rho(\infty) = \rho'(\infty)$: Consider the sequence of geodesic segments $\overline{p\rho(n)}$ as $n \to \infty$. Then the thin triangles property implies that these segments are contained in a δ-neighborhood of $\rho \cup \overline{p\rho(0)}$. Properness of X implies that this sequence subconverges to a geodesic ray ρ' as required.

LEMMA 1.59. *(Asymptotic rays are uniformly close). Let ρ_1, ρ_2 be asymptotic geodesic rays in X such that $\rho_1(0) = \rho_2(0) = p$. Then for each t,*

$$d(\rho_1(t), \rho_2(t)) \le 2\delta.$$

PROOF. Suppose that the rays ρ_1, ρ_2 are within distance $\le C$ from each other. Take $T \gg t$. Then (since the rays are asymptotic) there exists $S \in \mathbb{R}_+$ such that

$$d(\rho_1(T), \rho_2(S)) \le C.$$

By δ-thinness of the triangle $\Delta(p\rho_1(T)\rho_2(S))$, the point $\rho_1(t)$ is within distance $\le \delta$ from a point either on $\overline{p\rho_2(S)}$ or on $\overline{\rho_1(T)\rho_2(S)}$. Since the length of $\overline{\rho_1(T)\rho_2(S)}$ is $\le C$ and $T \gg t$, it follows that there exists t' such that

$$d(\rho_1(t), \rho_2(t')) \le \delta.$$

By the triangle inequality, $|t - t'| \le \delta$. It follows that $d(\rho_1(t), \rho_2(t)) \le 2\delta$. □

Pick a base-point $p \in X$. Given a number $k > 2\delta$, define the topology τ_k on $\partial_\infty X$ with the basis of neighborhoods of a point $\rho(\infty)$ given by

$$U_{k,n}(\rho) := \{\rho' : d(\rho'(t), \rho(t)) < k, t \in [0, n]\}, n \in \mathbb{R}_+$$

where the rays ρ' satisfy $\rho'(0) = p = \rho(0)$.

LEMMA 1.60. *Topologies τ and τ_k coincide.*

PROOF. 1. Suppose that ρ_j is a sequence of rays emanating from p such that $\rho_j \notin U_{k,n}(\rho)$ for some n. If $\lim_j \rho_j = \rho'$ then $\rho' \notin U_{k,n}$ and by the previous lemma, $\rho'(\infty) \neq \rho(\infty)$.

2. Conversely, if for each n, $\rho_j \in U_{k,n}(\rho)$ (provided that j is large enough), then the sequence ρ_j subconverges to a ray ρ' which belongs to each $U_{k,n}(\rho)$. Hence $\rho'(\infty) = \rho(\infty)$. □

EXAMPLE 25. Suppose that $X = \mathbb{H}^n$ is the hyperbolic n-space realized in the unit ball model. Then the ideal boundary of X is S^{n-1}.

LEMMA 1.61. *Let X be a proper geodesic Gromov-hyperbolic space. Then:*

1. $\partial_\infty X$ is compact.

2. For each pair of distinct points $\xi, \eta \in \partial_\infty X$ there exists a geodesic γ in X which is asymptotic to both ξ and η.

PROOF. 1. Compactness of $\partial_\infty X$ immediately follows from compactness of the set of rays emanating from a base-point $p \in X$.

2. Consider geodesic rays ρ, ρ' emanating from the same point $p \in X$ and asymptotic to ξ, η respectively. Since $\xi \neq \eta$, for each $R < \infty$ the set

$$K(R) := \{x \in X : d(x, \rho) \le R, d(x, \rho') \le R\}$$

is compact. Consider the sequences $x_n := \rho(n), x'_n := \rho'(n)$ on ρ, ρ' respectively. Since the triangles $\Delta p x_n x'_n$ are δ-thin, each segment $\gamma_n := \overline{x_n x'_n}$ contains a point within distance $\le \delta$ from both $\overline{px_n}, \overline{px'_n}$, i.e. $\gamma_n \cap K(\delta) \neq \emptyset$. Therefore the sequence of geodesic segments γ_n subconverges to a complete geodesic γ in X. Since $\gamma \subset N_\delta(\rho \cup \rho')$ it follows that γ is asymptotic to ξ and η. □

DEFINITION 1.62. We say that a sequence $x_n \in X$ converges to a point $\xi = \rho(\infty) \in \partial_\infty X$ in the **cone topology** if there is a constant C such that $x_n \in B_C(\rho)$ and the geodesic segments $\overline{x_1 x_n}$ converge to a geodesic ray asymptotic to ξ.

For instance, suppose that $X = \mathbb{H}^m$ in the upper half-space model, $\xi = 0 \in \mathbb{R}^{m-1}$, L is the vertical geodesic from the origin. Then a sequence $x_n \in X$ converges ξ in the cone topology iff all the points x_n belong to the Euclidean cone with the axis L and the Euclidean distance from x_n to 0 tends to zero. See Figure 9. This explains the name *cone topology.*

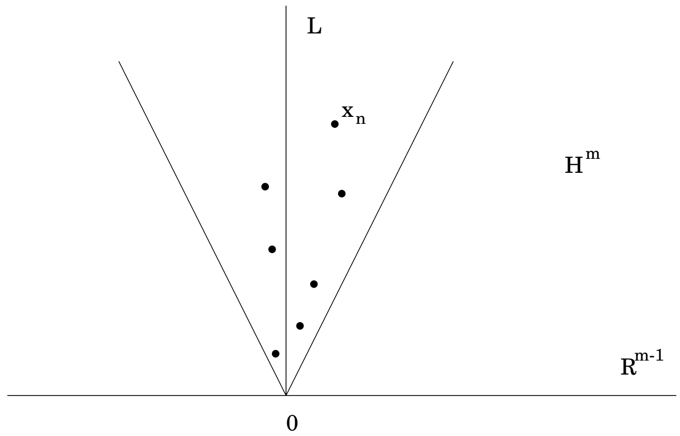

FIGURE 9. *Convergence in the cone topology.*

THEOREM 1.63. *Suppose that G is a hyperbolic group. Let $\partial_\infty G$ denote the ideal boundary of its Cayley graph*[1]*. Then:*

1. The group G acts by homeomorphisms on $\partial_\infty G$ as a **uniform convergence group***, i.e. the action of G on $Trip(\partial_\infty G)$ is properly discontinuous and cocompact, where $Trip(\partial_\infty G)$ consists of triples of distinct points in $\partial_\infty G$.*

2. $\partial_\infty G$ consists of 0, 2 or continuum of points.

PROOF. Let X be a δ-hyperbolic Cayley graph of G with the ideal boundary $\partial_\infty G = \partial_\infty X$. Then X is a proper geodesic metric space.

1. Consider distinct points $\xi, \eta, \zeta \in \partial_\infty G$. Then we have a geodesic triangle $\Delta = \Delta(xi\eta\zeta)$ formed by the geodesics $\overline{\xi\eta}, \overline{\eta\zeta}, \overline{\zeta\xi}$. Define δ-center of Δ, a point $c(\Delta) \in X$ which is within distance $\leq \delta$ from all three sides of Δ. Since X is δ-hyperbolic, it follows that δ-center exists.

$\square$

13. Further properties of hyperbolic groups

1. Linear isoperimetric inequality. Solution of WP.
2. Morse lemma.
3. Contractibility of Rips complex w/o proof.
4. Ubiquity of hyperbolic groups.
5. Rips construction.

14. Central extensions and quasi-isometries

Central extensions and 2-nd cohomology. Our first goal is to relate equivalence central extensions of groups and 2-nd group cohomology. We restrict ourselves to the case of central extensions: The same results hold for general extensions with abelian kernels, see e.g. [**16**].

Let G be a group and A an abelian group. A *central extension* of G by A is a group $\tilde{G}$ so that $G \cong \tilde{G}/A$, where A is contained in the center of $\tilde{G}$. Two central extensions are said to be equivalent if there exist an isomorphism τ making the following diagram commutative:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "id"'] & \tilde{G}_1 \arrow[r] \arrow[d, "\tau"'] & G \arrow[r] \arrow[d, "id"'] & 1 \\ 1 \arrow[r] & A \arrow[r] & \tilde{G}_2 \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

[1]Later on, we will see that all such boundaries are equivariantly homeomorphic.

Equivalence classes of central extensions of G by A are identified with the 2-nd cohomology group $H^2(G, A)$ as follows. Take a presentation $G = \langle X|R\rangle$ and let Y^2 denote the corresponding presentation complex. Embed Y to a 3-connected cell complex Y by attaching appropriate 3-cells to Y^2. Then $H^2(Y) \cong H^2(G, A)$. Each cocycle $\zeta \in Z^2(G, A)$ assigns elements of A to each 2-cell in Y. Recall that 2-cells c_i of Y are indexed by the relators $R_i, i \in I$, of G. By abusing the notation, we set $\zeta(R_i) := \zeta(c_i)$. We then define the group $\tilde{G}$ by the presentation

$$\tilde{G} = \langle X \cup A | [a, x] = 1, a \in A, x \in X, R_i\zeta(R_i) = 1, i \in I\rangle.$$

The presentation complex $\tilde{Y}$ for G admits an obvious map $f : \tilde{Y} \to Y$ which collapses each loop corresponding to $a \in A$ to the vertex of Y. So far we did not use the assumption that ζ is a cocycle, i.e., that $\zeta(\sigma) = 0$ whenever σ is a 2-boundary. To see how it is being used, let's check that the group A embeds in $\tilde{G}$. Suppose that $a \in A$ is trivial in $\tilde{G}$. Then the corresponding loop α in $\tilde{Y}$ bounds a 2-disk. The image of this disk under f is a spherical 2-cycle σ in Y. Since ζ is a cocycle, it vanishes on σ. Therefore, $a = \zeta(\sigma) = 0$ in A.

Example 1.64. Let G be the fundamental group of a genus $p \geq 2$ closed oriented surface S. Take the standard presentation of G, so that S is the (aspherical) presentation complex. Let $A = \mathbb{Z}$ and take $[\zeta] \in H^2(G, \mathbb{Z}) \cong H^2(S, \mathbb{Z})$ be the class Poincaré dual to the fundamental class of S. Then for the unique 2-cell c in S corresponding to the relator

$$R = [a_1, b_1]...[a_p, b_p],$$

we have $\zeta(c) = 1$. The corresponding group $\tilde{G}$ has the presentation

$$\langle a_1, b_1, ..., a_p, b_p, t| \quad [a_1, b_1]...[a_p, b_p]t, [a_i, t], [b_i, t], i = 1, ..., p\rangle.$$

Let's check that if ζ is a coboundary then it determines a trivial central extension. Let $\zeta = \delta\eta$, $\eta \in C^1(Y, A)$. Since each generator $x_j, j \in J$ of G determines a 1-cycle ξ_j in Y, we obtain $a_j := \eta(\xi_j)$. We then define a homomorphism $\psi : G \to \tilde{G}$ by $\psi(x_j) = x_j a_j$. The reader will check that ϕ is indeed a homomorphism which, therefore, determines a splitting of the exact sequence

$$1 \to A \to \tilde{G} \to G \to 1.$$

Conversely, suppose that a cocycle ζ determines a split extension. Thus we have a homomorphism $\phi : G \to \tilde{G}$ which sends x_j to $x_j a_j$ for some $a_j \in A$. Then, define the 1-cochain η by $\eta(x_j) = a_j$. Since $\phi(R_i) = 1$ for each relator, it is immediate that $\delta\eta = \zeta$.

A similar argument shows that $[\zeta] = [\zeta']$ in $H^2(G, A)$ if and only if the extensions determined by ζ, ζ' are equivalent.

The conclusion, thus, is that a group G with nontrivial 2-nd cohomology group $H^2(G, A)$ admits nontrivial central extensions with the kernel A. How does one construct groups with nontrivial H^2? Suppose that G admits an aspherical presentation complex Y so that $\chi(G) = \chi(Y) > 0$. Then for any A which, say, maps onto $\mathbb{Z}$, $H^2(G, A) \neq 0$.

Central extensions of hyperbolic groups.

We now consider a central extension

$$1 \to A \to \tilde{G} \xrightarrow{\pi} G \to 1 \tag{26}$$

with A a finitely-generated abelian group and G hyperbolic.

Theorem 1.65. *$\tilde{G}$ is QI to $A \times G$.*

Proof. In the case when $A \cong \mathbb{Z}$, the first published proof belongs to S. Gersten [**42**], although, it appears that D.B.A. Epstein and G. Mess also knew this result. Our proof follows the one in [**80**]. First of all, since an epimorphism with finite kernel is a quasi-isometry, it suffices to consider the case when A is free abelian of finite rank.

Our main goal is to construct a Lipschitz section $\rho : G \to \tilde{G}$ of the sequence (26). We first consider the case when $A \cong \mathbb{Z}$. Each fiber $\pi^{-1}(g), g \in G$, is a copy of $\mathbb{Z}$ and, therefore, has a natural order denoted $\leq$. We let ι denote the embedding $\mathbb{Z} \to A \to \tilde{G}$. We let X denote a symmetric generating set of $\tilde{G}$ and use the same name for its image under π. Let $|w|$ denote the word length with respect to this generating set, for $w \in X^*$, where X^* is the set of words in X. Lastly, let $\tilde{w}$ and $\bar{w}$ denote the elements of $\tilde{G}$ and G represented by $w \in X^*$.

LEMMA 1.66. *There exists $C \in \mathbb{N}$ so that for every $g \in G$ there exists*

$$\rho(g) := \max\{\tilde{w}\iota(-C|w|) : w \in X^*, \bar{w} = g\}.$$

Here the maximum is taken with respect to the natural order on $\pi^{-1}(g)$.

PROOF. We will use the fact that G satisfies the linear isoperimetric inequality

$$Area(\alpha) \leq K|\alpha|$$

for every $\alpha \in X^*$ representing the identity in G. We will assume that $K \in \mathbb{N}$. For each $R \in X^*$ so that $R^{\pm 1}$ is a defining relator for G, the word R represents some $\tilde{R} \in A$. Therefore, since G is finitely-presented, we define a natural number T so that

$$\iota(T) = \max\{\tilde{R} : R^{\pm 1} \text{ is a relator of } G\}.$$

We then claim that for each $s \in X^*$ representing the identity in G,

$$\tilde{s} \leq \iota(TArea(s)). \tag{27}$$

It suffices to verify that for $s = h^{-1}Rh$,

$$\tilde{s} \leq \iota(T)$$

where R is a defining relator of G and $h \in X^*$. The latter inequality follows from the fact that the multiplication by $\bar{h}$ and $\bar{h}^{-1}$ determine an order isomorphism and its inverse between $\pi^{-1}(1)$ and $\pi^{-1}(\bar{h})$.

Set $C := TK$. We are now ready to prove lemma. Let w, v be in X^* representing the same element $g \in G$. Set $u := v^{-1}$. Then $s = wu$ represents the identity and, hence, by (27),

$$\tilde{s} = \tilde{w}\tilde{u} \leq \iota(C|s|) = \iota(C|w|) + \iota(C|u|).$$

Thus,

$$w - v \leq \iota(C|w|) + \iota(C|v|),$$

and

$$w - \iota(C|w|) \leq v + \iota(C|v|).$$

Therefore, taking v to be a fixed word representing g, we conclude that all the differences $w - \iota(C|w|)$ are bounded from above. Hence their maximum exists. $\square$

Thus, we have a section ρ (given by Lemma 1.66) of the exact sequence (26). A word $w = w_g$ realizing the maximum in the definition of ρ is called *maximizing*. The section ρ, of course, need not be a group homomorphism. We will see nevertheless that it is not far from being one. Set

$$\sigma(g_1, g_2) := \rho(g_1)\rho(g_2) - \rho(g_1 g_2)$$

where the difference is taking place in $\pi^{-1}(g_1 g_2)$. Thus we can think of σ as measuring the deviation of ρ from being a homomorphism. The next lemma does not use hyperbolicity of G, only the definition of ρ.

LEMMA 1.67. *The set $\sigma(G, X)$ is finite.*

PROOF. Let $x \in X$, $g \in G$. We have to estimate the difference

$$\rho(g)x - \rho(gx).$$

Let w_1 and w_2 denote maximizing words for g and gx respectively. Note that the word $w_1 x$ also represents gx. Therefore, by the definition of ρ,

$$\widetilde{w_1 x}\iota(-C(|w_1| + 1)) \leq \tilde{w}_2\iota(-C|w_2|).$$

Hence, there exists $a \in A, a \geq 0$, so that

$$\widetilde{w_1}\iota(-C(|w_1|)\tilde{x}\iota(-C)a = \tilde{w}_2\iota(-C|w_2|)$$

and, thus

$$\rho(g)\widetilde{x}\iota(-C)a = \rho(gx). \tag{28}$$

Since $w_2 x^{-1}$ represents g, we similarly obtain

$$\rho(gx)\widetilde{x}^{-1}\iota(-C)b = \rho(g), b \geq 0, b \in A. \tag{29}$$

By combining equations (28) and (29) and switching to the additive notation for the group operation in A we get

$$a + b = \iota(2C).$$

Since $a \geq 0, b \geq 0$, we conclude that $-\iota(C) \leq a - \iota(C) \leq \iota(C)$. Therefore, (28) implies that

$$|\rho(g)x - \rho(gx)| \leq C.$$

Since the finite interval $[-\iota(C), \iota(C)]$ in A is finite, lemma follows. $\square$

REMARK 1.68. Actually, more is true: There exists a section $\rho : G \to \tilde{G}$ so that $\sigma(G, G)$ is a finite set. This follows from the fact that all (degree ≥ 2) cohomology classes of hyperbolic groups are *bounded*. However, the proof is more difficult and we will not need this fact.

Letting L denote the maximum of the word lengths (with respect to X) of the elements in the sets $\sigma(G, X), \sigma(X, G)$, we conclude that the map $\rho : G \to \tilde{G}$ is $(L + 1)$-Lipschitz. Given a section $\rho : G \to \tilde{G}$, we define the projection $\phi = \phi_\rho : \tilde{G} \to A$ by

$$\phi(\tilde{g}) = \tilde{g} - \rho \circ \pi(\tilde{g}). \tag{30}$$

It is immediate that ϕ is Lipschitz provided that ρ is Lipschitz.

We now extend the above construction to the case of central extensions with free abelian kernel of finite rank. Let $A = \prod_{i=1}^n A_i, A_i \cong \mathbb{Z}$. Consider a central extension (26). The homomorphisms $A \to A_i$ induce quotient maps $\eta_i : \tilde{G} \to \tilde{G}_i$ with the kernels $\prod_{j \neq i} A_j$. Each $\tilde{G}_i$, in turn, is a central extension

$$1 \to A_i \to \tilde{G}_i \to G \xrightarrow{\pi_i} 1. \tag{31}$$

Assuming that each central extension (31) has a Lipschitz section ρ_i, we obtain the corresponding Lipschitz projection $\phi_i : \tilde{G}_i \to A_i$ given by (30). This yields a Lipschitz projection

$$\Phi : \tilde{G} \to A, \Phi = (\phi_1 \circ \eta_1, ..., \phi_n \circ \eta_n).$$

We now set

$$\rho(\pi(\tilde{g})) := \tilde{g} - \Phi(\tilde{g}).$$

It is straightforward to verify that ρ is well-defined and that it is Lipschitz provided that each ρ_i is. We thus obtain

COROLLARY 1.69. *Given a finitely-generated free abelian group A and a hyperbolic group G, each central extension (26) admits a Lipschitz section $\rho : G \to \tilde{G}$ and a Lipschitz projection $\Phi : \tilde{G} \to A$ given by*

$$\Phi(\tilde{g}) = \tilde{g} - \rho(\pi(\tilde{g})).$$

We then define the map

$$h : G \times A \to \tilde{G}, \quad h(g, a) = \rho(g) + \iota(a)$$

and its inverse

$$h^{-1} : \tilde{G} \to G \times A, \quad \hat{h}(\tilde{g}) = (\pi(\tilde{g}), \Phi(\tilde{g})).$$

Since homomorphisms are 1-Lipschitz while the maps ρ and Φ are Lipschitz, we conclude that h is a bi-Lipschitz quasi-isometry. $\square$

Remark 1.70. The above proof easily generalizes to the case of an arbitrary finitely-generated group G and a central extension (26) given by a bounded 2-nd cohomology class: One has to observe only that each cyclic central extension

$$1 \to A_i \to \tilde{G}_i \to G \to 1$$

is still given by a bounded cohomology class.

Example 1.71. Let $G = \mathbb{Z}^2$, $A = \mathbb{Z}$. Since $H^2(G, \mathbb{Z}) = H^2(T^2, \mathbb{Z}) \cong \mathbb{Z}$, the group G admits nontrivial central extensions with the kernel A. The group $\tilde{G}$ for such an extension is nilpotent but not virtually abelian. Hence, it is not quasi-isometric to $G \times A = \mathbb{Z}^3$.

One can ask if Theorem 1.65 generalizes to other normal extensions of hyperbolic groups G:

We note that Theorem 1.65 does not extend, say, to the case where A is a non-elementary hyperbolic group and the action $G \curvearrowright A$ is trivial. The reason is the *quasi-isometric rigidity* for products of certain types of groups proven in **[64]**. A special case of this theorem says that if $G_1, ..., G_n$ are nonelementary hyperbolic groups, then quasi-isometries of the product $G = G_1 \times ... \times G_n$ quasi-preserve the product structure:

Theorem 1.72. *Let $\pi_j : G \to G_j, j = 1, ..., n$ be natural projections. Then for each (L, A)-quasi-isometry $f : G \to G$, there is $C = C(G, L, A) < \infty$, so that, up to a composition with a permutation of quasi-isometric factors G_k, the map f has the property that for each i and $g_j \in G_j, j \neq i$, the set*

$$H := \bigcap_{j \neq i} \pi_j^{-1}(g_j)$$

satisfies

$$d(f(H), H) \leq C.$$

CHAPTER 2

Coarse topology

The goal of this section is to provide tools of algebraic topology for studying quasi-isometries and other concepts of the geometric group theory. The class of *bounded geometry metric cell complexes* provides a class of spaces for which application of algebraic topology is possible.

1. Metric cell complexes

A metric space X *has bounded geometry* if there is a function $\phi(r)$ such that each ball $B_r(x) \subset X$ contains at most $\phi(r)$ points. For instance, if G is a finitely generated group with word metric then G has bounded geometry.

In particular, every metric space of bounded geometry necessarily has discrete topology. Our next goal is to extend the above definition to allow spaces with interesting topology.

A *metric cell complex* is a cell complex X together with a metric defined on its 0-skeleton X^0.

By abusing the notation, we will refer to metric concepts for X, but have in mind that the metric is defined only on X^0. For instance, an r-ball in X is an r-ball in X^0. The *diameter* of a cell σ in X is the diameter of its intersection with X^0, etc.

A reader uncomfortable with this, can instead consider *metric simplicial complexes* defined as follows. Let X be a connected simplicial complex. Metrize each k-simplex of X to be isometric to the standard k-simplex Δ^k in the Euclidean space:

$$\Delta^k = (\mathbb{R}_+)^{k+1} \cap \{x_0 + ... + x_n = 1\}.$$

Thus, for each m-simplex σ^m and its face σ^k, the inclusion $\sigma^k \to \sigma^m$ is an isometric embedding. This allows us to define a path-metric on X so that each simplex is isometrically embedded in X. The result is a *metric simplicial complex.*

A metric cell complex X is said to have *bounded geometry* if:

(a) Each ball $B_r(x) \subset X$ intersects at most $\phi(r, k)$ cells of dimension $\leq k$.

(b) Diameter of each k-cell is at most c_k, $k = 1, 2, 3, ...$

Note that we allow X to be infinite-dimensional.

Example 32. Let G be a finitely-generated group with its word metric, X be a Cayley graph of G. Then X is a metric cell complex of bounded geometry.

The corresponding concept for the metric simplicial complexes is that there exists a number $N < \infty$ so that every vertex of X is incident to at most N edges. In particular, such X is necessarily finite-dimensional. Therefore, the concept of metric cell complexes is more flexible, which is why we use it. However, within the realm of finite-dimensional complexes, both concepts are essentially equivalent. We leave it to the reader to work out the correspondence.

Example 2.1. Let M be a compact simplicial complex and X be its universal cover. Metrize each simplex in X to be isometric to the standard simplex and construct a path-metric on X as above. Then X is a metric simplicial complex of bounded geometry.

Given a bounded geometry metric cell complex, one can define certain metric concepts for X. For instance, the (coarse) R-ball $\mathbf{B}_R(x)$ centered at a vertex $x \in X^0$ is the union of the cells in X which intersect the R-ball $B_R(x) \subset X^0$. Therefore, we can talk, say, about L-Lipschitz cellular maps $f : X \to Y$ between metric cell complexes of bounded geometry by requiring that

$$f(\mathbf{B}_R(x)) \subset \mathbf{B}_{LR}(f(x)), \forall x \in X^0, R \in \mathbb{R}_+.$$

DEFINITION 2.2. A metric cell complex X is said to be *uniformly contractible* if there exists a continuous function $\psi(R)$ so that for every $x \in X^0$ the map

$$\mathbf{B}_R(x) \to \mathbf{B}_{\psi(R)}(x)$$

be null-homotopic.

Recall that quasi-isometries are not necessarily continuous. In order to use algebraic topology, we, thus, have to approximate quasi-isometries by continuous maps.

LEMMA 2.3. *Suppose that X, Y are bounded geometry metric cell complexes, X is finite-dimensional, Y is uniformly contractible, and $f : X^0 \to Y^0$ is a coarse (L, A)-Lipschitz map. Then f admits a (continuous) cellular extension $g : X \to Y$, which is an L'-Lipschitz map, where L' depends on L, A and geometric bounds on X and Y.*

PROOF. The proof of this lemma is a prototype of most of the proofs which appear in this chapter. We extend f by induction on skeleta of X.

If x, x' belong to the boundary of a 1-cell in X then $d(f(x), f(x')) \le LConst_1 + A$, where $Const_1$ is an upper bound on the diameter of 1-cells in X.

Inductively, assume that f was defined on X^k, so that $\mathrm{diam}(f(\partial\tau)) \le Const_{k+1}$ for every $k+1$-cell τ in X. Let σ be a $k+1$-cell in X. Then, using uniform contractibility of Y, we extend f to σ so that $\mathrm{diam}(f(\sigma)) \le C_{k+1}$ where $C_{k+1} = \psi(Const_k)$. Suppose that τ is a $k+2$-cell in X. Then, since X has bounded geometry, $\mathrm{diam}(\tau) \le c_{k+2}$. In particular, $\partial\tau$ is connected and is contained in the union of at most $\phi(c_{k+2}, k+1)$ cells of dimension $k+1$. Therefore,

$$\mathrm{diam}(f(\partial\tau)) \le C_{k+1} \cdot \phi(c_{k+2}, k+1) = Const_{k+1}.$$

Since X is n-dimensional the induction terminates after n many steps. The resulting map $f : X \to Y$ satisfies

$$\mathrm{diam}(f(\sigma)) \le C_n = L'$$

for every cell in X. Therefore, $f : X \to Y$ is L'-Lipschitz. □

2. Ends of spaces

In this section we review (historically) the first coarse topological notion. Let X be a locally compact connected topological space (e.g. a proper geodesic metric space). Given a compact subset $K \subset X$ we consider its complement K^c. Recall that 0-th homotopy set $\pi_0(K^c)$ counts the number of path-connected components of K^c.

The collection of sets $\pi_0(K^c)$ determines an inverse system:

$$K \subset L \Rightarrow \pi_0(L^c) \to \pi_0(K^c).$$

Then the set of ends $\epsilon(X)$ is defined as the inverse limit

$$\lim_{K \subset X} \pi_0(K^c).$$

The elements of $\epsilon(X)$ are called *ends of* X.

To be more concrete, from now on, assume that X admits an exhaustion by a countable collection of compact subsets. For instance, if X is a proper metric space take compacts $K_R := \bar{B}_R(p)$, $R \in \mathbb{N}$, for some fixed $p \in X$.

Then the ends of X are represented by decreasing sequences of nonempty noncompact path-connected sets $C_i, i \in \mathbb{N}$,

$$... \subset C_i \subset C_{i-1} \subset ... \subset C_1 \subset X,$$

where each C_i has compact frontier in X.

Here is another description of the ends of X. Consider a nested sequence of compacts $K_i \subset X, i \in \mathbb{N}$. For each i pick a connected component $U_i \subset K_i^c$ so that $U_i \supset U_{i+1}$. Then the nested sequence (U_i) represents a single point in $\epsilon(X)$. Even more concretely, pick a point $x_i \in U_i$ for each i and connect x_i, x_{i+1} by a curve $\gamma_i \subset U_i$. The concatenation of the curves γ_i defines a proper map $\gamma : [0, \infty) \to X$. Call two proper curves $\gamma, \gamma' : \mathbb{R}_+ \to X$ equivalent if for each compact $K \subset X$ there are points $x \in \gamma(\mathbb{R}_+), x' \in \gamma'(\mathbb{R}_+)$ which belong to the same connected component of K^c. The equivalence classes of such curves are in bijective correspondence with the ends of X, the map $(U_i) \mapsto \gamma$ was described above.

See Figure 1 as an example. The space X in this picture has 5 visibly different ends: $\epsilon_1, ..., \epsilon_5$. We have $K_1 \subset K_2 \subset K_3$. The compact K_1 separates the ends ϵ_1, ϵ_2. The next compact K_2 separates ϵ_3 from ϵ_4. Finally, the compact K_3 separates ϵ_4 from ϵ_5.

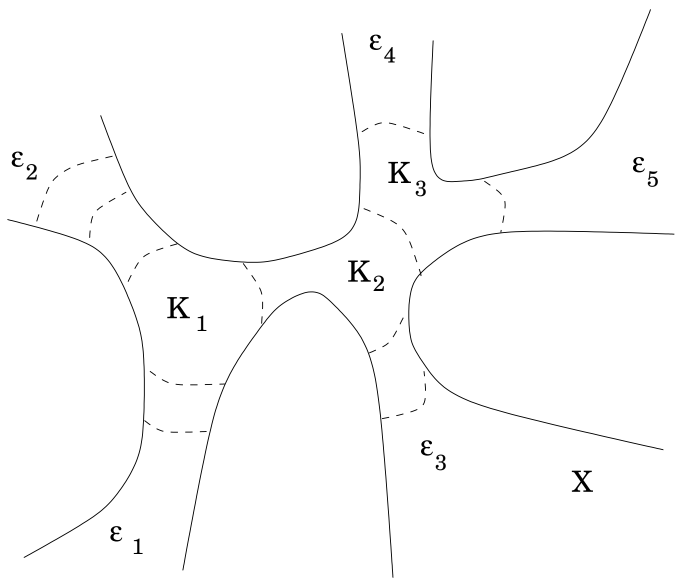

FIGURE 1. *Ends of X.*

Analogously, one defines *higher homotopy groups* $\pi_i^\infty(X, x_\bullet)$ at infinity of X, by considering inverse systems of higher homotopy groups: This requires a choice of a system of base-points $x_k \in K^c$ representing a single element of $\epsilon(X)$. The inverse limit of this sequence of base-points, $x_\bullet \in \epsilon(X)$, serves as a "base-point" for the homotopy group $\pi_i^\infty(X, x_\bullet)$. The elements of $\pi_i^\infty(X, x_\bullet)$ are elements of the inverse limit

$$\lim_{K_k \subset X} \pi_i(K_k^c, x_k).$$

Topology on $\epsilon(X)$. Let $\eta \in \epsilon(X)$ be represented by a decreasing sequence (U_i). Each U_i defines a neighborhood $N_i(\eta)$ of η consisting of all $\eta' \in \epsilon(X)$ which are represented by nested sequences (U'_j) such that $U'_j \subset U_i$ for all but finitely many $j \in \mathbb{N}$.

EXERCISE 33. Verify that this indeed defines a topology on $\epsilon(X)$.

LEMMA 2.4. *If $f : X \to Y$ is an (L, A)-quasi-isometry of proper geodesic metric spaces then f induces a homeomorphism $\epsilon(X) \to \epsilon(Y)$.*

PROOF. For geodesic metric spaces, path-connectedness is equivalent to connectedness. Since f is a quasi-isometry, for each bounded subset $K \subset X$, the image $f(K)$ is again bounded. Although for a path-connected subset $C \subset X$ the image $f(C)$ is not necessarily path-connected, for $R := 1+A$, the R-neighborhood $B_R(f(C))$ is connected. Indeed, for points $x, x' \in C$, and $\delta > 0$ take a *chain*

$x_0 = x, x_1, ..., x_n = x'$, so that $x_i \in C$ and $d(x_i, x_{i+1}) \leq \delta$, $i = 0, ..., n-1$. Then we obtain a chain $y_i = f(x_i)$, $i = 0, ..., n$, so that

$$d(y_i, y_{i+1}) \leq L\delta + A$$

It follows that a geodesic segment $\overline{y_i y_{i+1}}$ is contained in

$$B_{L\delta+A}(f(C)).$$

Hence, the $L\delta + A$-neighborhood of $f(C)$ is path-connected for every δ. Taking $\delta = L^{-1}$ we conclude that $B_R(f(C))$ is connected.

Thus we define a map $\epsilon(f) : \epsilon(X) \to \epsilon(Y)$ as follows. Set $R := A + 1$. Suppose that $\eta \in \epsilon(X)$ is represented by a nested sequence (U_i), where U_i is a path-connected component of $X \setminus K_i$, $K_i \subset X$ is compact. Without loss of generality we may assume that for each i, $B_R(U_i) \subset U_{i-1}$. Thus we get a nested sequence of connected subsets $B_R(f(U_i)) \subset Y$ each of which is contained in a connected component V_i of the complement to the bounded subset $f(K_{i-1}) \subset Y$. Thus we send η to the end $\epsilon(f)(\eta)$ represented by (V_i). By considering the quasi-inverse $\bar{f}$ to f, we see that $\epsilon(f)$ has the inverse map $\epsilon(\bar{f})$. It is also clear from the construction that both $\epsilon(f)$ and $\epsilon(\bar{f})$ are continuous. $\square$

If G is a finitely generated group then the space of ends $\epsilon(G)$ is defined to be the set of ends of its Cayley graph. The previous lemma implies that $\epsilon(G)$ does not depend on the choice of a finite generating set.

Theorem 2.5. *Properties of $\epsilon(X)$:*

1. $\epsilon(X)$ is compact, Hausdorff and totally disconnected.

2. Suppose that G is a finitely-generated group. Then $\epsilon(G)$ consists of 0, 1, 2 points or of continuum of points. In the latter case the set $\epsilon(G)$ is perfect: Each point is a limit point.

3. $\epsilon(G)$ is empty iff G is finite. $\epsilon(G)$ consists of 2-points iff G is virtually (infinite) cyclic.

4. $|\epsilon(G)| > 1$ iff G splits nontrivially over a finite subgroup.

Properties 1–3 are relatively trivial, see for instance [**15**, Theorem 8.32] for the detailed proofs. The hard part of this theorem is

Theorem 2.6. *If $|\epsilon(G)| > 1$ then G splits nontrivially over a finite subgroup.*

This theorem is due to Stallings [**97**] (in the torsion-free case) and Bergman [**8**] for groups with torsion. To this day, there is no simple proof of this result. We refer to [**59**] for a proof of Stallings' theorem using harmonic functions, along the lines sketched by Gromov; another geometric proof could be found in Niblo's paper [**81**]. For finitely-presented groups, there is an easy combinatorial proof due to Dunwoody using minimal tracks, [**28**]. A combinatorial argument could be found in [**27**].

Corollary 2.7. *1. Suppose that G is quasi-isometric to $\mathbb{Z}$ then G contains $\mathbb{Z}$ as a finite index subgroup.*

2. Suppose that G splits nontrivially as $A * B$ and G' is quasi-isometric to G. Then G' splits nontrivially as $H *_F E$ (amalgamated product) or as $H *_F$ (HNN splitting) where F is a finite group.

For hyperbolic groups one relates ends and the boundary as follows:

Theorem 2.8. *Suppose that G is a hyperbolic group. Then there exists a continuous G-equivariant surjection*

$$\sigma : \partial_\infty G \to \epsilon(G)$$

such that the preimages $\sigma^{-1}(\xi)$ are connected components of $\partial_\infty G$.

3. Rips complexes and coarse connectedness

Let X be a metric space of bounded geometry, $R \in \mathbb{R}_+$. Then the *R-Rips complex* $\mathrm{Rips}_R(X)$ is the simplicial complex whose vertices are points of X; vertices $x_1, ..., x_n$ span a simplex iff $d(x_i, x_j) \le R$ for each i, j. Note that the system of Rips complexes of X is a direct system $\mathrm{Rips}_\bullet(X)$ of simplicial complexes:

For each pair $0 \le r \le R < \infty$ we have a natural simplicial embedding $\iota_{r,R} : \mathrm{Rips}_r(X) \to \mathrm{Rips}_R(X)$, so that $\iota_{r,\rho} = \iota_{R,\rho} \circ \iota_{r,R}$ provided that $r \le R \le \rho$.

One can metrize $\mathrm{Rips}_R(X)$ by declaring each simplex to be isometric to the standard Euclidean simplex. Note that the assumption that X has bounded geometry implies that $\mathrm{Rips}_R(X)$ is finite-dimensional for each R. Moreover, $\mathrm{Rips}_R(X)$ is a metric simplicial complex of bounded geometry.

If one omits the assumption that X has bounded geometry, then the above construction yields only a metric cell complex.

The following simple observation explains why Rips complexes are useful for analyzing quasi-isometries:

LEMMA 2.9. *Let $f : X \to Y$ be an (L, A)-coarse Lipschitz map. Then f induces a simplicial map $\mathrm{Rips}_d(X) \to \mathrm{Rips}_{Ld+A}(Y)$ for each $d \ge 0$.*

PROOF. Consider an m-simplex σ in $\mathrm{Rips}_d(X)$, the vertices of σ are distinct points $x_0, x_1, ..., x_m \in X$ within distance $\le d$ from each other. Since f is (L, A)-coarse Lipschitz, the points $f(x_0), ..., f(x_m) \in Y$ are within distance $\le Ld + A$ from each other, hence they span a simplex σ' of dimension $\le m$ in $\mathrm{Rips}_{Ld+A}(Y)$. The map f sends vertices of σ to vertices of σ', extend this map linearly to the simplex σ. It is clear that this extension defines a simplicial map of simplicial complexes $\mathrm{Rips}_d(X) \to \mathrm{Rips}_{Ld+A}(Y)$. ☐

The idea behind the next definition is that the "coarse homotopy groups" of a metric space X are the homotopy groups of the Rips complexes $\mathrm{Rips}_R(X)$ of X. Literally speaking, this does not make much sense since the above homotopy groups depend on R. To eliminate this dependence, we have to take into account the maps $\iota_{r,R}$.

DEFINITION 2.10. A metric space X is **coarsely k-connected** if for each r there exists $R \ge r$ so that the mapping $\mathrm{Rips}_r(X) \to \mathrm{Rips}_R(X)$ induces trivial maps of the i-th homotopy groups π_i for $0 \le i \le k$.

For instance, X is coarsely 0-connected if there exists a number R such that each pair of points $x, y \in X$ can be connected by an R-chain of points $x_i \in X$, i.e., a finite sequence of points x_i, where $d(x_i, x_{i+1}) \le R$ for each i. Note that for $k \le 1$ coarse k-connectedness of X is equivalent to the property that $\mathrm{Rips}_R(X)$ is k-connected for sufficiently large R.

The above definition is not quite satisfactory since it only deals with "vanishing" of coarse homotopy groups without actually defining these groups. One way to deal with this issue is to consider *pro-groups* which are direct systems

$$\pi_i(\mathrm{Rips}_r(X)), r \in \mathbb{N}$$

of groups. Given such algebraic objects, one can define their *pro-homomorphisms*, *pro-monomorphisms*, etc., see **[62]** where this is done in the category of abelian groups (which appear as homology groups).

Properties of the direct system of Rips complexes:

LEMMA 2.11. *For $r, C < \infty$, each simplicial spherical cycle σ of diameter $\le C$ in Rips_r bounds a disk of diameter $\le C + d$ within Rips_{r+C}.*

PROOF. Pick a point $x \in \sigma$. Then Rips_{r+C} contains a simplicial cone $\beta(\sigma)$ over σ with the origin at x. Clearly, $\mathrm{diam}(\beta) \le r + C$. ☐

Corollary 2.12. *Let $f, g : X \to Y$ be maps within distance $\leq C$ from each other, which extend to simplicial maps*

$$f, g : \mathrm{Rips}_{d_1}(X) \to \mathrm{Rips}_{d_2}(Y)$$

Then for $d_3 = d_2 + C$, the maps $f, g : \mathrm{Rips}_{d_1} \to \mathrm{Rips}_{d_3}(Y)$ are homotopic via a homotopy whose tracks have lengths $\leq D = D(X, d_1)$.

Proof. We give the product $\mathrm{Rips}_{d_1}(X) \times I$ the *standard* structure of a simplicial complex with the vertex set $X \times \{0, 1\}$ (by triangulating the prisms $\sigma \times I$).

We now construct a homotopy F on $\mathrm{Rips}_{d_1}(X) \times I$ by induction on skeleta (cf. proof of Lemma 2.3). The map of the zero-skeleton is, of course, $F(x, 0) = f(x), F(x, 1) = g(x)$. Suppose that a map of the i-skeleton

$$F : (\mathrm{Rips}_{d_1}(X))^i \times I \to \mathrm{Rips}_{r_i}(Y)$$

is constructed. Let $\sigma \subset \mathrm{Rips}_{d_1}(X)$ be an $i+1$-simplex. Then $\mathrm{diam}(F(\sigma^0)) \leq d_3 = d_2 + C$. Therefore, F extends (linearly) to a map $F : \sigma \to \mathrm{Rips}_{d_3}(Y)$ whose image is a simplex. To estimate the lengths of the tracks of the homotopy F, we note that for each $x \in X$, $\mathrm{length}(F(x \times I)) \leq 1$ since $F(x \times I)$ is a point or the 1-simplex in $\mathrm{Rips}_{d_3}(Y)$ connecting $f(x), g(x)$. Now, the assertion follows from the fact that each prism $\sigma \times I \subset \mathrm{Rips}_{d_1}(X) \times I$ contains at most $D = D(X, d_1)$ simplices. □

In view of the above lemma, we say that maps $f, g : X \to Y$ are *coarsely homotopic* if they satisfy the conclusion of the above lemma. We then say that a map $f : X \to Y$ determines a *coarse homotopy equivalence* between the direct systems of Rips complexes of X, Y if there exists a map $g : Y \to X$ so that the compositions $g \circ f, f \circ g$ are coarsely homotopic to the identity maps, i.e., for each simplicial map f : ...

Corollary 2.13. *Suppose that $f, g : X \to Y$ be L-Lipschitz maps within finite distance from each other. Then they induce coarsely homotopic maps $\mathrm{Rips}_d(X) \to \mathrm{Rips}_{Ld}(Y)$ for each $d \geq 0$.*

Corollary 2.14. *if $f : X \to Y$ is a quasi-isometry, then f induces a coarse homotopy-equivalence of the Rips complexes: $\mathrm{Rips}_\bullet(X) \to \mathrm{Rips}_\bullet(Y)$.*

Corollary 2.15. *Coarse k-connectedness is a QI invariant.*

Proof. Suppose that X' is coarsely k-connected and $f : X \to X'$ is an L-Lipschitz quasi-isometry with L-Lipschitz quasi-inverse $\bar{f} : X' \to X$. Let γ be a spherical i-cycle in $\mathrm{Rips}_d(X)$, $0 \leq i \leq k$. Then we have the induced spherical i-cycle $f(\gamma) \subset \mathrm{Rips}_{Ld}(X')$. Since X' is coarsely k-connected, there exists $d' \geq Ld$ such that $f(\gamma)$ bounds a singular $i+1$-disk β within $\mathrm{Rips}_{d'}(X')$. Consider now $\bar{f}(\beta) \subset \mathrm{Rips}_{L^2 d}(X)$. The boundary of this singular disk is a singular i-sphere $\bar{f}(\gamma)$. Since $\bar{f} \circ f$ is homotopic to id within $\mathrm{Rips}_{d''}(X)$, $d'' \geq L^2 d$, there exists a singular cylinder σ in $\mathrm{Rips}_{d''}(X)$ which cobounds γ and $\bar{f}(\gamma)$. Note that d'' does not depend on γ. By combining σ and $\bar{f}(\beta)$ we get a singular $i+1$-disk in $\mathrm{Rips}_{d''}(X)$ whose boundary is γ. Hence X is coarsely k-connected. □

Our next goal is to find a large supply of examples of metric spaces which are coarsely k-connected.

Definition 2.16. A bounded geometry metric cell complex X is said to be *uniformly k-connected* if there is a function $\psi(k, r)$ such that for each $i \leq k$, each singular i-sphere of diameter $\leq r$ in $X^{(i+1)}$ bounds a singular $i+1$-disk of diameter $\leq \psi(k, r)$.

For instance, if X is a finite-dimensional contractible complex which admits a cocompact cellular group action, then X is uniformly k-connected for each k.

Here is an example of a simply-connected complex which is not uniformly simply-connected. Take $S^1 \times \mathbb{R}_+$ with the product metric and attach to this complex a 2-disk along the circle $S^1 \times \{0\}$.

Theorem 2.17. *Suppose that X is a metric cell complex of bounded geometry such that X is uniformly n-connected. Then $Z := X^{(0)}$ is coarsely n-connected.*

Proof. Let $\gamma : S^k \to \text{Rips}_R(Z)$ be a spherical m-cycle in $\text{Rips}_R(Z)$, $0 \le k \le n$. Without loss of generality (using simplicial approximation) we can assume that γ is a simplicial cycle, i.e. the sphere S^k is given a triangulation τ so that γ sends simplices of S^k to simplices in $\text{Rips}_R(Z)$ so that the restriction of γ to each simplex is a linear map. Let Δ_1 be a k-simplex in S^k. Then $\gamma(\Delta_1)$ is spanned by points $x_1, ..., x_{k+1} \in Z$ which are within distance $\le R$ from each other. Since X is uniformly k-connected, there is a singular k-disk $\gamma_1(\Delta_1)$ containing $x_1, ..., x_{k+1}$ and having diameter $\le R'$, where R' depends only on R. Namely, we construct γ_1 by induction on skeleta: First connect each pair of points x_i, x_j by a path in X (of length bounded in terms of R), this defines the map γ_1 on the 1-skeleton of Δ_1. Then continue inductively. This construction ensures that if Δ_2 is a k-simplex in S^k which shares an m-face with Δ_1 then γ_2 and γ_1 agree on $\Delta_1 \cap \Delta_2$. As the result, we have "approximated" γ by a singular spherical k-cycle $\gamma' : S^k \to X^{(k)}$ (the restriction of γ' to each Δ_i equals γ_i). See figure 2 in the case $k = 1$.

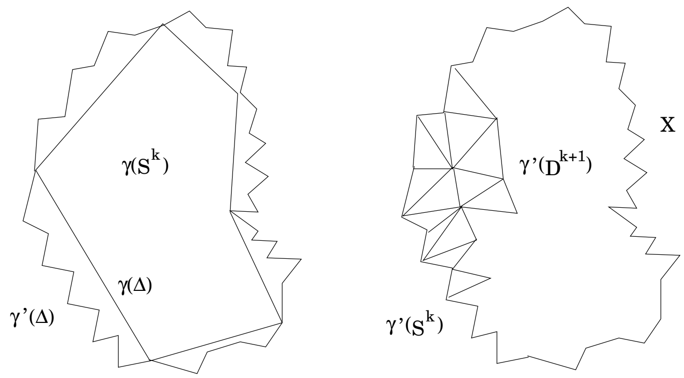

Figure 2.

Since X is k-connected, the map γ' extends to a cellular map $\gamma' : D^{k+1} \to X^{(k+1)}$. Let D denote the maximal diameter of a $k+1$-cell in X. For each simplex $\sigma \subset D^{k+1}$ the diameter of $\gamma'(\sigma)$ is at most D. We therefore can "push" the singular disk $\gamma'(D^{k+1})$ into $\text{Rips}_D(Z)$ by replacing each linear map $\gamma' : \sigma \to \gamma'(\sigma) \subset X$ with the linear map $\gamma'' : \sigma \to \gamma''(\sigma) \subset \text{Rips}_D(Z)$ where $\gamma''(\sigma)$ is the simplex spanned by the vertices of $\gamma'(\sigma)$. This yields a map $\gamma'' : D^{k+1} \to \text{Rips}_D(Z)$. Observe that the map γ'' is a cellular map with respect to a subdivision τ' of the initial triangulation τ of S^k.

Note however that γ and $\gamma''|S^k$ are different maps. Let V denote the vertices of a k-simplex $\Delta \subset S^k$; let V'' denote the set of vertices of τ' within the simplex Δ. Then the diameter of $\gamma''(V'')$ is at most R'. Hence $\gamma(V) \subset \gamma''(V)$ is contained in a simplex in $\text{Rips}_{R+R'}(Z)$. Therefore, by taking $\rho = R + D + R'$ we conclude that the maps $\gamma, \gamma'' : S^k \to \text{Rips}_\rho(Z)$ are homotopic. See Figure 3. Thus the map γ is nil-homotopic within $\text{Rips}_\rho(Z)$. $\square$

Corollary 2.18. *Suppose that G is a finitely-presented group with the word metric. Then G is coarsely simply-connected.*

Corollary 2.19. *(See for instance* [**15**, Proposition 8.24]*) Finite presentability is a QI invariant.*

Proof. It remains to show that each coarsely 1-connected group G is finitely presentable. The Rips complex $X := \text{Rips}_R(G)$ is 1-connected for large R. The group G acts on X properly discontinuously and cocompactly. Therefore G is finitely presentable. $\square$

Definition 2.20. A group G is said to be of type F_n ($n \le \infty$) if its admits a cellular action on a cell complex X such that for each $k \le n$: (1) $X^{(k+1)}/G$ is compact. (2) $X^{(k+1)}$ is k-connected. (3) The action $G \curvearrowright X$ is free.

Example 34. (See [**9**].) Let $\mathbb{F}_2$ be free group on 2 generators a, b. Consider the group $G = \mathbb{F}_2^n$ which is the direct product of $\mathbb{F}_2$ with itself n times. Define a homomorphism $\phi : G \to \mathbb{Z}$ which sends each generator a_i, b_i of G to the same generator of $\mathbb{Z}$. Let $K := Ker(\phi)$. Then K is of type F_{n-1} but not of type F_n.

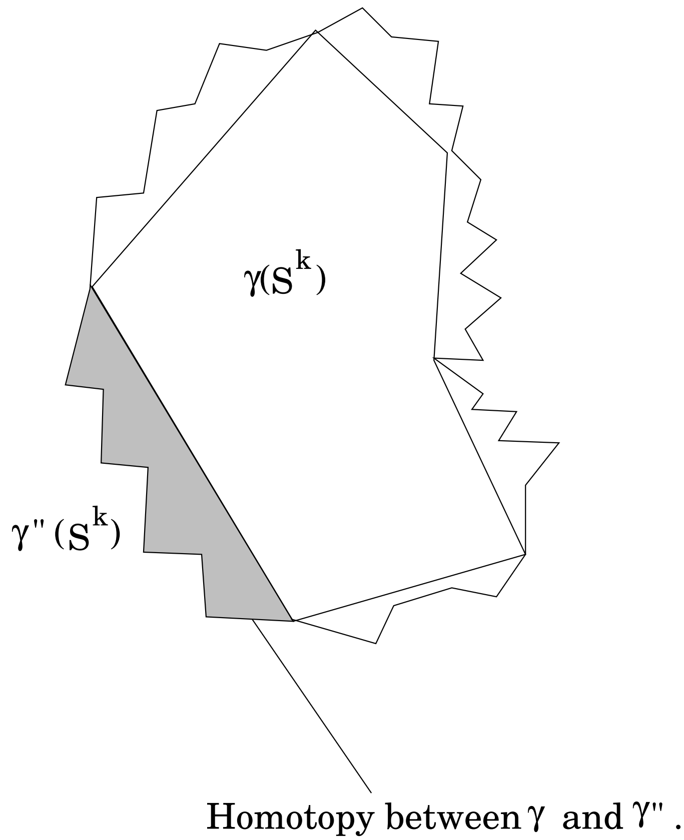

FIGURE 3.

Thus, analogously to Corollary 2.19 we get:

THEOREM 2.21. *(See* [**50**, 1.C2]*) Type F_n is a QI invariant.*

PROOF. It remains to show that each coarsely n-connected group has type F_n. The proof below follows [**61**]. We build the complex X on which G would act as required by the definition of type F_n. We build this complex and the action by induction on skeleta.

(0). $X^{(1)}$, is a Cayley graph of G; the action of G is cocompact, free, cellular.

(i⇒ i+1). Suppose that $X^{(i)}$ has been constructed. Using i-connectedness of $\mathrm{Rips}_\bullet(G)$ we construct (by induction on skeleta) a G-equivariant cellular map $f : X^{(i)} \to \mathrm{Rips}_D(G)$ for a sufficiently large D. If G were torsion-free, the action $G \curvearrowright \mathrm{Rips}_D(G)$ is free; this allows one to we construct (by induction on skeleta) a G-equivariant "retraction" $\rho : \mathrm{Rips}_D(G)^{(i)} \to X^{(i)}$, i.e. a map such that the composition $\rho \circ f$ is G-equivariantly homotopic to the identity.

However, if G contains nontrivial elements of finite order, we have to use a more complicated construction.

Suppose that $2 \le i \le n$ and an $i-1$-connected complex $X^{(i)}$ together with a free discrete cocompact action $G \curvearrowright X^{(i)}$ was constructed. Let $x_0 \in X^{(0)}$ be a base-point.

LEMMA 2.22. *There are finitely many spherical i-cycles $\sigma_1, ..., \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_1(X^{(i)})$, in the sense that the normal closure of the cycles $\{g\hat{\sigma}_j : j = 1, ..., k, g \in G\}$ is $\pi_i(X^{(i)})$, where each $\hat{\sigma}_j$ is obtained from σ_j by attaching a "tail" from x_0.*

PROOF. Without loss of generality we can assume that $X^{(i)}$ is a (metric) simplicial complex. Let $f : X^{(i)} \to Y := \mathrm{Rips}_D(Z)$ be a G-equivariant continuous map as above.

Here is the construction of σ_j's:

Let $\tau_\alpha : S^i \to Y^{(i)}, \alpha \in \mathbb{N}$, denote the attaching maps of the $i+1$-cells in Y, these maps are just simplicial homeomorphic embeddings from the boundary S^i of the standard $i+1$-simplex into $Y^{(i)}$. Starting with a G-equivariant projection $Y^{(0)} \to X^{(0)}$ one inductively constructs a (non-equivariant!) map $\bar{f} : Y^{(i)} \to X^{(i)}$ so that $f \circ \bar{f} : Y^{(i)} \to Y^{(i+1)}$ is within distance $\leq Const$ from the identity. Hence (by coarse connectedness of Z) this composition is homotopic to the identity inclusion within $\mathrm{Rips}_{D'}(Z)$. The homotopy H is such that its tracks have "uniformly bounded complexity", i.e. the compositions

$$H \circ (\tau_\alpha \times id) : S^i \times I \to \mathrm{Rips}_{D'}(Z)$$

are simplicial maps with a uniform upper bound on the number of simplices in a triangulation of $S^i \times I$. Let $B \subset X^{(i)}$ denote a compact subset such that $GB = X^{(i)}$. We let σ_j denote the composition $g_\alpha \circ \bar{f} \circ \tau_\alpha$ where $g_\alpha \in G$ are chosen so that the image of σ_j intersects B. $\square$

We now equivariantly attach $i+1$-cells along G-orbits of the cycles σ_j: for each j and $g \in G$ we attach an $i+1$-cell along $g(\sigma_j)$. Note that if σ_j is stabilized by a subgroup of order $m = m(j)$ in G, then we attach m copies of the $i+1$-dimensional cell along σ_j. We let $X^{(i+1)}$ denote the resulting complex and we extend the G-action to $X^{(i+1)}$ in obvious fashion. It is clear that $G \curvearrowright X^{(i+1)}$ is free, discrete and cocompact. $\square$

4. Coarse separation

Suppose that X is a metric cell complex and $Y \subset X$ is a subset. We let $N_R(Y)$ denote the metric R-neighborhood of Y in X. Let C be a complementary component of $N_R(Y)$ in Y. Define the *inradius*, $inrad(C)$, of C to be the supremum of radii of metric balls in X contained in C. A component C is called *shallow* if $inrad(C)$ is $< \infty$ and *deep* if $inrad(C) = \infty$.

EXAMPLE 35. Suppose that Y is compact. Then deep complementary components of $X \setminus N_R(Y)$ are those components which have infinite diameter.

A subcomplex Y is said to *coarsely separate* X if there is R such that $N_R(Y)$ has at least two distinct deep complementary components.

EXAMPLE 36. The curve Γ in $\mathbb{R}^2$ does not coarsely separate $\mathbb{R}^2$. A straight line in $\mathbb{R}^2$ coarsely separates $\mathbb{R}^2$.

THEOREM 2.23. *Suppose that Y, X be uniformly contractible metric cell complexes of bounded geometry which are homeomorphic to $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ respectively. Then for each uniformly proper map $f : Y \to X$, the image $f(Y)$ coarsely separates X. Moreover, the number of deep complementary components is 2.*

PROOF. Actually, our proof will use the assumption on the topology of Y only weakly: to get coarse separation it suffices to assume that $H^{n-1}_c(Y, \mathbb{R}) \neq 0$.

Let $W := f(Y)$. Given $R \in \mathbb{R}_+$ we define a *retraction* $\rho : N_R(W) \to Y$, so that $d(\rho \circ f, id_Y) \leq$ *const*, where *const* depends only on the distortion function of f and on the geometry of X and Y. Here $\mathcal{N}_R(W)$ is the smallest subcomplex in X containing the R-neighborhood of W in X. We define ρ by induction on skeleta of $N_R(W)$. For each vertex $x \in \mathcal{N}_R(W)$ we pick a vertex $\rho(x) := y \in Y$ such that the distance $d(x, f(y))$ is the smallest possible. If there are several such points y, we pick one of them arbitrarily. The fact that f is a uniform proper embedding ensures that

$$d(\rho \circ f, id_{Y^0}) \leq const_0.$$

Note also that for any 1-cell σ in $\mathcal{N}_R(W)$, $\mathrm{diam}(\rho(\partial\sigma)) \leq Const_0$. Suppose that we have constructed ρ on $\mathcal{N}^{(k)}_R(W)$. Inductively we assume that:

$$d(\rho \circ f, id_{Y^k}) \leq const_k, \mathrm{diam}(\rho(\partial\sigma)) \leq Const_k, \tag{37}$$

for each $k+1$-cell σ. We extend ρ to the $k+1$-skeleton by using uniform contractibility of Y: For each $k+1$-cell σ there exists a singular disk $\eta : D^{k+1} \to Y$ in Y^{k+1} of diameter $\leq \psi(Const_k)$ whose boundary is $\rho(\partial\sigma)$. Then we extend ρ to σ via η. It is clear that the extension satisfies the inequalities (37) with k replaced with $k+1$.

Since Y is uniformly contractible we get a homotopy $\rho \circ f \cong id_Y$, whose tracks are uniformly bounded (construct it by induction on skeleta the same way as before).

Recall that we have a system of isomorphisms

$$P : H_c^{n-1}(\mathcal{N}_r) \cong H_1(X, X \setminus \mathcal{N}_r)$$

given by the Poincaré duality in $\mathbb{R}^n$. This isomorphism moves support sets of $n-1$-cocycles by a uniformly bounded amount (to support sets of 1-cycles). Let ω be a generator of $H_c^{n-1}(Y)$. Given $R > 0$ consider "retraction" ρ as above and the pull-back $\omega_R := \rho^*(\omega)$. If for some $0 < r < R$ the restriction ω_r of ω_R to $\mathcal{N}_r(W)$ is zero then we get a contradiction, since $f^* \circ \rho^* = id$ on the compactly supported cohomology of Y. Thus ω_r is nontrivial. Applying the Poincaré duality operator P to the cohomology class ω_r we get a nontrivial relative homology class

$$P(\omega_r) \in H_1(X, X \setminus \mathcal{N}_r) \cong \tilde{H}_0(X \setminus \mathcal{N}_r).$$

We note that for each $R \geq r$ the class $P(\omega_r) \in H_1(\mathcal{N}_r, \partial \mathcal{N}_r)$ is represented by "restriction" of the class $P(\omega_R) \in H_1(\mathcal{N}_R, \partial \mathcal{N}_R)$ to $\mathcal{N}_r$, see Figure 4. In particular, the images α_r, α_R of $P(\omega_r), P(\omega_R)$ in $\tilde{H}_0(X \setminus \mathcal{N}_r)$, $\tilde{H}_0(X \setminus \mathcal{N}_R)$ are homologous in $\tilde{H}_0(X \setminus \mathcal{N}_r)$. Moreover, α_R restricts nontrivially to $\alpha_1 \in \tilde{H}_0(X \setminus \mathcal{N}_1)$.

Therefore, we get sequences of points

$$x_i, x_i' \in \partial \mathcal{N}_i, i \in \mathbb{N},$$

such that x_i, x_i' belong to the support sets of α_i for each i, x_i, x_{i+1} belong to the same component of $X \setminus \mathcal{N}_i$, x_i', x_{i+1}' belong to the same component of $X \setminus \mathcal{N}_i$, but the points x_i, x_i' belong to distinct components C, C' of $X \setminus \mathcal{N}_1$. It follows that C, C' are distinct deep complementary components of W. The same argument run in the reverse implies that there are exactly two deep complementary components (although we will not use this fact). $\square$

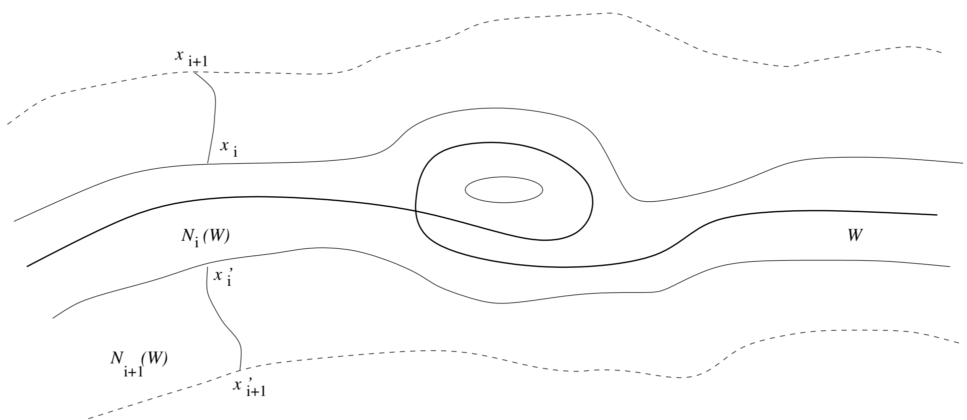

Figure 4. *Coarse separation.*

I refer to **[38]**, **[62]** for further discussion and generalization of coarse separation and coarse Poincare/Alexander duality.

5. Other notions of coarse equivalence

6. Topological coupling

We first introduce Gromov's interpretation of quasi-isometry between groups using the language of topological actions.

Given groups G_1, G_2, a *topological coupling* of these groups is a metrizable locally compact topological space X together with two commuting cocompact properly discontinuous topological actions $\rho_i : G_i \curvearrowright X, i = 1, 2$. (The actions commute iff $\rho_1(g_1)\rho_2(g_2) = \rho_2(g_2)\rho_1(g_1)$ for all $g_i \in G_i$, $i = 1, 2$.) Note that the actions ρ_i are not required to be isometric.

THEOREM 2.24. *(M. Gromov, [**50**], see also de la Harpe [**25**, page 98]) If G_1, G_2 are finitely-generated groups, then G_1 is QI to G_2 iff there exists a topological coupling between these groups.*

PROOF. 1. Suppose that G_1 is QI to G_2. Then there exists an (L, A) quasi-isometry $q : G_1 \to G_2$. Without loss of generality, we may assume that q is L-Lipschitz. Consider the space X of such maps $G_1 \to G_2$. We will give X the topology of pointwise convergence. This topology is metrizable. By Arcela-Ascoli theorem, X is locally compact. The groups G_1, G_2 act on X as follows:

$$\rho_1(g_1)(f) := f \circ g_1^{-1}, \quad \rho_2(g_2)(f) := g_2 \circ f, \quad f \in X.$$

It is clear that these actions commute and are topological. For each $f \in X$ there exist $g_1 \in G_1, g_2 \in G_2$ so that

$$g_2 \circ f(1) = 1, f \circ g_1^{-1}(1) \in B_A(1).$$

Therefore, by Arcela-Ascoli theorem, both actions are cocompact. We will check that ρ_2 is properly discontinuous as the case of ρ_1 is analogous. Let $K \subset X$ be a compact subset. Then there exists $R < \infty$ so that for every $f \in K$, $f(1) \in B_R(1)$. If $g_2 \in G_2$ is such that $g_2 \circ f \in K$ for some $f \in K$, then

$$g_2(B_R(1)) \cap B_R(1) \neq \emptyset. \tag{38}$$

Since the action of G_2 on itself is free, it follows that the collection of $g_2 \in G_2$ satisfying (38) is finite. Hence, ρ_2 is properly discontinuous.

2. Suppose that X is a topological coupling of G_1 and G_2. If X were a geodesic metric space and the actions of G_1, G_2 were isometric, we would not need commutation of these action. However, there are examples of QI groups which do not act geometrically on the same geodesic metric space, see see below. However, the construction of a quasi-isometry is pretty much the same as in the proof of Milnor-Schwarz lemma.

Pick a point $p \in X$ and a compact $K \subset X$ so that $G_i \cdot K = X$ and $p \in K$. Then for each $g_i \in G_i$ there exists $\phi_i(g_i) \in G_{i+1}$ so that $g_i(p) \in \phi_i(g_i)(K)$, here and below i is taken mod 2. We have maps $\phi_i : G_i \to G_{i+1}$.

a. Let's check that these maps are Lipschitz. Let $s \in S_i$, a finite generating set of G_i, we will use the word metric on G_i with respect to S_i, $i = 1, 2$. Define C to be the union

$$\cup_{i=1,2} \bigcup_{s \in S_i} s(K).$$

Since ρ_i are properly discontinuous actions, the sets $G_i^C := \{h \in G_i : h(C) \cap C \neq \emptyset\}$ are finite for $i = 1, 2$. Therefore, the word-lengths of the elements of these sets are bounded by some $L < \infty$. Suppose now that $g_{i+1} = \phi_i(g_i)$, $s \in S_i$. Then $g_i(p) \in g_{i+1}(K), sg_i(p) \in g'_{i+1}(K)$ for some $g'_{i+1} \in G_{i+1}$. Therefore, $sg_{i+1}(K) \cap g'_{i+1}(K) \neq \emptyset$ and, hence, $g_{i+1}^{-1} g'_{i+1}(K) \cap s(K) \neq \emptyset$. (This is where we are using the fact that the actions of G_1 and G_2 on X commute.) Therefore, $g_{i+1}^{-1} g'_{i+1} \in G_{i+1}^C$ and, hence, $d(g_{i+1}, g'_{i+1}) \leq L$. Hence, ϕ_i is L-Lipschitz.

b. Let $\phi_i(g_i) = g_{i+1}, \phi_{i+1}(g_{i+1}) = g'_i$. Then $g_i(K) \cap g'_i(K) \neq \emptyset$ and, hence, $g_i^{-1} g'_i \in G_i^C$. Therefore, $d(\phi_{i+1}\phi_i, Id) \leq L$ and $\phi_i : G_i \to G_{i+1}$ is a quasi-isometry. ☐

The more useful direction of this theorem is, of course, from QI to a topological coupling, see e.g. [**96, 92**].

DEFINITION 2.25. Groups G_1, G_2 are said to **have a common geometric model** if there exists a proper geodesic metric space X such that G_i, G_2 both act geometrically on X.

In view of Lemma 1.43, if groups have a common geometric model then they are quasi-isometric. The following theorem shows that the converse is false:

THEOREM 2.26. *(L. Mosher, M. Sageev, K. Whyte, [**76**]) Let $G_1 := \mathbb{Z}_p * \mathbb{Z}_p, G_2 := \mathbb{Z}_q * \mathbb{Z}_q$, where p, q are distinct primes. Then the groups G_1, G_2 do not have a common geometric model.*

This theorem in particular implies that in Theorem 2.24 one cannot assume that both group actions are isometric.

Spaces (or finitely generated groups) X_1, X_2 are *bilipschitz equivalent* if there exists a bilipschitz bijection $f : X_1 \to X_2$.

THEOREM 2.27. *(K. Whyte, [**109**]) Suppose that G_1, G_2 are non-amenable finitely generated groups which are quasi-isometric. Then G_1, G_2 are bilipschitz equivalent.*

On the other hand, there are examples of T. Dymarz [**29**] of solvable groups which are quasi-isometric but not bilipschitz equivalent. See also section 8.

CHAPTER 3

Ultralimits of Metric Spaces

Let (X_i) be a sequence of metric spaces. One can describe the limiting behavior of the sequence (X_i) by studying limits of sequences of finite subsets $Y_i \subset X_i$. Ultrafilters are an efficient technical device for simultaneously taking limits of all such sequences of subspaces and putting them together to form one object, namely an ultralimit of (X_i).

1. Ultrafilters

Let I be an infinite set, $\mathcal{S}$ is a collection of subsets of I. A *filter based on* $\mathcal{S}$ is a nonempty family ω of members of $\mathcal{S}$ with the properties:

- $\emptyset \notin \omega$.
- If $A \in \omega$ and $A \subset B$, then $B \in \omega$.
- If $A_1, \dots, A_n \in \omega$, then $A_1 \cap \cdots \cap A_n \in \omega$.

If $\mathcal{S}$ consists of *all* subsets of I we will say that ω is a filter on I. Subsets $A \subset I$ which belong to a filter ω are called ω-large. We say that a property (P) holds for ω-*all* i, if (P) is satisfied for all i in some ω-large set. An *ultrafilter* is a maximal filter. The maximality condition can be rephrased as: For every decomposition $I = A_1 \cup \cdots \cup A_n$ of I into finitely many disjoint subsets, the ultrafilter contains exactly one of these subsets.

For example, for every $i \in I$, we have the *principal* ultrafilter δ_i defined as $\delta_i := \{A \subset I \mid i \in A\}$. An ultrafilter is principal if and only if it contains a finite subset. The interesting ultrafilters are of course the non-principal ones. They cannot be described explicitly but exist by Zorn's lemma: Every filter is contained in an ultrafilter. Let $\mathcal{Z}$ be the *Zariski filter* which consists of complements to finite subsets in I. An ultrafilter is a nonprincipal ultrafilter, if and only if it contains $\mathcal{Z}$.

Here is an alternative interpretation of ultrafilters. An ultrafilter is a finitely additive measure defined on all subsets of I so that each subset has measure 0 or 1. An ultrafilter is nonprincipal iff the measure contains no atoms: The measure of each point is zero.

Given an ultrafilter ω on I and a collection of sets $X_i, i \in I$, define the *ultraproduct*

$$\prod_{i \in I} X_i / \omega$$

to be the collection of equivalence classes of maps $f : I \to X$ such that $f \sim g$ iff $f(i) = g(i)$ for ω-all i.

Given a function $f : I \to Y$ (where Y is a topological space) define the ω-limit

$$\omega\text{-}\lim_i f(i)$$

to be a point $y \in Y$ such that for every neighborhood U of y the preimage $f^{-1}U$ belongs to ω.

LEMMA 3.1. *Suppose that Y is compact and Hausdorff. Then for each function $f : I \to Y$ the ultralimit exists and is unique.*

PROOF. To prove existence of a limit, assume that there is no point $y \in Y$ satisfying the definition of the ultralimit. Then each point $z \in Y$ possesses a neighborhood U_z such that $f^{-1}U_z \notin \omega$. By compactness, we can cover Y with finitely many of these neighborhoods. It follows that $I \notin \omega$. This contradicts the definition of a filter. Uniqueness of the point y follows, because Y is Hausdorff. $\square$

Note that if y is an accumulation point of $\{f(i)\}_{i\in I}$ then there is a non-principal ultrafilter ω with ω-$\lim f = y$, namely an ultrafilter containing the pullback of the neighborhood basis of y.

2. Ultralimits of metric spaces

Let $(X_i)_{i\in I}$ be a family of metric spaces parameterized by an infinite set I. For an ultrafilter ω on I we define the ultralimit

$$X_\omega = \omega\text{-}\lim_i X_i$$

as follows. Let $\prod_i X_i$ be the product of the spaces X_i, i.e. it is the space of sequences $(x_i)_{i\in I}$ with $x_i \in X_i$. The distance between two points $(x_i), (y_i) \in \prod_i X_i$ is given by

$$d_\omega\big((x_i),(y_i)\big) := \omega\text{-}\lim\big(i \mapsto d_{X_i}(x_i,y_i)\big)$$

where we take the ultralimit of the function $i \mapsto d_{X_i}(x_i,y_i)$ with values in the compact set $[0,\infty]$. The function d_ω is a pseudo-distance on $\prod_i X_i$ with values in $[0,\infty]$. Set

$$(X_\omega, d_\omega) := (\prod_i X_i, d_\omega)/\sim$$

where we identify points with zero d_ω-distance.

Exercise 39. Let $X_i = Y$ for all i, where Y is a compact metric space. Then $X_\omega \cong Y$ for all ultrafilters ω.

If the spaces X_i do not have uniformly bounded diameter, then the ultralimit X_ω decomposes into (generically uncountably many) components consisting of points of mutually finite distance. We can pick out one of these components if the spaces X_i have base-points x_i^0. The sequence $(x_i^0)_i$ defines a base-point x_ω^0 in X_ω and we set

$$X_\omega^0 := \big\{x_\omega \in X_\omega \mid d_\omega(x_\omega, x_\omega^0) < \infty\big\}.$$

Define the *based ultralimit* as

$$\omega\text{-}\lim_i (X_i, x_i^0) := (X_\omega^0, x_\omega^0).$$

Example 40. For every locally compact space Y with a base-point y_0, we have:

$$\omega\text{-}\lim_i (Y, y_0) \cong (Y, y_0).$$

Lemma 3.2. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of geodesic δ_i-hyperbolic spaces with δ_i tending to 0. Then for every non-principal ultrafilter ω each component of the ultralimit X_ω is a metric tree.*

Proof. We first verify that between any pair of points $x_\omega, y_\omega \in X_\omega$ there is a unique geodesic segment. Let γ_ω denote the ultralimit of the geodesic segments $\gamma_i := \overline{x_i y_i} \subset X_i$; it connects the points x_ω, y_ω. Suppose that β is another geodesic segment connecting x_ω to y_ω. Pick a point $p_\omega \in \beta$. Then

$$\omega\text{-}\lim_i (x_i, y_i)_{p_i} = \omega\text{-}\lim_i \frac{1}{2}[d(x_i,p_i) + d(y_i,p_i) - d(x_i,y_i)] = 0.$$

Since, by Lemma 1.50,

$$d(p_i, \gamma_i) \le (x_i, y_i)_{p_i} + 2\delta_i,$$
$$d(p_\omega \gamma_\omega) = 0.$$

Now, suppose that $\Delta(x_\omega y_\omega z_\omega)$ is a geodesic triangle in X_ω. By uniqueness of geodesics in X_ω, this triangle appears as ultralimit of the δ_i-thin triangles $\Delta(x_i y_i z_i)$. It follows that $\Delta(x_\omega y_\omega z_\omega)$ is zero-thin, i.e. each component of X_ω is zero-hyperbolic. □

Exercise 41. If T is a metric tree, $-\infty < a < b < \infty$ and $f : [a,b] \to T$ is a continuous embedding then the image of f is a geodesic segment in T. (Hint: use PL approximation of f to show that the image of f contains the geodesic segment connecting $f(a)$ to $f(b)$.)

Lemma 3.3. *(Morse Lemma) Let X be a δ–hyperbolic geodesic space, k, c be positive constants, then there is a function $\theta = \tau(k,c)$ such that for any (k,c)-quasi-isometric embedding $f : [a,b] \to X$ the Hausdorff distance between the image of f and the geodesic segment $[f(a)f(b)] \subset X$ is at most θ.*

PROOF. Suppose that the assertion of lemma is false. Then there exists a sequence of (k, c)-quasi-isometric embeddings $f_n : [-n, n] \to X_n$ to $CAT(-1)$-spaces X_n such that

$$\lim_{n\to\infty} d_{Haus}(f([-n, n]), [f(-n), f(n)]) = \infty$$

where d_{Haus} is the Hausdorff distance in X_n.

Let $d_n := d_{Haus}(f([-n, n]), [f(-n), f(n)])$. Pick points $t_n \in [-n, n]$ such that

$$|d(t_n, [f(-n), f(n)]) - d_n| \leq 1.$$

Consider two sequences of pointed metric spaces

$$(\frac{1}{d_n} X_n, f_n(t_n)), \quad (\frac{1}{d_n}[-n, n], t_n).$$

It is clear that $\omega\text{-}\lim n/d_n > 1/k > 0$ (but this ultralimit could be infinite). Let

$$(X_\omega, x_\omega) = \omega\text{-}\lim(\frac{1}{d_n} X_n, f_n(t_n))$$

and

$$(Y, y) := \omega\text{-}\lim(\frac{1}{d_n}[-n, n], t_n).$$

The metric space Y is either a nondegenerate segment in $\mathbb{R}$ or a closed geodesic ray in $\mathbb{R}$ or the whole real line. Note that the Hausdorff distance between the image of f_n in $\frac{1}{d_n} X_n$ and $[f_n(-n), f_n(n)] \subset \frac{1}{d_n} X_n$ is at most $1 + 1/d_n$. Each map

$$f_n : \frac{1}{d_n}[-n, n] \to \frac{1}{d_n} X_n$$

is a $(k, c/n)$-quasi-isometric embedding. Therefore the ultralimit

$$f_\omega = \omega\text{-}\lim f_n : (Y, y) \to (X_\omega, x_\omega)$$

is a $(k, 0)$-quasi-isometric embedding, i.e. it is a k-bilipschitz map:

$$|t - t'|/k \leq d(f_\omega(t), f_\omega(t')) \leq k|t - t'|.$$

In particular this map is a continuous embedding. On the other hand, the sequence of geodesic segments $[f_n(-n), f_n(n)] \subset \frac{1}{d_n} X_n$ also ω-converges to a nondegenerate geodesic $\gamma \subset X_\omega$, this geodesic is either a finite geodesic segment or a geodesic ray or a complete geodesic. In any case the Hausdorff distance between the image L of f_ω and γ is exactly 1, it equals the distance between x_ω and γ which is realized as $d(x_\omega, z) = 1$, $z \in \gamma$. I will consider the case when γ is a complete geodesic, the other two cases are similar and are left to the reader. Then $Y = \mathbb{R}$ and by Exercise 41 the image L of the map f_ω is a complete geodesic in X_ω which is within Hausdorff distance 1 from the complete geodesic γ. This contradicts the fact that X_ω is a metric tree. $\square$

Historical Remark. Morse **[75]** proved a special case of this lemma in the case of $\mathbb{H}^2$ where the quasi-geodesics in question where geodesics in another Riemannian metric on $\mathbb{H}^2$, which admits a cocompact group of isometries. Busemann, **[18]**, proved a version of this lemma in the case of $\mathbb{H}^n$, where metrics in question were not necessarily Riemannian. A version in terms of quasi-geodesics is due to Mostow **[77]**, in the context of negatively curved symmetric spaces, although his proof is general.

COROLLARY 3.4. *Suppose that X, X' are quasi-isometric geodesic metric spaces and X is Gromov-hyperbolic. Then X' is also Gromov-hyperbolic.*

PROOF. Let $f : X' \to X$ be a (L, A)-quasi-isometry. Pick a geodesic triangle $\Delta ABC \subset X'$. Its image is a quasi-geodesic triangle whose sides are (L, A)-quasi-geodesic. Therefore each of the quasi-geodesic sides of $f(\Delta ABC)$ is within distance $\leq c = c(L, A)$ from a geodesic connecting the end-points of this side. See Figure 1. The geodesic triangle $\Delta f(A)f(B)f(C)$ is δ-thin, it follows that the quasi-geodesic triangle $f(\Delta ABC)$ is $(2c+\delta)$-thin. Thus the triangle ΔABC is $L(2c+\delta)+A$-thin. $\square$

Here is another example of application of asymptotic cones to study quasi-isometries.

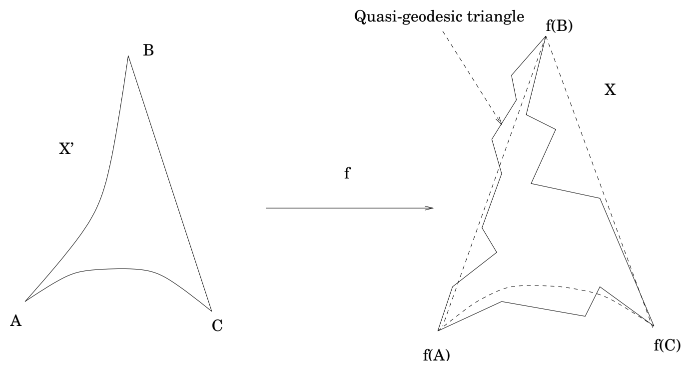

FIGURE 1. *Image of a geodesic triangle.*

LEMMA 3.5. *Suppose that $X = \mathbb{R}^n$ or $\mathbb{R}_+$, $f : X \to X$ is an (L, A)-quasi-isometric embedding. Then $N_C(f(X)) = X$, where $C = C(L, A)$.*

PROOF. I will give a proof in the case of $\mathbb{R}^n$ as the other case is analogous. Suppose that the assertion is false, i.e. there is a sequence of (L, A)-quasi-isometries $f_j : \mathbb{R}^n \to \mathbb{R}^n$, sequence of real numbers r_j diverging to infinity and points $y_j \in \mathbb{R}^n \setminus Image(f)$ such that $d(y_j, Image(f)) = r_j$. Let $x_j \in \mathbb{R}^n$ be a point such that $d(f(x_j), y_j) \le r_j + 1$. Using x_j, y_j as base-points on the domain and target to f_j rescale the metrics on the domain and the target by $1/r_j$ and take the corresponding ultralimits. In the limit we get a bi-Lipschitz embedding

$$f_\omega : \mathbb{R}^n \to \mathbb{R}^n,$$

whose image misses the point $y_\omega \in \mathbb{R}^n$. However each bilipshitz embedding is necessarily proper, therefore by the invariance of domain theorem the image of f_ω is both closed and open. Contradiction. □

REMARK 3.6. Alternatively, one can prove the above lemma as follows: Approximate f by a continuous mapping g. Then, since g is proper, it has to be onto.

3. The asymptotic cone of a metric space

Let X be a metric space and ω be a non-principal ultrafilter on $I = \mathbb{N}$. Suppose that we are given a sequence λ_i so that $\omega\text{-}\lim \lambda_i = 0$ and a sequence of base-points $x_i^0 \in X$. Given this data the *asymptotic cone* $\mathrm{Cone}_\omega(X)$ of X is defined as the based ultralimit of rescaled copies of X:

$$\mathrm{Cone}_\omega(X) := X_\omega^0, \qquad \text{where} \quad (X_\omega^0, x_\omega^0) = \omega\text{-}\lim_i (\lambda_i \cdot X, x_i^0).$$

The discussion in the previous section implies:

PROPOSITION 3.7.
(1) $\mathrm{Cone}_\omega(X \times Y) = \mathrm{Cone}_\omega(X) \times \mathrm{Cone}_\omega(Y)$.
(2) *$Cone_\omega \mathbb{R}^n \cong \mathbb{R}^n$.*
(3) *The asymptotic cone of a geodesic space is a geodesic space.*
(4) *The asymptotic cone of a CAT(0)-space is CAT(0).*
(5) *The asymptotic cone of a space with a negative upper curvature bound is a metric tree.*

REMARK 3.8. Suppose that X admits a cocompact discrete action by a group G of isometries. The problem of dependence of the topological type of $Cone_\omega X$ on the ultrafilter ω and the scaling sequence λ_i was open until recently counterexamples were constructed in [**99**], [**31**]. However in the both examples the group G is not finitely presentable. Moreover, if a finitely-presentable group has *one asymptotic cone* which is a tree, then the group is hyperbolic and hence *every* asymptotic cone is a tree, see [**61**] and [**63**].

To get an idea of the size of the asymptotic cone, we will see below that in the most interesting cases it is homogeneous.

We call an isometric action $G \curvearrowright X$ *cobounded* if there exists $D < \infty$ such that for some point $x \in X$,

$$\bigcup_{g \in G} g(B_D(x)) = X,$$

i.e. $G \cdot x$ is a D-net in X. Equivalently, given any pair of points $x, y \in X$, there exists $g \in G$ such that $d(g(x), y) \le 2D$. We call a metric space X *quasi-homogeneous* if the action $\mathrm{Isom}(X) \curvearrowright X$ is cobounded.

Suppose that X is a metric space and $G \subset \mathrm{Isom}(X)$ is a subgroup. Given a nonprincipal ultrafilter ω define the group G^* to be the ultraproduct

$$G^* = \prod_{i \in I} G/\omega.$$

By abusing notation we will refer to points in G^* as *sequences*. Given a sequence λ_i so that $\omega\text{-}\lim \lambda_i = 0$ and a sequence of base-points $x_i^0 \in X$, let $\mathrm{Cone}_\omega(X)$ be the corresponding asymptotic cone. It is clear that G^* acts isometrically on the ultralimit

$$U := \omega\text{-}\lim_i (\lambda_i \cdot X).$$

Let $G_\omega \subset G^*$ denote the stabilizer in G^* of the component $\mathrm{Cone}_\omega(X) \subset U$. In other words,

$$G_\omega = \{(g_i) \in G^* : \omega\text{-}\lim_i \lambda_i d(g_i(x_i^0), x_i^0) < \infty\}.$$

Thus $G_\omega \subset \mathrm{Isom}(\mathrm{Cone}_\omega(X))$. Observe that if (x_i^0) is a bounded sequence in X then the group G has a diagonal embedding in G_ω.

Proposition 3.9. *Suppose that $G \subset \mathrm{Isom}(X)$ and the action $G \curvearrowright X$ is cobounded. Then for every asymptotic cone $\mathrm{Cone}_\omega(X)$ the action $G_\omega \curvearrowright \mathrm{Cone}_\omega(X)$ is transitive. In particular, $\mathrm{Cone}_\omega(X)$ is a homogeneous metric space.*

Proof. Let $D < \infty$ be such that $G \cdot x$ is a D-net in X. Given two sequences $(x_i), (y_i)$ of points in X there exists a sequence (g_i) of elements of G such that

$$d(g_i(x_i), y_i) \le 2D.$$

Therefore, if $g_\omega := (g_i) \in G^*$, then $g_\omega((x_i)) = (y_i)$. Hence the action

$$G^* \curvearrowright X_\omega = \omega\text{-}\lim_i (\lambda_i \cdot X)$$

is transitive. It follows that the action $G_\omega \curvearrowright \mathrm{Cone}_\omega(X)$ is transitive as well.

Example 42. Construct an example of a metric space X and an asymptotic cone $\mathrm{Cone}_\omega(X)$ so that for the isometry group $G = \mathrm{Isom}(X)$ the action $G_\omega \curvearrowright \mathrm{Cone}_\omega(X)$ is not effective (i.e. has nontrivial kernel). Construct an example when the kernel of $G_\omega \to \mathrm{Isom}(\mathrm{Cone}_\omega(X))$ contains the entire group G embedded diagonally in G_ω.

Lemma 3.10. *Let X be a quasi-homogeneous δ–hyperbolic space with uncountable number of ideal boundary points. Then for every nonprincipal ultrafilter ω the asymptotic cone $\mathrm{Cone}_\omega(X)$ is a tree with uncountable branching.*

Proof. Let $x^0 \in X$ be a base-point and $y, z \in \partial_\infty X$. Denote by γ the geodesic in X with the ideal endpoints z, y. Then $\mathrm{Cone}_\omega([x^0, y))$ and $\mathrm{Cone}_\omega([x^0, z))$ are geodesic rays in $\mathrm{Cone}_\omega(X)$ emanating from x_ω^0. Their union is equal to the geodesic $\mathrm{Cone}_\omega \gamma$. This produces uncountably many rays in $\mathrm{Cone}_\omega(X)$ so that any two of them have precisely the base-point in common. The homogeneity of $\mathrm{Cone}_\omega(X)$ implies the assertion. $\square$

LEMMA 3.11. *Suppose that X is a proper δ-hyperbolic geodesic space. Let $Q \subset X$ be a (L, A)-quasigeodesic ray or a complete (L, A)-quasigeodesic. Then there is Q^* which is either a geodesic ray (or a complete geodesic) in X so that the Hausdorff distance between Q and Q^* is $\leq C(L, A, \delta)$.*

PROOF. We will consider only the case of quasigeodesic rays $\rho : [0, \infty) \to Q \subset X$ as the other case is similar. Consider the sequence of geodesic segments $\gamma_i = \overline{\rho(0)\rho(i)}$. By Morse lemma, each γ_i is contained within $N_c(Q)$, where $c = c(L, A, \delta)$. By local compactness, the geodesic segments γ_i subconverge to a complete geodesic ray $Q^* = \gamma(\mathbb{R}_+)$ which is contained in $N_c(Q)$.

It remains to show that Q is contained in $N_D(Q^*)$, where $D = D(L, A, \delta)$. Consider the nearest-point projection $p : Q^* \to Q$. This projection is clearly a quasi-isometric embedding with the constants depending only on L, A, δ. Lemma 3.5 shows that the image of p is ϵ-dense in Q with $\epsilon = \epsilon(L, A, \delta)$. Hence each point of Q is within distance $\leq D = \epsilon + c$ from a point of Q^*. □

Observe that this lemma implies that for any divergent sequence $t_j \in \mathbb{R}_+$, the sequence of points $\rho(t_j)$ on a quasi-geodesic ray in X, converges to a point $\eta \in \partial_\infty X$, $\eta = \gamma(\infty)$. Indeed, if γ, γ' are geodesic rays Hausdorff-close to Q then γ, γ' are Hausdorff-close to each other as well, therefore $\gamma(\infty) = \gamma'(\infty)$.

We will refer to the point η as $\rho(\infty)$. Note that if ρ' is another quasi-geodesic ray which is Hausdorff-close to ρ then $\rho(\infty) = \rho'(\infty)$.

THEOREM 3.12. *Suppose that X and X' are Gromov-hyperbolic proper geodesic metric spaces. Let $f : X \to X'$ be a quasi-isometry. Then f admits a homeomorphic extension $f_\infty : \partial_\infty X \to \partial_\infty X'$. This extension is such that the map $f \cup f_\infty$ is continuous at each point $\eta \in \partial_\infty X$.*

PROOF. First, we construct the extension f_∞. Let $\eta \in \partial_\infty X$, $\eta = \rho(\infty)$ where ρ is a geodesic ray in X. The image of this ray $\rho' := f \circ \rho : \mathbb{R}_+ \to X'$ is a quasi-geodesic ray, hence we set $f_\infty(\eta) := \rho'(\infty)$. Observe that $f_\infty(\eta)$ does not depend on the choice of a geodesic ray asymptotic to η. Let $\bar{f}$ be quasi-inverse of f. It is clear from the construction that $(\bar{f})_\infty$ is inverse to f_∞. It remains therefore to verify continuity.

Suppose that $x_n \in X$ is a sequence which converges to η in the cone topology, $d(x_n, \rho) \leq c$. Then $d(f(x_n), \rho') \leq Lc + A$ and $d(f(x_n), (\rho')*) \leq C(Lc + A)$, where $(\rho')*$ is a geodesic ray in X' asymptotic to $\rho'(\eta)$. Thus $f(x_n)$ converges to $f_\infty(\eta)$ in the cone topology.

Finally, let $\eta_n \in \partial_\infty X$ be a sequence which converges to η. Let ρ_n be a sequence of geodesic rays asymptotic to η_n with $\rho_n(0) = \rho(0) = x_0$. Then, for each $T \in \mathbb{R}_+$ there exists n_0 such that for all $n \geq n_0$ and $t \in [0, T]$ we have

$$d(\rho(t), \rho_n(t)) \leq 2\delta,$$

where δ is the hyperbolicity constant of X. Hence

$$d(f(\rho_n(t)), \rho(t)) \leq 2L\delta + A.$$

Set $\rho'_n := f \circ \rho_n$. Then

$$(\rho'_n)^*([0, L^{-1}T - A]) \subset N_C((\rho')^*([0, LT + A])),$$

for all $n \geq n_0$. Thus the geodesic rays $(\rho'_n)^*$ converge to a ray within finite distance from $(\rho')^*$. It follows that the sequence $f_\infty(\eta_n)$ converges to $f_\infty(\eta)$. □

LEMMA 3.13. *Let X and X' be proper geodesic δ-hyperbolic spaces. In addition we assume that X is quasi-homogeneous and that $\partial_\infty X$ consists of at least four points. Suppose that $f, g : X \to X'$ are (L, A)-quasi-isometries such that $f_\infty = g_\infty$. Then $d(f, g) \leq D$, where D depends only on L, A, δ and the geometry of X.*

PROOF. Let γ_1, γ_2 be complete geodesics in X which are asymptotic to the points ξ_1, η_1, ξ_2, η_2 respectively, where all the points ξ_1, η_1, ξ_2, η_2 are distinct. There is a point $y \in X$ which is within distance $\leq r$ from both geodesics γ_1, γ_2. Let G be a group acting isometrically on X so that the $GB = X$ for an R-ball B in X. Pick a point $x \in X$: Our goal is to estimate $d(gf(x), g(x))$. By applying an element of G to x we can assume that $d(x, y) \leq R$, in particular, $d(x, \gamma_1) \leq R + r, d(x, \gamma_2) \leq R + r$. Thus the distance from $f(x)$ to the quasi-geodesics $f(\gamma_1), f(\gamma_2)$ is at most $L(R + r) + A$. We now apply the quasi-inverse $\bar{g}$ the to quasi-isometry g: $\bar{g}f(\gamma_i)$ is an $(L^2, LA + A)$-quasi-geodesic in X; since $f_\infty = g_\infty$, these quasi-geodesics are asymptotic to the points ξ_i, η_i, $i = 1, 2$. Since the Hausdorff distance from $\bar{g}f(\gamma_i)$ to γ_i is at most $C + 2\delta$ (where $C = C(L^2, LA + A, \delta)$ is the constant from Lemma 3.11) we conclude that

$$d(\bar{g}f(x), \gamma_i) \leq C' := C + 2\delta.$$

See Figure 2.

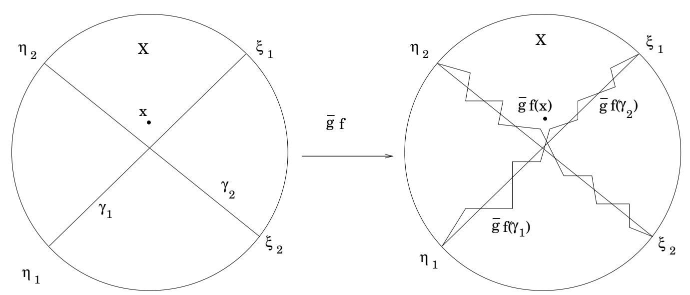

FIGURE 2.

Since the geodesics γ_1, γ_2 are asymptotic to distinct points in $\partial_\infty X$, it follows that the diameter of the set $\{z \in X : d(z, \gamma_i) \leq \max(C', r + R), i = 1, 2\}$ is at most C'', where C'' depends only on the geometry of X and the fixed pair of geodesics γ_1, γ_2. Hence $d(\bar{g}f(x), x) \leq C''$. By applying g to this formula we get:

$$d(g(x), g\bar{g}f(x)) \leq L(C'' + A) + A,$$
$$d(f(x), g\bar{g}f(x)) \leq A.$$

Therefore

$$d(f(x), g(x)) \leq 2A + L(C'' + A). \quad \square$$

REMARK 3.14. The line $X = \mathbb{R}$ is 0-hyperbolic, its ideal boundary consists of 2 points. Take a translation $f : X \to X$, $f(x) = x + a$. Then f_∞ is the identity map of $\{-\infty, \infty\}$ but there is no bound on the distance from f to the identity.

CHAPTER 4

Tits alternative

THEOREM 4.1 (Tits alternative, [**100**]). *Let L be a Lie group with finitely many components and $\Gamma \subset L$ be a finitely generated subgroup. Then either Γ is virtually solvable or Γ contains a free nonabelian subgroup.*

We will prove this theorem in the next two sections.

Outline of the proof of Tits' alternative. The first step is to reduce the problem to the case of subgroups of the general linear group.

Since L has only finitely many components, the connected component of the identity $L_0 \subset L$ is a finite index subgroup. Thus $\Gamma \cap L_0$ has finite index in Γ. Therefore we can assume that L is connected.

LEMMA 4.2. *There exists a homomorphism $\phi : \Gamma \to GL_n(\mathbb{R})$ whose kernel is contained in the center of Γ.*

PROOF. Recall that a Lie group L is a differentiable manifold with group operations which are smooth maps. Actually, L can be made into a real analytic manifold with real analytic group operations. The group G acts on itself via conjugations

$$\rho(g)(h) = ghg^{-1}.$$

This action is smooth but nonlinear. However the point e is fixed by the entire group L. Therefore L acts linearly on the tangent space $V = T_eL$ at the identity $e \in L$. The action of L on V is called *the adjoint representation* Ad of the group L. Therefore we have the homomorphism

$$\psi : L \to GL(V).$$

What is the kernel of ψ? There is a local diffeomorphism

$$exp : V \to L$$

called *the exponential map* of the group L. (In the case when $G = GL(n, \mathbb{R})$ this map is the ordinary matrix exponential.) It satisfies the identity

$$g \exp(v) g^{-1} = exp(Ad(g)v), \quad \forall v \in V, g \in L.$$

Thus, if $Ad(g) = Id$ then g commutes with every element of L of the form $exp(v), v \in V$. The set of such elements is open in L. Now, if we are willing to use real analytic structure on L then it would follow that g belongs to the center of L. Alternatively, we can argue that the centralizer $Z(g)$ in G is a closed subgroup in L which has nonempty interior (containing e). Sine $Z(g)$ acts transitively on itself by, say, left multiplication, $Z(g)$ is open. As L is connected, we conclude that $Z(g) = L$. Therefore we get a homomorphism $\phi : \Gamma \to GL(V) = GL(n, \mathbb{R})$ whose kernel is in the center of Γ. $\square$

Observe that

1. Γ is virtually solvable if and only if $\phi(\Gamma)$ is virtually solvable.
2. Γ contains a free subgroup if and only if $\phi(\Gamma)$ contains a free subgroup.

Therefore we can assume that Γ is a linear group, $\Gamma \subset GL(n, \mathbb{R})$.

1. Unbounded subgroups of $SL(n)$

In this section we consider subgroups Γ of $SL(n, \mathbf{k})$ where $\mathbf{k}$ is either $\mathbb{R}$ or $\mathbb{C}$. For technical reasons, in order to prove Tits' Alternative, one should also consider the case of other *local fields* $\mathbf{k}$. Recall that a local field is a field with a norm $|\cdot|$ which determines a locally compact topology on $\mathbf{k}$. The most relevant for us examples are when $\mathbf{k} = \mathbb{R}, \mathbf{k} = \mathbb{C}$, $\mathbf{k} = \mathbb{Q}_p$ and, more generally, $\mathbf{k}$ is the completion of a finite extension of $\mathbb{Q}$.

In what follows, V is an n-dimensional vector space over a local field $\mathbf{k}$, $n = dim(V) > 1$. We fix a basis $e_1, ..., e_n$ in V. The norm $|\cdot|$ on $\mathbf{k}$ determines the Euclidean norms $\|\cdot\|$ on V and on its exterior powers. We let $P(V)$ denote the projective space of V. We will use the notation E^c to denote the complement of a subset $E \subset P$. Given a nonzero vector $v \in V$ let $[v]$ denote the projection of v to $\mathbb{P}^{n-1}$; similarly, if $W \subset V$ is a nonzero vector subspace, $[W]$ will denote the projection of W to P.

We give $P(V)$ the metric

$$d([v], [w]) = \frac{\|v \wedge w\|}{\|v\| \cdot \|w\|}.$$

This metric is clearly invariant under the maximal compact subgroup $K \subset GL(n, \mathbf{k})$, $K = O(n)$ if $\mathbf{k} = \mathbb{R}$, $K = SU(n)$ if $\mathbf{k} = \mathbb{C}$.

Remark 4.3. It is clear from the description that d is a continuous function on $P \times P$ and $d([v], [w]) = 0$ if and only if $[v] = [w]$. What is less clear however is that d satisfies the triangle inequality. The reader can either verify this by a direct calculation or note that the proofs below will use only the above properties of d.

Exercise 43. In the case of $\mathbf{k} = \mathbb{R}$, derive the triangle inequality for the function d from the following elementary inequality

$$\forall \alpha, \beta, \gamma \in [0, \pi/2], \alpha + \beta \geq \gamma \Rightarrow \sin(\alpha) + \sin(\beta) \geq \sin(\gamma).$$

Zariski topology. Before proceeding further we need a bit of the algebraic geometry terminology. The group $GL(V)$ has the *Zariski topology*, where closed subsets are given by (systems of) polynomial equations. An *algebraic subgroup* of $GL(V)$ is a subgroup which is Zariski-closed, i.e., is given by a system of polynomial equations. For instance, the subgroup $SL(V)$ is given by the equation $det(g) = 1$.

A subset $Y \subset X$ is called *Zariski-dense* if its Zariski closure is the entire X.

Lemma 4.4. *Suppose that $\Gamma \subset SL(V)$ is a subgroup. Then its Zariski closure $\bar{\Gamma}$ in $SL(V)$ is also a subgroup.*

Proof. Consider the map $f : \bar{\Gamma} \to SL(V)$ given by $f(\gamma) = \gamma^{-1}$. Then f is a polynomial isomorphism and hence $f(\bar{\Gamma})$ is Zariski closed. Since Γ is a subgroup, $f(\bar{\Gamma})$ contains Γ. Thus $\bar{\Gamma} \cap f(\bar{\Gamma})$ is a Zariski closed set containing Γ. It therefore follows that $\bar{\Gamma} = f(\bar{\Gamma})$ and hence $\bar{\Gamma}$ is stable under the inversion. The argument for the multiplication is similar. $\square$

Separating sets. A subset $F \subset PGL(V)$ is called m-separating if for every choice of points $p_1, ..., p_m \in P = P(V)$ and hyperplanes $H_1, ..., H_m \subset P$, there exists $f \in F$ so that

$$f^{\pm 1}(v_i) \notin H_j, \forall i, j = 1, ..., m.$$

A 1-separating subset is called simply *separating.*

A group of projective transformations $G \subset PGL(V)$ is said to act irreducibly on $P(V)$ if G does not leave any proper subspace invariant

Proposition 4.5. *Let $G \subset PGL(V)$ be a connected algebraic subgroup acting irreducibly. Then for every m, the every Zariski-dense subgroup $\Gamma \subset G$ contains a finite m-separating subset F.*

Proof. Let $P^\vee$ denote the space of hyperplanes in P. For each $g \in G$ let $M_g \subset P^m \times (P^\vee)^m$ denote the collection of $2m$-tuples

$$(p_1, ..., p_m, H_1, ..., H_m)$$

so that

$$g(p_i) \in H_j \text{ or } g^{-1}(p_i) \in H_j$$

for some $i, j = 1, ..., m$.

LEMMA 4.6.

$$\bigcap_{g \in \Gamma} M_g = \emptyset.$$

PROOF. Since Γ is Zariski-dense in G, it suffices to prove the assertion for G itself. Suppose to the contrary that the intersection is nonempty. Thus, there exists a $2m$-tuple

$$(p_1, ..., p_m, H_1, ..., H_m)$$

so that for every $g \in G$, there exist i, j so that

$$g(p_i) \in H_j \text{ or } g^{-1}(p_i) \in H_j.$$

Let $G^{\pm}_{p_i,H_j}$ denote the set of $g \in G$ so that

$$g^{\pm 1}(p_i) \in P_j.$$

Clearly, these subsets are Zariski-closed and cover the group G. Since G is connected, it is irreducible as a variety; it follows that one of these sets, say, $G^{+}_{p_i,H_j}$ is the entire G. Therefore, for every $g \in G$, $g(p_i) \in H_j$. Thus, minimal subspace $L \subset P$ containing the orbit $G \cdot p_i$ is contained in H_j. Hence, L is a proper G-invariant subspace in P. Contradiction. □

We can now finish the proof of Proposition. Let M_g^c denote the complement of M_g in $P^m \times (P^\vee)^m$. This set is Zariski open. By Lemma, the sets M_g^c cover the space $P^m \times (P^\vee)^m$. Since $\mathbf{k}$ is a local field, the product $P^m \times (P^\vee)^m$ is compact and, thus, the above open cover contains a finite subcover. Hence, there exists a finite set $F \subset G$ so that

$$\bigcup_{f \in F} M_f^c = P^m \times (P^\vee)^m.$$

This set satisfies the assertion of Proposition. □

REMARK 4.7. The above proposition holds even if the field $\mathbf{k}$ is not local. Then the point is that by Hilbert's Nulstullenzats, there exists a finite subset $F \subset G$ so that

$$\bigcap_{f \in F} M_f = \bigcap_{g \in G} M_g = \emptyset.$$

With this modification, the above proof goes through.

Diverging sequences of projective transformations. Suppose that $n = dim(V)$. Recall that the group $GL(V)$ has the singular value (or Cartan) decomposition $GL(V) = KAK$, where A is the group of diagonal matrices and K is a maximal compact subgroup. For instance, if $\mathbf{k} = \mathbb{R}$ then $K = O(n)$; if $\mathbf{k} = \mathbb{C}$ then $K = U(n)$, if $\mathbf{k} = \mathbb{Q}_p$ then $K = GL(\mathbb{Z}_p)$. Hence, every matrix $g \in GL(V)$ decomposes as the product $g = kak'$, where $k, k' \in K$ and $a \in A$. Let $a_1(g), ..., a_n(g) \in \mathbf{k}$ denote the diagonal entries of the matrix a. Since permutation matrices belong to K, we can choose a so that

$$|a_1(g)| \geq ... \geq |a_n(g)| > 0.$$

Although the numbers $a_1(g), ..., a_n(g) \in \mathbf{k}$ are not unique, their norms are uniquely determined by the matrix g. These norms will be called *singular values* of g. For an element $g \in PGL(V)$ we will always choose a representative in $GL(V)$, also denoted g, so that $|a_1(g)| = 1$. With this convention, singular values of $g \in PGL(V)$ are well-defined and again denoted $|a_1(g)|, ..., |a_n(g)|$.

It is immediate from compactness of K that a sequence (g_i) in $PGL(V)$ is bounded (equivalently, is relatively compact) iff the singular values of g_i's are bounded away from zero.

DEFINITION 4.8. We say that a sequence (g_i) in $PGL(V)$ is M-contracting if M is the largest among numbers m so that the sequence $(|a_m(g_i)|)$ are bounded away from zero. A 1-contracting sequence is simply called *contracting*. A sequence is *very contracting* if both (g_i) and (g_i^{-1}) are contracting.

Clearly, every unbounded sequence in $PGL(V)$ contains an m-contracting subsequence for some $m < n$. We will assume now that (g_i) is a sequence which is m-contracting for some $m < n$. Our next goal is to analyze the dynamics of m-contracting sequences on $P = P(V)$. First of all, it suffices to consider only diagonal sequences (g_i) in $GL(V)$. If (g_i) is an m-contracting diagonal sequence then for any m-dimensional subspace $U \subset V$ which is disjoint from $E = Span(e_{m+1}, ..., e_n)$, we have

$$\lim_{i\to\infty} g_i(U) = A = Span(e_1, ..., e_n).$$

The subspace A is called the *attracting* subspace of (g_i) and the subspace E the *exceptional* subspace of the sequence (g_i). More generally, we say that a point A in $Gr_m(V)$ is *attracting* of (g_i) and a point $E \in Gr_{n-m}(V)$ is *exceptional* for (g_i) if the sequence (g_i) converges to A uniformly on compacts away from the set

$$\widehat{E} = \{U \in Gr_m(V) : U \cap E \neq 0\}.$$

In the case of a contracting sequence (g_i), we will use the notation E for the projection of the exceptional set to $P(V)$. In this case, A is just a point, called the *attracting point* of (g_i).

Example 4.9. To make things more concrete, consider the case $n = 2$ and $\mathbf{k} = \mathbb{R}$. Then $P = \mathbb{P}^1$ is the circle on which the group $PSL(2, \mathbb{R})$ acts by linear-fractional transformations. Since $0 < m < n = 2$, it follows that $m = 1$ and, hence, every divergent sequence is contracting. It is easy to see that it has to be very contracting as well. Moreover, the exceptional hyperplanes in P are again points. Thus for each unbounded sequence $g_i \in PSL(2, \mathbb{R})$ there exists a pair of points A and E in P such that

$$\lim_{i\to\infty} g_i|_{P\setminus E} = A$$

uniformly on compacts. For instance, if $g_i = g^i$, and g is parabolic, then $A = E$ is the fixed point of g. If g is hyperbolic then A is the attractive and E is the repulsive fixed point of g. Thus in general (unlike in the diagonal case), $A(g_i)$ may belong to $E(g_i)$.

The following lemma is easy and is left as an exercise to the reader:

Lemma 4.10. *Let (g_i) be a m-contracting sequence in $PGL(V)$, $f, h \in PGL(V)$. Then the sequence $g_i' := (f g_i h)$ is also m-contracting and*

$$A(g_i') = f(A(g_i)), \quad E(g_i') = h^{-1}E(g_i).$$

Lemma 4.11. *The following are equivalent for a sequence (g_i) of projective transformations:*

1) (g_i) is contracting.

2) For some ball $B \subset P$, the sequence of restrictions $g_i|_B$ converges uniformly to a constant.

3) $\lim_{i\to\infty} \frac{a_2(g_i)}{a_1(g_i)} = 0$.

Proof. As before, we lift the elements g_i to matrices in $GL(V)$ so that a_1's of the lifts equal 1. In view of the Cartan decomposition, it suffices to consider the case when g_i are diagonal matrices.

The implication (1)⇒(2) is obvious. Suppose that (2)⇒(3) fails. Then there is a constant $C > 0$ so that

$$1 \leq \frac{|a_1(g_i)|}{|a_2(g_i)|} \leq C$$

Take a small round ball D in the preimage of B in V. Then, after passing to a subsequence in (g_i) if necessary, we conclude that the sequence of images $g_i(D)$ converge to an ellipsoid in V of dimension at least 2. This contradicts the assumption that the sequence $(g_i|_B)$ converges to a point.

Consider the implication (3)⇒(1). Recall that $g_i(e_1) = e_1$. Since

$$\lim_{i\to\infty} a_k(g_i) = 0, k = 2, ..., n,$$

we have

$$\lim_i g_i = 0$$

uniformly on compacts in V away from $E = Span(e_2, ..., e_n)$. (1) follows. ☐

LEMMA 4.12. *Let (g_i) be an m-contracting sequence with the exceptional subspace $E \subset P$. Then for each compact $K \subset E^c$ there exists L so that the sequence (g_i) is L-Lipschitz on K.*

PROOF. Without loss of generality, we may assume that the sequence (g_i) in $GL(V)$ is diagonal and that $|a_1(g_i) = 1|$ for each i. Since (g_i) is m-contracting, there exists $\epsilon > 0$ so that

$$1 = |a_1(g_i)| \geq ... \geq |a_m(g_i)| \geq \epsilon \quad \forall i.$$

Let $v, w \in V$ be nonzero vectors. Then

$$\|g(v) \wedge g(w)\|^2 = \sum_{p<q} |a_p a_q||v_p w_q| \leq \sum_{p<q} |v_p w_q| = \|v \wedge w\|^2.$$

Since K is a compact in E^c, there exists $\delta > 0$ so that for each unit vector $v = (v_1, ..., v_n) \in V$ whose projection to P is in K, we have

$$\sum_{k=1}^{m} |v_k|^2 \geq \delta^2.$$

Therefore, for such vector v,

$$\|g_i(v)\| \geq (\sum_{k=1}^{m} |a_k|^2 |v_k|^2)^{1/2} \geq \epsilon\delta.$$

Hence, for unit vectors v, w whose projections are in K, we obtain

$$d(g(v), g(w)) = \frac{\|g(v) \wedge g(w)\|}{\|g(v)\| \cdot \|g(w)\|} \leq \frac{\|v \wedge w\|}{(\epsilon\delta)^2} = \frac{d(v,w)}{(\epsilon\delta)^2}.$$

□

LEMMA 4.13. *Let (g_i) be an unbounded sequence in $PGL(V)$. Then there exists an embedding $\rho : PGL(V) \subset PGL(W)$ so that a subsequence in $(\rho(g_i))$ is contracting in $PGL(W)$.*

PROOF. After passing to a subsequence, we may assume that (g_i) is m-contracting for some $0 < m < n$. Without loss of generality, the sequence (g_i) is diagonal and

$$A(g_i) = Span(e_1, ..., e_m), E = Span(e_{m+1}, ..., e_n)$$

are its attractive and exceptional subspaces. We let

$$W := \Lambda^m V.$$

The group $GL(V)$ acts naturally on V and we obtain the embedding $\rho : GL(V) \subset GL(W)$. Clearly, for a diagonal matrix $g \in GL(V)$, the singular values of $\rho(g) \in GL(W)$ are the products $\rho(g), g \in GL(V)$ are the products

$$\prod_{j_1 < ... < j_m} |a_{j_1} ... a_{j_m}(g)|.$$

Then $|a_1(\rho(g_i))| = |a_1 ... a_m(g_i)|$ and it is immediate that

$$\lim_{i \to \infty} \frac{a_l(\rho(g_i))}{a_1(\rho(g_i))} = 0, \forall l > 1.$$

□

Proximal sequences. We say that a sequence (g_i) in $PGL(V)$ is *very proximal* if it is very contracting and

$$A(g_i) \notin E(g_i), \quad A(g_i^{-1}) \notin E(g_i^{-1}).$$

Furthermore, a pair of sequences $(g_i), (h_i)$ is said to be a *ping-pong* pair if they are both very proximal and

$$A(g_i^{\pm 1}) \notin E(h_i) \cup E(h_i^{-1}), \quad A(h_i^{\pm 1}) \notin E(g_i) \cup E(g_i^{-1})$$

Let $G \curvearrowright P$ be an irreducible projective action of a connected algebraic group and $\Gamma \subset G$ be a Zariski-dense subgroup which contains a contracting sequence.

PROPOSITION 4.14. *The group Γ contains ping-pong pairs of sequences.*

PROOF. The idea of the proof is to use separating sets in Γ in order to produce first very contracting sequences, then proximal sequences and, finally, ping-pong pairs of sequences. We fix a 4-separating subset $F \subset \Gamma \subset PGL(V)$ (see Proposition 4.5).

Let (g_i) be a contracting sequence in Γ which converges to a point $p \in P$ away from a hyperplane H.

LEMMA 4.15. *There exists $f \in F$ so that (after passing to a subsequence in (g_i)) both sequences $h_i := g_i f g_i^{-1}$ and $g_i f^{-1} g_i^{-1}$ are contracting; in other words, the sequence (h_i) is very contracting.*

PROOF. After passing to a subsequence, we can assume that the sequence (g_i^{-1}) is m-attractive with the exceptional subspace $E \subset P = P(V)$. Pick a point $q \in E^c \subset P$. After passing to a subsequence in (g_i) again we can assume that $\lim_i g_i^{-1}(q) = u \in P$. Since F is a separating subset of G, there exists $f \in F$ so that $f^{\pm 1}(u) \notin H$. Take a small closed ball $B \subset P$ centered at q and disjoint from E. Then the restrictions $g_i^{-1}|_B$ are L-Lipschitz for some $L < \infty$ (Lemma 4.12). Therefore, if B is sufficiently small and i is sufficiently large, the distance from $f^{\pm 1} g_i^{-1}(B)$ to H is $\geq \epsilon$ for some $\epsilon > 0$. However, the restrictions $g_i|_{B_\epsilon(H)^c}$ converge to p uniformly. Therefore, the sequence $g_i f^{\pm 1} g_i^{-1}$ also converges to p uniformly on B. Now the assertion follows from Lemma 4.11. ☐

Let (h_i) be the very contracting sequence constructed in the above lemma. We

LEMMA 4.16. *There exists $f \in F$ so that the sequence $y_i = f h_i$ is very proximal.*

PROOF. By Lemma 4.10, for any choice $f \in F$, the sequence $f h_i$ is very attracting and

$$A(fh_i) = f(A(h_i)), E(fh_i) = E(h_i),$$

$$A((fh_i)^{-1}) = A(h_i^{-1}), E((fh_i)^{-1}) = fE(h_i^{-1}).$$

Now, the assertion follows from the fact that F is a 4-separating set. ☐

Let (y_i) be the very proximal sequence from the previous lemma.

LEMMA 4.17. *There exists $f \in F$ so that the sequence $(y_i), (z_i) = (f y_i f^{-1})$ form a ping-pong pair. conversely, the*

PROOF. By Lemma 4.10, (z_i) is very attractive and $A(z_i^{\pm 1}) = fA(y_i^{\pm 1})$, while $E(z_i^{\pm 1}) = fE(y_i^{\pm 1})$. Now, the assertion follows from the fact that F is 4-separating. ☐

This finishes the proof of Proposition as well. ☐

At last, we can prove Tits' alternative for unbounded subgroups of $GL(V)$ for finite-dimensional vector spaces over local fields $\mathbf{k}$. Let $\Gamma \subset PGL(V)$ be an unbounded subgroup. Let G be the Zariski-closure of Γ on $PGL(V)$. Although G need not be connected, it has only finitely many connected components, so, after passing to a finite-index subgroup in Γ, we can assume that G is connected. If $G \curvearrowright V$ is reducible, we consider a proper G-invariant subspace $W \subset V$ and the actions $G \curvearrowright W, G \curvearrowright V/W$. We continue this process inductively until we either obtain a complete G-invariant flag

$$0 \subset V_1 \subset ... \subset V_{n-1} \subset V = V_n, dim(V_i) = i,$$

in which case G (and, hence Γ) is solvable, or we obtain an irreducible action of G on a vector space V' of dimension > 1. Without loss of generality, we may assume that G is semisimple. We retain the notation V for V'. Our goal is to show that in this case Γ contains a free subgroup on 2 generators. The key tool is the following "ping-pong lemma," which goes back (in some form) to Felix Klein:

LEMMA 4.18. [**100**, Proposition 1.1] *Let P be a set, I an index set, $\Gamma_i, i \in I$, a collection of groups acting on P, $P_i, i \in I$, a collection of subsets of P and $p \in P \setminus \cup_{i \in I} P_i$. Assume that for all pairs of distinct elements $i, j \in I$, we have*

$$g(P_j \cup \{p\}) \subset P_i, \forall g \in \Gamma_i \setminus \{1\}.$$

Then the group Λ generated by $\Gamma_i, i \in I$, is the free product of $\Gamma_i, i \in I$.

PROOF. Consider words $w = g_{i_n}....g_{i_1}$, $g_{i_s} \in \Gamma_{i_s} \setminus \{1\}$, $i_{s-1} \neq i_s$ for all s. We claim that $w(p) \neq p$ provided that $n > 0$. The proof is by induction on n. If $n = 1$ then $w(p) \in P_{i_1}$, hence, $w(p) \neq p$. Assume that for a word

$$w = g_{i_n}....g_{i_1}$$

we have $w(p) \in P_{i_n}$. Then for $u = g_{i_{n+1}}w$, we have

$$u(p) \in g_{i_{n+1}}(P_{i_n}) \subset P_{i_{n+1}}.$$

Hence, $w(p) \neq 1$ thus, $w \neq 1$ in Λ. $\square$

Our goal is to produce a projective space P, subsets P_1, P_2 and two cyclic subgroups Γ_1, Γ_2 satisfying the above lemma. First of all, since Γ is unbounded, there exists a faithful linear finite-dimensional representation $\rho : G \to GL(W)$, so that $\rho(\Gamma)$ contains a contracting sequence $(\rho(g_i))$, Lemma 4.13. In case if the action $G \curvearrowright W$ is reducible, we take a direct sum decomposition

$$W = \oplus_{i=1}^{s} W_s$$

into G-invariant subspaces, so that the restriction of $\rho(G)$ to each is irreducible. Without loss of generality, we can assume that each subspace has dimension > 1.

LEMMA 4.19. *For some s, the sequence $(\rho(g_i))$ restricted to W_s is contracting.*

PROOF. Let $p \in P(W), E \subset P(W)$ be the attracting point and the exceptional hyperplane of the sequence $(\rho(g_i))$. Since the subspaces W_t are G-invariant, the for each t either $p \in P(W_t)$ or $P(W_t) \subset E$. Since $E \neq P(W)$, $p \in P(W_s)$ for some s. The restriction of $(\rho(g_i))$ to $P(W_s)$ converges to p away from $E \cap P(W_s)$. Since $dim(W_s) > 1$, we are done. $\square$

We now set $U := W_s$, where W_s is as in the above lemma. We retain the notation ρ for the representation $G \to GL(U)$. By Proposition 4.14, the group $\rho(\Gamma) \subset GL(U)$ contains a ping-pong pair of sequences $(\alpha_i), (\beta_i)$. Let $p^{\pm}, Y^{\pm}$ denote the attracting points and exceptional hyperplanes of $\alpha_i^{\pm 1}$ and $z^{\pm}, Z^{\pm}$ the attracting points and exceptional hyperplanes of $\beta_i^{\pm 1}$.

Let $P_1^{\pm}$, $P_2^{\pm}$ be small closed balls centered at $p^{\pm}$ and $q^{\pm}$ respectively. Since $\{y^+, z^+, z^-\}$ is disjoint from Z^+ etc., we can choose the balls $P_1^{\pm}$, $P_2^{\pm}$ so that

$$P_1^+ \cap (Y^- \cup Z^+ \cup Z^-) = \emptyset,$$

$$P_1^- \cap (Y^+ \cup Z^+ \cup Z^-) = \emptyset,$$

$$P_2^+ \cap (Y^+ \cup Y^- \cup Z^-) = \emptyset,$$

$$P_2^- \cap (Y^+ \cup Y^- \cup Z^+) = \emptyset.$$

(Note that it can easily happen that, say, $y^+ \in Y^-$.) Set $P_1 := P_1^+ \cup P_1^-$, $P_2 := P_2^+ \cup P_2^-$. Thus, the sequences $\rho(\alpha_i^{\pm 1})$, $\rho(\beta_i^{\pm 1})$ converge to $y^{\pm}, z^{\pm}$ on $P_1^{\mp} \cup P_2$ and $P_1 \cup P_2^{\mp}$ respectively. It therefore follows that for large i and all $m \neq 0$, we have

$$\rho(\alpha_i^m)(P_2) \subset P_1, \rho(\beta_i^m)(P_1) \subset P_2.$$

Take a point

$$p \in P(U) \setminus \left(Y^{\pm} \cup Z^{\pm} \cup P_1 \cup P_2\right).$$

By the same reasoning, if we take i large enough then for all $m \neq 0$, $\rho(\alpha_i)(p) \in P_1, \rho(\beta_i)(p) \in P_2$. Now, (for some large i) set $\Gamma_1 := \langle \rho(\alpha_i) \rangle, \Gamma_2 := \langle \rho(\beta_i) \rangle$. Then Lemma 4.18 implies that Γ_1, Γ_2 generate a free group on two generators. $\square$

We obtain

THEOREM 4.20. *Let $\Gamma \subset GL(V)$ be a finitely-generated group which is not relatively compact. Then either Γ is virtually solvable, or it contains a free nonabelian subgroup.*

2. Free subgroups in compact Lie groups

The compact case is more complicated. Let Γ be a relatively compact finitely-generated subgroup of $G = SL(n, \mathbb{C})$. Without loss of generality, we may assume that this subgroup is irreducible, i.e., it does not preserve a proper subspace of $\mathbb{C}^n$. Let $\gamma_1, ..., \gamma_m$ denote generators of Γ and consider the subfield F in $\mathbb{C}$ generated by the matrix entries of the generators.

Reduction to a number field case. Consider the *representation variety* $R(\Gamma, G) = Hom(\Gamma, G)$. This space can be described as follows. Let $\Gamma = \langle \gamma_1, ..., \gamma_m | r_1, ... \rangle$ be a presentation of Γ (the number of relators could be infinite). Each homomorphism $\rho : \Gamma \to G$ is determined by the images of the generators of Γ. Hence, $R(\Gamma, G)$ is a subset of G^m. A map $\rho : \gamma_i \to G, i = 1, ..., n$ extends to a homomorphism of Γ if and only if

$$\rho(r_i) = 1, i = 1, ... \qquad (44)$$

Since the relators r_i are words in $\gamma_1^{\pm 1}, ..., \gamma_m^{\pm 1}$, the equations (44) amount polynomial equations on G^m. Hence, $R(\Gamma, G)$ is given by a system of polynomial equations and has a natural structure of an algebraic variety. Since the formula for the inverses in $SL(n)$ involves only integers linear combinations of products of matrix entries, it follows that the above equations have integer (in particular, rational) coefficients. Thus, the representation variety $R(\Gamma, G)$ is *defined over* $\mathbb{Q}$.

Lemma 4.21. *Algebraic points are dense in $R = R(\Gamma, G)$ with respect to the usual topology. In other words, for every homomorphism $\rho : \Gamma \to G$, there exists a sequence of homomorphisms $\rho_j : \Gamma \to G$ converging to ρ so that the matrix entries of the images of generators $\rho_j(\gamma_i)$ are in $\bar{\mathbb{Q}}$.*

Proof. This is a general fact about affine varieties defined over $\mathbb{Q}$. Let $V \subset \mathbb{R}^N$ be such a variety. We are claiming that points with coordinates in $\bar{\mathbb{Q}}$ are dense in V. The proof by induction on N. The assertion is clear for $N = 1$: Either $V = \mathbb{R}$ or V is a finite set of roots of a polynomial with rational coefficients: These roots are algebraic numbers. Suppose the assertion holds for subvarieties in $\mathbb{R}^{N-1}$. Pick a point $x \in V$ and let q_i be a sequence of rational numbers converging to its first coordinate x_1. For each rational number q_i, the intersection $V \cap \{x_1 = q_i\}$ is again an affine variety defined over $\mathbb{Q}$ which sits inside $\mathbb{R}^{N-1}$. Now the claim follows by the induction hypothesis and taking a diagonal sequence. $\square$

We now let $\rho_i \in R(\Gamma, G)$ be a sequence which converges to the identity representation $\rho : \Gamma \to \Gamma \subset G$. Recall that in section 5, we proved that for every finitely-generated subgroup $\Gamma \subset GL(n, \mathbb{C})$ which is not virtually solvable, there exists a neighborhood Σ of id in $Hom(\Gamma, GL(n, \mathbb{C}))$ so that every $\rho \in \Xi$ has image which is not virtually solvable. Therefore, without loss of generality, we may assume that $\rho_j(\Gamma)$ constructed above is not virtually solvable.

Lemma 4.22. *If Γ_j contains a free subgroup Λ_j of rank 2 then so does Γ.*

Proof. Let $g_1, g_2 \in \Gamma$ be the elements which map to the free generators of Λ_j under ρ_j. Let Λ be the subgroup of Γ generated by g_1, g_2. We claim that Λ is free of rank 2. Indeed, since Λ is 2-generated and Λ_j is free of rank 2, there exists a homomorphism $\phi_j : \Lambda_j \to \Lambda$ sending $\rho_j(g_k)$ to $g_k, k = 1, 2$. If ρ_j is not injective, we obtain an surjective homomorphism $\phi_j \circ \rho_j : \Lambda \to \Lambda$ with nontrivial kernel. Since free groups are Hopfian, this is impossible. $\square$

Thus it suffices to consider the case when F is a number field, i.e., is contained in $\bar{\mathbb{Q}}$. The Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on F and, hence, on $SL(n, F)$. Every $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ induces (a discontinuous!) automorphism σ of the complexification $SL(n, \mathbb{C})$, and therefore it will send the groups $\Gamma \subset SL(n, F)$ to $\sigma(\Gamma) \subset G(\sigma(F)) \subset SL(n, \mathbb{C})$. The homomorphism $\sigma : \Gamma \to \Gamma' := \sigma(\Gamma)$ is $1 - 1$ and therefore, if for some σ the group $SL(n, \sigma(F))$ happens to be a non-relatively compact subgroup of $SL(n, \mathbb{C})$ we are done by Theorem 4.20.

However it could happen that for each σ the group $G(\sigma(F))$ is relatively compact and thus we seemingly have gained nothing. There is a remarkable construction which saves the proof.

Adeles. (See [**69**, Chapter 6].) The ring of adeles was introduced by A. Weil in 1936. For the field F consider various norms $|\cdot| : F \to \mathbb{R}_+$. A norm is called *nonarchimedean* if instead of the usual triangle inequality one has:

$$|a+b| \leq \max(|a|,|b|).$$

For each norm ν we define F_ν to be the completion of F with respect to this norm. For each nonarchimedean norm ν the *ring of integers* $O_\nu := \{x : |x|_\nu \leq 1\}$ is an open subset of F_ν: If $|x|_\nu = 1$, $|y|_\nu < 1/2$, then for $z = x+y$ we have: $|z|_\nu \leq \max(1, |y|_\nu) = 1$. Therefore, if z belongs to a ball of radius $1/2$ centered at x, then $z \in O_\nu$.

Example 45. (A). **Archimedean norms.** Let $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, then the embedding $\sigma : F \to \sigma(F) \subset \mathbb{C}$ defines a norm ν on F by restriction of the norm (the usual absolute value) from $\mathbb{C}$ to $\sigma(F)$. Then the completion F_ν is either isomorphic to $\mathbb{R}$ or to $\mathbb{C}$. Such norms (and completions) are *archimedean* and each archimedean norm of F appears in this way.

(B). **Nonarchimedean norms.** Let $F = \mathbb{Q}$, pick a prime number $p \in \mathbb{N}$. For each number $x = q/p^n \in \mathbb{Q}$ (where both numerator and denominator of q are not divisible by p) let $\nu_p(x) := p^n$. One can check that ν is a nonarchimedean norm and the completion of $\mathbb{Q}$ with respect to this norm is the field of p-adic numbers.

Let $Nor(F)$ denote the set of all norms on F which restrict to either standard or one of the p-adic norms on $\mathbb{Q} \subset F$. Note that for each $x \in \mathbb{Q}$, $x \in O_p$ (i.e. p-adic norm of x is ≤ 1) for all but finitely many p's, since x has only finitely many primes in its denominator. The same is true for elements of F: For all but finitely many $\nu \in Nor(F)$, $\nu(x) \leq 1$.

Product formula: For each $x \in \mathbb{Q} \setminus \{0\}$

$$\prod_{\nu \in Nor(\mathbb{Q})} \nu(x) = 1.$$

Indeed, if $x = p$ is prime then $|p| = p$ for the archimedean norm, $\nu(p) = 1$ if $\nu \neq \nu_p$ is a nonarchimedean norm and $\nu_p(p) = 1/p$. Thus the product formula holds for prime numbers x. Since norms are multiplicative functions from $\mathbb{Q}^*$ to $\mathbb{R}_+$, the product formula holds for arbitrary $x \neq 0$. A similar product formula is true for an arbitrary algebraic number field F:

$$\prod_{\nu \in Nor(F)} (\nu(x))^{N_\nu} = 1,$$

where $N_\nu = [F_\nu : \mathbb{Q}_\nu]$, see [**69**, Chapter 6].

Definition 4.23. The ring of *adeles* is the *restricted product*

$$\mathbb{A}(F) := \prod_{\nu \in Nor(F)} F_\nu,$$

i.e. the subset of the direct product which consists of points whose projection to F_ν belongs to O_ν for all but finitely many ν's.

We topologize $\mathbb{A}(F)$ via the product topology. For instance, if $F = \mathbb{Q}$ then $\mathbb{A}(\mathbb{Q})$ is the restricted product

$$\mathbb{R} \times \prod_{p \text{ is prime}} \mathbb{Q}_p.$$

Now a miracle happens:

Theorem 4.24. *(See* [**69**, Chapter 6, Theorem 1]*.) The image of the diagonal embedding $F \hookrightarrow \mathbb{A}(F)$ is a discrete subset in $\mathbb{A}(F)$.*

Proof. It suffices to verify that 0 is an isolated point. Take the archimedean norms $\nu_1, ..., \nu_m$ (there are only finitely many of them) and consider the open subset

$$U = \prod_{i=1}^{m} \{x \in F_{\nu_i} : \nu_i(x) < 1/2\} \times \prod_{\mu \in Nor(F) \setminus \{\nu_1, ..., \nu_m\}} O_\mu$$

of $\mathbb{A}(F)$. Then for each $(x_\nu) \in U$,

$$\prod_{\nu \in Nor(F)} \nu(x_\nu) < 1/2 < 1.$$

Hence, by the product formula, the intersection of U with the image of F in $\mathbb{A}(F)$ consists only of $\{0\}$. $\square$

Thus the embedding $F \hookrightarrow \mathbb{A}(F)$ induces a discrete embedding

$$\Gamma \subset G(F) \hookrightarrow G(\mathbb{A}(F)).$$

For each norm $\nu \in Nor(F)$ we have the projection $p_\nu : \Gamma \to G(F_\nu)$. If the image $p_\nu(\Gamma)$ is relatively compact for each ν then Γ is a discrete compact subset of $G(\mathbb{A}(F))$, which implies that Γ is finite, a contradiction! Thus there exists a norm $\nu \in Nor(F)$ such that the image of Γ in $G(F_\nu)$ is not relatively compact. If ν happens to be archimedean we are done as before. The more interesting case occurs if ν is nonarchimedean. Then the field $F_\nu = \mathbf{k}$ is a local field (just like the p-adic completion of the rational numbers) and one appeals Theorem 4.20 to conclude that Γ contains a free subgroup. This concludes the proof of the Tits Alternative (Theorem 4.1). $\square$

REMARK 4.25. 1. The above proof works only if Γ is finitely generated. The general case was treated in ???.

2. Tits Alternative also applies to algebraic groups over fields of positive characteristic. See ???

3. The arguments in the above proof mostly follow the ones of Breuillard and Gelander in [**14**].

The Hausdorff Example. Below we describe the historically first example of a free subgroup of $SO(3)$ due to Hausdorff (constructed in 1914, [**54, 53**]). This example also illustrates the proof of Tits Alternative for relatively compact groups. Let g be the order 2 rotation in $\mathbb{R}^3$ and h be the order 3 rotation so that the angle between the axes of these rotations is $\pi/4$.

LEMMA 4.26. *The group generated by g, h is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/3$.*

PROOF. In coordinates, we can describe the rotations g, h by the matrices

$$g = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Let Γ be the group generated by g, h. This group is contained in $SO(3, \mathbb{Q}(\sqrt{3}))$....

CHAPTER 5

Growth of groups and Gromov's theorem

Let X be a metric space of bounded geometry and $x \in X$ is a base-point. We define the *growth function*

$$\beta_{X,x}(R) := |B(x,R)|,$$

the cardinality of R-ball centered at x. We introduce the following *asymptotic inequality* between functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\beta \prec \alpha,$$

if there exist constants C_1, C_2 such that $\beta(R) \le C_1 \alpha(C_2 R)$ for sufficiently large R. We say that two functions are equivalent, $\alpha \sim \beta$, if

$$\alpha \prec \beta \quad \text{and} \quad \beta \prec \alpha.$$

LEMMA 5.1. *(Equivalence class of growth is QI invariant.) Suppose that $f : (X,x) \to (Y,y)$ is a quasi-isometry. Then $\beta_{X,x} \sim \beta_{Y,y}$.*

PROOF. Let $\bar f$ be a coarse inverse to f, assume that $f, \bar f$ are L-Lipschitz. Then both f, $\bar f$ have multiplicity $\le m$ (since X and Y have bounded geometry). Then

$$f(B(x,R)) \subset B(y, LR).$$

It follows that $|B(x,R)| \le m|B(y,LR)|$ and $|B(y,R)| \le m|B(x,LR)|$. □

COROLLARY 5.2. *$\beta_{X,x} \sim \beta_{X,x'}$ for all $x, x' \in X$.*

Henceforth we will suppress the choice of the base-point in the notation for the growth function.

DEFINITION 5.3. X has polynomial growth if $\beta_X(R) \prec R^d$ for some d. X has exponential growth if $e^R \prec \beta_X(R)$. X has subexponential growth if for each $c > 0$, $\beta_X(R) \le e^{cR}$ for all sufficiently large R.

EXAMPLE 46. Show that for each (bounded geometry) space X, $\beta_X(R) \prec e^R$.

For a group G with finite generating set S we sometimes will use the notation $\beta_S(R)$ for $\beta_G(R)$, where S is used to metrize the group G. Since G acts transitively on itself, this definition does not depend on the choice of a base-point.

EXAMPLE 47. Suppose that $G = \mathbb{F}_r$ is a free nonabelian group. Show that G has exponential growth.

Suppose that H is a subgroup of G. It is then clear that

$$\beta_H \prec \beta_G.$$

Note that if $\phi : G \to \mathbb{F}_r$ is an epimorphism, then its admits a left inverse $\iota : \mathbb{F}_r \to G$. Hence G contains $\mathbb{F}_r$ and if $r \ge 2$ it follows that G has exponential growth.

The main objective of this chapter is to prove

THEOREM 5.4. *(Gromov, [**48**]) If G is a finitely generated group of polynomial growth then G is virtually nilpotent.*

We will also verify that all virtually nilpotent groups have polynomial growth.

COROLLARY 5.5. *Suppose that G is a finitely generated group which is quasi-isometric to a nilpotent group. Then G is virtually nilpotent.*

Proof. Follows directly from Gromov's theorem since polynomial growth is a QI invariant. $\square$

Remark 5.6. An alternative proof of the above corollary (which does not use Gromov's theorem) was recently given by Y. Shalom [**96**].

1. Nilpotent and solvable groups

Suppose that G is a finitely-generated nilpotent group with the lower central series

$$G = G_0 \supset G_1 = [G_0, G_0] \supset ... \supset G_s = 1,$$

and ψ is an automorphism of G. According to our discussion in section 3, ψ induces automorphisms $\bar{\psi}_i$ of the quotients

$$A_i/A_i^f$$

where $A_i = G_{i-1}/G_i$. Each $\bar{\psi}_i$ is represented by a matrix $M_i \in GL(m_i, \mathbb{Z})$.

Our goal is to analyze the semidirect product $G \rtimes \psi\mathbb{Z}$ according to the dichotomy established in in Corollary 1.9.

Case a. For each i all eigenvalues of M_i are roots of unity. Then, as in Corollary 1.9, $G \rtimes \psi\mathbb{Z}$ is virtually nilpotent. In particular, it has polynomial growth.

Case b. For some i, at least one eigenvalue of M_i is not a root of unity.

Theorem 5.7. *Under the above assumptions, the group $\tilde{G} := G \ltimes_\psi \mathbb{Z}$ has exponential growth.*

Proof. As in Corollary 1.9, we replace ψ with $\phi := \psi^N$ so that the matrix $M_i(\phi)$ admits an eigenvalue ρ so that $|\rho| \geq 2$.

Let $x \in G_i$ be an element which projects to $a \in A_i$ under the homomorphism $G_i \to G_i/G_{i+1}$. Let $z \in \tilde{G}$ denote the generator corresponding to the automorphism ϕ. Define the elements

$$x^{\epsilon_0}(zx^{\epsilon_1}z^{-1})...(z^m x^{\epsilon_m} z^{-m}) \in G_i, \epsilon_i \in \{0, 1\}.$$

After cancelling out z's we get:

$$x^{\epsilon_0} z x^{\epsilon_1} z x^{\epsilon_2} z ... z x^{\epsilon_m} z^{-m}$$

The norm of each of these elements in $\tilde{G}$ is at most $3(m+1)$. These elements are distinct for different choices of (ϵ_i)'s, since their projections to A_i are distinct according to Lemma 1.8. Thus we get 2^m distinct elements of $\tilde{G}$ whose word norm is at most $3(m+1)$. This implies that $\tilde{G}$ has exponential growth. $\square$

Proposition 5.8. *Suppose that G is a group of growth less than $2^{\sqrt{n}}$, which fits into a short exact sequence*

$$1 \to K \to G \xrightarrow{\pi} \mathbb{Z} \to 1.$$

Then K is finitely generated. Moreover, if $\beta_G(R) \prec R^d$ then $\beta_K(R) \prec R^{d-1}$.

Proof. Let $\gamma \in G$ be an element which projects to the generator 1 of $\mathbb{Z}$. Let $\{f_1, ..., f_k\}$ denote a set of generators of G. Then for each i there exists $s_i \in \mathbb{Z}$ such that $\pi(f_i\gamma^{s_i}) = 0 \in \mathbb{Z}$. Define elements $g_i := f_i\gamma^{s_i}$, $i = 1, ..., k$. Clearly, the set $\{g_1, ..., g_k, \gamma\}$ generates G. Without loss of generality we may assume that each generator g_i is nontrivial. Define

$$S := \{\gamma_{m,i} := \gamma^m g_i \gamma^{-m}, m \in \mathbb{Z}, i = 1, ..., k\}.$$

Then the (infinite) set S generates K. Given i consider products of the form:

$$\gamma_{0,i}^{\epsilon_1} \cdots \gamma_{m,i}^{\epsilon_m}, \quad \epsilon_i \in \{0, 1\}, m \geq 0.$$

We have 2^{m+1} words like this, each of length $\leq m^2$. Hence the fact that the growth of G is less than $2^{\sqrt{n}}$ implies that for a certain $m = m(i)$, two of these words are equal:

$$\gamma_{0,i}^{\epsilon_1} \cdots \gamma_{m,i}^{\epsilon_m} = \gamma_{0,i}^{\delta_1} \cdots \gamma_{m,i}^{\delta_m},$$

$\epsilon_m \neq \delta_m$. It follows that

$$\gamma_{m,i} = w(\gamma_{0,i}, ..., \gamma_{m-1,i}) \in \langle \gamma_{0,i}, ..., \gamma_{m-1,i} \rangle,$$

where w is a certain word in the generators $\gamma_{0,i}, ..., \gamma_{m-1,i}$. Consider

$$\gamma_{m+1,i} = \gamma\gamma_{m,i}\gamma^{-1} = \gamma w(\gamma_{0,i}, ..., \gamma_{m-1,i})\gamma^{-1} = w'(\gamma_{1,i}, ..., \gamma_{m,i}).$$

Here w' is the word in the generators $\gamma_{1,i}, ..., \gamma_{m,i}$ which is obtained from w by inserting the products $\gamma^{-1} \cdot \gamma$ between each pair of letters in the word w and then using the fact that

$$\gamma_{j+1,i} = \gamma\gamma_{j,i}\gamma^{-1}, j = 0, ..., m-1.$$

However $w' \in \langle\gamma_{0,i}, ..., \gamma_{m-1,i}\rangle$, since

$$\gamma_{m,i} \in \langle\gamma_{0,i}, ..., \gamma_{m-1,i}\rangle.$$

Thus

$$\gamma_{m+1,i} \in \langle\gamma_{0,i}, ..., \gamma_{m-1,i}\rangle$$

as well. We continue by induction: It follows that

$$\gamma_{n,i} \in \langle\gamma_{0,i}, ..., \gamma_{m-1,i}\rangle$$

for each $n \geq 0$. The same argument works for the negative values of m and therefore there exists $M(i)$ so that each $\gamma_{j,i}$ is contained in the subgroup of K generated by

$$\{\gamma_{l,i}, |l| \leq M(i)\}.$$

Hence the subgroup K is generated by the finite set

$$\{\gamma_{l,i}, |l| \leq M(i), i = 1, ..., k\}.$$

This proves the first assertion of the Proposition.

Now let us prove the second assertion which estimates the growth function of K. Take a finite generating set Y of the subgroup K and set $X := Y \cup \{\gamma\}$, where γ is as above. Then X is a generating set of G. Given $n \in \mathbb{N}$ let $N := \beta_Y(n)$, where β_Y is the growth function of K with respect to the generating set Y. Thus there exists a subset

$$H := \{h_1, ..., h_N\} \subset K$$

where $\|h_i\|_Y \leq n$ and $h_i \neq h_j$ for all $i \neq j$. Then we get a set T of $(2n+1) \cdot N$ pairwise distinct elements

$$h_i\gamma^j, \quad -n \leq j \leq n, \quad i = 1, ..., N.$$

It is clear that $\|h_i\gamma^j\|_X \leq 2n$ for each $h_j\gamma^j \in T$. Therefore

$$n\beta_Y(n) \leq (2n+1)\beta_Y(n) = (2n+1)N \leq \beta_X(2n) \leq C(2n)^d = 2^d C \cdot n^d$$

It follows that

$$\beta_Y(n) \leq 2^d C \cdot n^{d-1} \prec n^{d-1}. \quad \square$$

2. Growth of nilpotent groups

Consider an s-step nilpotent group G with the lower central series

$$G_0 \supset G_1 \supset ...G_{s-1} \supset 1,$$

and the abelian quotients $A_i = G_i/G_{i+1}$. Let d_i denote the rank of A_i (or, rather, the rank of its free part). Define

$$d(G) := \sum_{i=0}^{s-1}(i+1)d_i.$$

The precise calculation of growth of nilpotent groups is due to H. Bass:

Theorem 5.9. *(Bass,* **[5]***)* $\beta_G(R) \sim R^{d(G)}$.

Example 48. Prove Bass' theorem for abelian groups.

Our goal is to prove only that G has at most polynomial growth without getting a sharp estimate.

For this proof we introduce the notion of *distortion* for subgroups which is another useful concept of the geometric group theory. Let H be a finitely generated subgroup of a finitely generated group G, let d_H, d_G denote the respective word metrics on H and G, let $B_G(e,r)$ denote r-ball centered at the origin in the group G.

DEFINITION 5.10. Define the *distortion function* $\delta(R) = \delta(H : G, R)$ as

$$\delta(R) := \max\{d_H(e,h) : h \in B_G(e,R)\}.$$

The subgroup H is called *undistorted* (in G) if $\delta(R) \sim R$.

EXAMPLE 49. Show that H is undistorted iff the embedding $\iota : H \to G$ is a quasi-isometric embedding.

In general, distortion functions for subgroups can be as bad as one can imagine, for instance, nonrecursive.

EXAMPLE 50. Let $G := \langle a, b : aba^{-1} = b^p\rangle$, $p \geq 2$. Then the subgroup $H = \langle b\rangle$ is exponentially distorted in G.

PROOF. To establish the lower exponential bound note that:

$$g_n := a^n b a^{-n} = b^{p^n},$$

hence $d_G(1, g_n) = 2n+1$, $d_H(1, g_n) = p^n$, hence

$$\delta(R) \geq p^{[(R-1)/2]}.$$

It will leave the upper exponential bound as a exercise (compare the proof of Theorem 5.12). □

Recall that each subgroup of a finitely generated nilpotent group is finitely generated itself.

The following theorem was originally proven by M. Gromov in [**50**] (see also [**106**]); later on, an explicit computation of the degrees of distortion was established by D. Osin in [**83**]:

THEOREM 5.11. *Let G be a finitely generated nilpotent group, then every subgroup $H \subset G$ has polynomial distortion.*

I will prove only a special case of this result which will suffice for our purposes:

THEOREM 5.12. *Let G be a finitely generated nilpotent group, then it commutator subgroup $G_1 := [G,G] \subset G_0 := G$ has at most polynomial distortion.*

PROOF. As the equivalence class of a distortion function is a commensurability invariant, it suffices to consider the case when $A = G/G_1$ is free abelian. Suppose that G is n-step nilpotent. We choose a generating set T of G as follows. Set $T := T_0 \sqcup T_1 \sqcup ... \sqcup T_n$, where T_0 projects to the set of free generators of A, $T_i \subset G_i$. Let $x_i, i = 1, ..., p$, denote the elements of T_0. We assume that each T_{i+1} contains all the commutators $[y_k^{\pm 1}, x_j^{\pm 1}]$, where $y_j \in T_{i-1}$, $i = 1, ..., n$.

For each word w in the generating set T define its i-length $\ell_i(w)$ to be the total number of the letters $y_j^{\pm 1} \in T_i$ which appear in w. Clearly,

$$\|w\| = \sum_{i=0}^{n} \ell_i(w).$$

Given an appearance of the letter $a = x_k^{\pm 1}$ in the word w let's "move" this letter through w so that the resulting word w' equals to w as an element of G and that the letter a appears as the first letter in the new word w'.

This involves at most $\|w\|$ "crossings" of the letters in w. Each "crossing" results in introducing a commutator of the corresponding generators:

$$y_j a \to a y_j [y_j^{-1}, a^{-1}].$$

Therefore,

$$\ell_{i+1}(w') \le \ell_{i+1}(w) + \ell_i(w). \tag{51}$$

We will apply this procedure inductively to each letter $a = x_k^{\pm 1}$ in the word w, so that the new word w_* starts with a power of x_1, then comes the power of x_2, etc, by moving first all appearances of x_1 to the left, then of x_2 to the left, etc. In other words

$$w_* = x_1^{\alpha_1} x_2^{\alpha_2} ... x_p^{\alpha_p} \cdot u,$$

where u is a word in the generators $S = T_1 \cup .. \cup T_n$ of the group G_1. We have to estimate the length of the word u. We have a sequence of words

$$w_0 = w, w_1, ..., w_m = w_*,$$

where each w_i is the result of moving a letter in w_{i-1} to the left and $m \le \ell_0(w)$. Clearly, $\ell_0(w_j) = \ell_0(w)$ for each j. By applying the inequality (51) inductively we obtain

$$\ell_{i+1}(w_j) \le \ell_{i+1}(w) + j\ell_i(w_{j-1})$$

and hence:

$$\ell_{i+1}(w_m) \le \ell_{i+1}(w) + m\ell_i(w) + m(m-1)\ell_{i-1}(w) + ... + \frac{m!}{(m-i-1)!}\ell_0(w)$$

$$\le m^{i+1} \sum_{j=0}^{i+1} \ell_j(w) \le m^{i+1} \|w\|.$$

Therefore, $\ell_i(w_*) \le \|w\|^{i+1}$ for each i. By adding up the results we get:

$$\|u\| = \sum_{i=1}^{n} \ell_i(w_*) \le \sum_{i=1}^{n} \|w\|^{i+1} \le n\|w\|^{n+1}.$$

Suppose now that w represents an element g of $H = G_1$. Then w projects to 0 in A and, hence, $\pi(u) = 0, \pi(x_1^{\alpha_1} x_2^{\alpha_2} ... x_p^{\alpha_p}) = 0$ where $\pi : G \to A$ is the quotient map. Since $\pi(x_1), ..., \pi(x_p)$ is a free generating set of A, it follows that the word $x_1^{\alpha_1} x_2^{\alpha_2} ... x_p^{\alpha_p}$ is empty, and, hence $w_* = u$ is a word in the generators of the group $H = G_1$. Thus for each $g \in H$ we obtain:

$$d_H(1, g) \le n d_G(1, g)^{n+1}.$$

Hence the distortion function δ of H in G satisfies $\delta(R) \prec R^{n+1}$. □

THEOREM 5.13. *(J. Wolf, [**112**]) Every nilpotent group has at most polynomial growth.*

PROOF. The proof is by induction on the number of steps in the nilpotent group. The assertion is clear is G is 1-step nilpotent (i.e. abelian). Suppose that each $s-1$-step nilpotent group has at most polynomial growth. Consider s-step nilpotent group G:

$$G = G_0 \supset G_1 \supset ... G_{s-1} \supset 1.$$

By the induction hypothesis, G_1 has growth $\prec R^d$ and, according to Theorem 5.12, the distortion of G_1 in G is at most R^D. Let r denote the rank of the abelianization of G.

Consider an element $\gamma \in B_G(e, R)$, then γ can be written down as a product $w_0 w_1$ where w_0 is a word on T_0 of the form:

$$x_1^{k_1} ... x_n^{k_n},$$

and w_1 is a word on T_1. Then $\|w_0\| \le R$, and $\|w_1\| \le \|w_0\| + \|\gamma\| \le 2R$. The number of the words w_0 of length $\le R$ is $\prec R^r$. Since G_1 has distortion $\prec R^D$ in G, the length of the word w_1 on the generators T_1 is $\prec (2R)^D$. Since $\beta_{G_1} \prec R^d$ we conclude that

$$\beta_G(R) \prec R^r \cdot (2R)^{dD} \sim R^{dD+r}. \quad □$$

COROLLARY 5.14. *A solvable group G has polynomial growth iff G is virtually nilpotent.*

Proof. It remains to show that if G is a solvable and has polynomial growth then G is virtually nilpotent group. By considering the derived series of G we get the short exact sequence

$$1 \to K \to G \to \mathbb{Z}.$$

Suppose that G has polynomial growth $\prec R^d$, then K is finitely generated, solvable and has growth $\prec R^{d-1}$. By induction, we can assume that K is virtually nilpotent. Then Theorem 5.7 implies that G is also virtually nilpotent. □

Corollary 5.15. *Suppose that G is a finitely generated linear group. Then G either has polynomial or exponential growth.*

Proof. By Tits alternative either G contains a nonabelian free subgroup (and hence G has exponential growth) or G is virtually solvable. For virtually solvable groups the assertion follows from Corollary 5.14. □

R. Grigorchuk [**47**] constructed finitely generated groups of *intermediate growth*, i.e. their growth is superpolynomial but subexponential. Existence of finitely-presented groups of this type is unknown. We note that all currently known groups of intermediate growth have growth faster than $2^{\sqrt{n}}$.

3. Elements of the nonstandard analysis

Our discussion here follows [**46**], [**105**].

Let I be a countable set. Recall that an *ultrafilter* on I is a finitely additive measure with values in the set $\{0,1\}$ defined on the power set 2^I. We will assume that ω is nonprincipal. Given a set S we have its *ultrapower*

$$S^* := S^I/\omega,$$

which is a special case of the ultraproduct. Note that if G is a group (ring, field, etc.) then G^* has a natural group (ring, field, etc.) structure. If S is totally ordered then S^* is totally ordered as well: $[f] \le [g]$ (for $f, g \in S^I$) iff $f(i) \le g(i)$ for ω-all $i \in I$.

For subsets $P \subset S$ we have the canonical embedding $P \hookrightarrow \hat{P} \subset S^*$ given by sending $x \in P$ to the constant function $f(i) = x$.

Thus we define the ordered semigroup $\mathbb{N}^*$ (the nonstandard natural numbers) and the ordered field $\mathbb{R}^*$ (the nonstandard real numbers). An element $R \in \mathbb{R}^*$ is called *infinitely large* if given any $r \in \mathbb{R} \subset \mathbb{R}^*$, one has $R \ge r$. Note that given any $R \in \mathbb{R}^*$ there exists $n \in \mathbb{N}^*$ such that $n > R$.

Definition 5.16. A subset $W \subset S_1^* \times ... \times S_n^*$ is called *internal* if "membership in W can be determined by coordinate-wise computation", i.e. if for each $i \in I$ there is a subset $W_i \subset S_1 \times ... \times S_n$ such that for $f_1 \in S_1^I, ..., f_n \in S_n^I$

$$([f_1], ..., [f_n]) \in W \iff (f_1(i), ..., f_n(i)) \in W_i \text{ for } \omega - \text{all } i \in I.$$

The sets W_i are called *coordinates* of W.

Using this definition we can also define *internal functions* $S_1^* \to S_2^*$ as functions whose graphs are internal subsets of $S_1^* \times S_2^*$. Clearly the image of an internal function is an internal subset of S_2^*.

Lemma 5.17. *Suppose that $A \subset S$ is infinite subset. Then $\hat{A} \subset S^*$ is not internal.*

Proof. Suppose that $A_i, i \in I$, are coordinates of $\hat{A}$. Let $a_1, a_2, ...$ be an infinite sequence of distinct elements of A. Define the following function $f \in S^I$:

Case 1. $f(n) = a_j$, where $j = \max\{j' : a_{j'} \in A_n\}$ if the maximum exists,

Case 2. $f(n) = a_{n+j}$, where $j = \min\{j' : a_{n+j'} \in A_n\}$ if the maximum above does not exist.

Note that for each $n \in I$, $f(n) \in A_n$, therefore $[f] \in \hat{A}$. Since $\hat{A}$ consists of (almost) constant functions, there exists $m \in \mathbb{N}$ such that $f(n) = a_m$ for ω-all $n \in I$.

It follows that the Case 2 of the definition of f cannot occur for ω-all $n \in I$. Thus for almost all $n \in I$ the function f is defined as in Case 1. It follows that for almost all $n \in I$, $a_{m+1} \notin A_n$. Thus $a_{m+1} \notin A$, which is a contradiction. □

Corollary 5.18. $\mathbb{N}$ *is not an internal subset of* $\mathbb{N}^*$.

Suppose that (X, d) is a metric space. Then X^* has a natural structure of $\mathbb{R}^*$-metric space where the "distance function" d takes values in $\mathbb{R}^*_+$:

$$d([f], [g]) := [i \mapsto d(f(i), g(i))].$$

We will regard d^* as a generalized metric, so we will talk about metric balls, etc. Note that the "metric balls" in X^* are internal subsets.

A bit of logic. Let Φ be a statement about elements and subsets of S. The *nonstandard interpretation* Φ^* of Φ is a statement obtained from Φ by replacing:

1. Each entry of the form "$x \in S$" with "$x \in S^*$".
2. Each entry of the form "$A \subset S$" with "A an internal subset of S^*".

Theorem 5.19. *(Los) A statement Φ about S is true iff its nonstandard interpretation Φ^* about S^* is true.*

As a corollary we get:

Corollary 5.20. *1. (Completeness axiom) Each nonempty bounded from about internal subset $A \subset \mathbb{R}^*$ has supremum. (Note that $\mathbb{R} \subset \mathbb{R}^*$ does not have supremum.)*

2. (Nonstandard induction principle.) Suppose that $S \subset \mathbb{N}^*$ is an internal subset such that $1 \in S$ and for each $n \in S$, one has $n+1 \in S$. Then $S = \mathbb{N}^*$. (Note that this fails for $S = \mathbb{N} \subset \mathbb{N}^*$.)

Example 52. 1. Give a direct proof of the completeness axiom for $\mathbb{R}^*$.

2. Use the completeness axiom to derive the nonstandard induction principle.

Suppose we are given $a_n \in \mathbb{R}^*$, where $n \in \mathbb{N}^*$. Using the nonstandard induction principle on can define the nonstandard products:

$$a_1 ... a_n, n \in \mathbb{N}^*,$$

as an internal function $f : \mathbb{N}^* \to \mathbb{R}^*$ given by $f(1) = a_1$, $f(n+1) = f(n)a_{n+1}$.

4. Regular growth theorem

A metric space X is called *doubling* if there exists a number N such that each R-ball in X is covered by N balls of radius $R/2$.

Exercise 53. Show that doubling implies polynomial growth for spaces of bounded geometry.

Although there are spaces of polynomial growth which are not doubling, the Regular Growth Theorem below shows that groups of polynomial growth exhibit doubling-like behavior.

Our discussion here follows [**105**].

Theorem 5.21 (Regular growth theorem). *Suppose that G is a finitely generated group such that $\beta_G(R) \prec R^d$. Then there exists an infinitely large $\rho \in \mathbb{R}^*$ such that for all $i \in \mathbb{N} \setminus \{1\}$ the following assertion $P(\rho, i)$ holds:*

If $x_1, ..., x_t \in B(e, \rho/2) \subset G^*$ and the balls $B(x_j, \rho/i)$ are pairwise disjoint ($j = 1, ..., t$) then $t \leq i^{d+1}$.

Here e is the identity in G^*.

Proof. Start with an arbitrary infinitely large $R \in \mathbb{R}^*$ (for instance, represented by the sequence n, $n \in \mathbb{N} = I$). I claim that the number ρ can be found within the interval $[\log R, R]$ (here logarithm is taken with the base 2). Suppose to the contrary, that for each $\rho \in [\log R, R]$ there exists $i \in \mathbb{N} \setminus \{1\}$ such that $P(\rho, i)$ fails. Observe that the assertion $P(\rho, i)$ also makes sense when $i \in \mathbb{N}^*$. Then we define the function

$$\iota : [\log R, R] \to \mathbb{N}^*, \ \iota(\rho) \text{ is the smallest } i \text{ for which } P(\rho, i) \text{ fails.}$$

Since i is less than any nonstandard natural number, it follows that the image of ι is contained in $\mathbb{N}$ (embedded in $\mathbb{N}^*$ diagonally). Since the nonstandard distance function is an internal function, the function ι is internal as well. Therefore, according to Lemma 5.17, the image of ι has to be finite. Thus there exists $K \in \mathbb{N}$ such that

$$\iota(\rho) \in [2, K], \quad \forall \rho \in [\log R, R].$$

We now define (using the nonstandard induction) the following elements of G^*:

1. $x_1(1), ..., x_{t_1}(1) \in B(e, R/2)$ such that $t_1 = i_1^{d+1}$, (for $i_1 = \iota(R)$) and the balls $B(x_j(1), R/i_1)$, contained in $B(e, R)$ are pairwise disjoint.

2. Each nonstandard ball $B(x_j(1), R/i_1)$ is isometric to $B(e, R/i_1)$. Therefore failure of $P(R/i_1, i_2)$ (where $i_2 = \iota(R/i_1)$) implies that in each ball $B(x_j(1), R/(2i_1))$ we can find points

$$x_1(2), ..., x_{t_2}(2), t_2 = i_2^{d+1},$$

so that the balls $B(x_j(2), R/(i_1 i_2)) \subset B(x_j(1), R/i_1)$ are pairwise disjoint.

We continue via the nonstandard induction. Given $u \in \mathbb{N}^*$ such that the points $x_1(u), ..., x_{t_u}(u)$ are constructed, we construct the next generation of points

$$x_1(u+1), ..., x_{t_{u+1}}(u+1)$$

within each ball $B(x_j(u), R/(2i_1...i_u))$ so that the balls

$$B(x_j(u+1), R/(i_1...i_{u+1}))$$

are pairwise disjoint and $t_{u+1} = i_u^{d+1}$. Here and below the product $i_1...i_{u+1}$ is understood via the nonstandard induction as in the end of the previous section.

Note that

$$B(x_j(u+1), R/(i_1...i_{u+1})) \subset B(x_j(u), R/(i_1...i_u)).$$

In particular,

$$B(x_j(u+1), R/(i_1...i_{u+1})) \cap B(x_k(u+1), R/(i_1...i_{u+1})) = \emptyset$$

when $j \neq k$.

Remark 5.22. Thus in the formulation of the Assertion $P(\rho, i)$ it is important to consider points in the ball $B(e, \rho/2)$ rather than in $B(e, \rho)$.

This induction process continues as long as $R/(i_1...i_{u+1}) \geq \log R$. Recall that $i_j \geq 2$, hence

$$R/(i_1...i_u) \leq 2^{-u} R.$$

Therefore, if $u > \log R - \log\log R$ then

$$R/(i_1...i_u) < \log R.$$

Thus there exists $u \in \mathbb{N}^*$ such that

$$R/(i_1...i_u) \geq \log R, \text{ but } R/(i_1...i_{u+1}) < \log R.$$

Let's count the "number" (nonstandard of course!) of points $x_i(k)$ we have constructed between the step 1 of induction and the u-th step of induction:

We get $i_1^{d+1} i_2^{d+1} ... i_u^{d+1}$ points; since

$$R/(K \log R) \leq R/(i_{u+1} \log R) < i_1...i_u,$$

we get:

$$(R/(K \log R))^{d+1} \leq (i_1 i_2 ... i_u)^{d+1}.$$

What does this inequality actually mean? Recall that R and u are represented by sequences of real and natural numbers R_n, u_n respectively. The above inequality thus implies that for ω-all $n \in \mathbb{N}$, one has:

$$\left(\frac{R_n}{K \log R_n}\right)^{d+1} \leq |B(e, R_n)|.$$

Since $|B(e, R)| \leq C R^d$, we get:

$$R_n \leq Const (\log(R_n))^{d+1},$$

for ω-all $n \in \mathbb{N}$. If $R_n = 2^{\lambda_n}$, we obtain

$$2^{\lambda_n/(d+1)} \leq Const \lambda_n,$$

where $\omega\text{-}\lim \lambda_n = \infty$. Contradiction. $\square$

5. An alternative proof avoiding the nonstandard analysis

The key to this proof is the following convexity property of the growth function:

THEOREM 5.23. *Let G be a finitely-generated group, $\beta = \beta_G$ be its growth function. Then β is log-concave:*

....

6. Topological group actions

The proof of Gromov's polynomial growth theorem relies heavily upon the work of Montgomery and Zippin on Hilbert's 5-th problem (characterization of Lie group as topological groups).[1] Therefore in this section we collect several *elementary* facts in point-set topology and review, *highly nontrivial* results of Montgomery and Zippin.

Recall that a *topological group* is a group G which is given topology so that the group operations (multiplication and inversion) are continuous. A continuous group action of a topological group G on a topological space X is a continuous map

$$\mu : G \times X \to X$$

such that $\mu(e, x) = x$ for each $x \in X$ and for each $g, h \in G$

$$\mu(gh, x) = \mu(g) \circ \mu(h)(x).$$

In particular, for each $g \in G$ the map $x \mapsto \mu(g)(x)$ is a homeomorphism $X \to X$. Thus each action μ defines a homomorphism $G \to Homeo(X)$. The action μ is called *effective* if this homomorphism is injective.

Throughout this section we will consider only *metrizable* topological spaces X. We will topologize the group of homeomorphisms $Homeo(X)$ via the compact-open topology, so that we obtain a continuous action $Homeo(X) \times X \to X$.

LEMMA 5.24. *Fix some $r > 0$ and let Y be a geodesic metric space where each metric r-ball is compact. Then Y is a proper metric space.*

PROOF. Pick a point $o \in Y$ and a number ϵ in the open interval $(0, r)$. We will prove inductively that for each $n \in \mathbb{N}$ the ball

$$B(o, n(r - \epsilon))$$

is compact. The assertion is clear for $n = 1$ since $B(o, r - \epsilon)$ is a closed subset of the compact $B(o, r)$. Suppose the assertion holds for some $n \in \mathbb{N}$. Then the metric sphere $S(o, n(r - \epsilon)) = \{y \in Y : d(y, o) = n(r - \epsilon)\}$ is compact. Let $\{x_j\}$ be a finite $\epsilon/2$-net in $S(o, n(r - \epsilon))$. Since Y is a geodesic metric space, for each point $y \in Y$ such that $d(y, o) = R > n(r - \epsilon)$, there exists a point $y' \in S_{n(r-\epsilon)}(o)$, which lies on a geodesic connecting o and y, such that $d(y', y) = R - n(r - \epsilon)$. Therefore, given each point $y \in B(o, (n + 1)(r - \epsilon))$, there exists a point x_j as above such that

$$d(y, x_j) \leq r - \epsilon + \frac{\epsilon}{2} \leq r.$$

Therefore $y \in B(x_j, r)$. Therefore, the finite union of compact metric balls $B(x_j, r)$ cover the ball $B(o, (n + 1)(r - \epsilon))$. Thus the ball $B(o, (n + 1)(r - \epsilon))$ is compact. $\square$

EXAMPLE 54. Construct an example of a *non-geodesic* metric space where the assertion of the above lemma fails.

DEFINITION 5.25 (Property A, section 6.2 of **[74]**). Suppose that H is a separable, locally compact topological group. Then H is said to satisfy Property A if for each neighborhood V of e in H there exists a compact subgroups $K \subset H$ so that $K \subset V$ and H/K is a Lie group.

[1] In 1990 J. Hirschfeld **[56]** gave an alternative solution of Hilbert's problem, based on nonstandard analysis. However, his proof does not seem to apply in the context of more general topological group actions needed for Gromov's proof.

In other words, the group H can be *approximated by* the Lie groups H/K.

According to [**74**, Chapter IV], each separable locally compact group H contains an open and closed subgroup $\hat{H} \subset H$ such that $\hat{H}$ satisfies Property A.

THEOREM 5.26 (Montgomery-Zippin, [**74**], Corollary on page 243, section 6.3). *Suppose that X is a topological space which is connected, locally connected, finite-dimensional and locally compact. Suppose that H is a separable locally compact group satisfying Property A, $H \times X \to X$ is a topological action which is effective and transitive. Then H is a Lie group.*

Suppose now that X is a metric space which is complete, proper, connected, locally connected. We give $Homeo(X)$ the compact-open topology.

Let $H \subset Homeo(X)$ be a closed subgroup for which there exists $L \in \mathbb{R}$ such that each $h \in H$ is L-Lipschitz. (For instance, $H = \mathrm{Isom}(X)$.) We assume that $H \curvearrowright X$ is transitive. Pick a point $x \in X$. It is clear that $H \times X \to X$ is a continuous effective action. It follows from the Arcela-Ascoli theorem that H is locally compact.

THEOREM 5.27. *Under the above assumptions, the group H is a Lie group with finitely many connected components.*

PROOF.

LEMMA 5.28. *The group H is separable.*

PROOF. Given $r \in \mathbb{R}_+$ consider the subset $H_r = \{h \in H : d(x, h(x)) \le r\}$. By Arcela-Ascoli theorem, each H_r is a compact set. Therefore

$$H = \bigcup_{r \in \mathbb{N}} H_r$$

is a countable union of compact subsets. Thus it suffices to prove separability of each H_r. Given $R \in \mathbb{R}_+$ define the map

$$\phi_R : H \to C_L(B(x,R), X)$$

given by the restriction $h \mapsto h|B(x,R)$. Here $C_L(B(x,R), X)$ is the space of L-Lipschitz maps from $B(x,R)$ to X. Observe that $C_L(B(x,R), X)$ is metrizable via

$$d(f,g) = \max_{y \in B(x,R)} d(f(y), g(y)).$$

Thus the image of H_r in each $C_L(B(x,R), X)$ is a compact metrizable space. Therefore $\phi_R(H_r)$ is separable. Indeed, for each $i \in \mathbb{N}$ take $\mathcal{E}_i \subset \phi_R(H_r)$ to be an $\frac{1}{i}$-net. The union

$$\bigcup_{i \in \mathbb{N}} \mathcal{E}_i$$

is a dense countable subset of $\phi_R(H_r)$. On the other hand, the group H (as a topological space) is homeomorphic to the *inverse limit*

$$\lim_{R \in \mathbb{N}} \phi_R(H),$$

i.e. the subset of the product $\prod_i \phi_i(H)$ (given the product topology) which consists of sequences (g_i) such that

$$\phi_j(g_i) = g_j, j \le i.$$

Let $E \subset \phi_i(H_r)$ be a dense countable subset. For each element $e_i \in E_i$ consider a sequence $(g_j) = \tilde{e}_i$ in the above inverse limit such that $g_i = e_i$. Let $\tilde{e}_i \in H$ be the element corresponding to this sequence (g_j). It is clear now that

$$\bigcup_{i \in \mathbb{N}} \{\tilde{e}_i \in H, e_i \in E_i\}$$

is a dense countable subset of H_r. □

COROLLARY 5.29. *Separability implies that for each open subgroup $U \subset H$, the quotient H/U is a countable set.*

LEMMA 5.30. *The orbit* $Y := \hat{H}x \subset X$ *is open.*

PROOF. If Y is not open then it has empty interior (since $\hat{H}$ acts transitively on Y). Since $\hat{H} \subset H$ is closed, the Arcela-Ascoli theorem implies that Y is closed as well. Since $\hat{H}$ is open, by the above corollary, the coset $S := H/\hat{H}$ is countable. Choose representatives g_i of S. Then

$$\bigcup_i g_i Y = X.$$

Therefore the space X is a countable union of closed subsets with empty interior. However, by Baire's theorem, each first category subset in the locally compact metric space X has empty interior. Contradiction. □

We now can conclude the proof of Theorem 5.27. Let $Z \subset Y$ be the connected component of Y containing x. Its stabilizer $F \subset \hat{H}$ again has the Property A. It is clear that F is an open subgroup of $\hat{H}$. Then the assumptions of Theorem 5.26 are satisfied by the action $F \curvearrowright Z$. Therefore F is a Lie group. However $F \subset H$ is an open subgroup; therefore the group H is a Lie group as well. Let K be the stabilizer of x in H. The subgroup K is a compact Lie group and therefore has only finitely many connected components. Since the action $H \curvearrowright X$ is transitive, X is homeomorphic to H/K. Connectedness of X now implies that H has only finitely many connected components. □

We now verify that the isometry groups of asymptotic cones corresponding to groups of polynomial growth satisfy the assumptions of Theorem 5.27.

PROPOSITION 5.31. *Let* G *be a group of growth* $\prec R^d$*. Suppose that* $\rho = (\rho_n)$ *is a sequence satisfying the assertion of the Regular Growth Theorem. Then the asymptotic cone* X_ω *constructed from the Cayley graph of* G *by rescaling via* ρ_n^{-1}*, is (a) a proper homogeneous metric space, (b) has the covering dimension* $\le d+1$*.*

PROOF. (a) Recall that X_ω is complete, geodesic and G_ω acts isometrically and transitively on X_ω, see Proposition 3.9.

Therefore, according to Lemma 5.24 it suffices to show that the metric ball $B(e_\omega, 1/2)$ is totally bounded. Let $\epsilon > 0$. Then there exists $i \in \mathbb{N}$, $i \ge 2$, such that $1/(2i) < \epsilon$. For the ball $B(e, \rho) \subset G^*$ consider a maximal collection of points $x_i \in B(e, \rho/2)$ so that the balls $B(x_j, \rho/i)$ are pairwise disjoint. Then, according to the regular growth theorem, the number t of such points x_j does not exceed i^{d+1}. Then the points $x_1, ..., x_t$ form a $2\rho/i$-net in $B(e, \rho/2)$. By passing to X_ω we conclude that the corresponding points $x_{1\omega}, ..., x_{t\omega} \in B(e_\omega, 1/2)$ form an ϵ-net. Since t is finite we conclude that $B(e_\omega, 1/2)$ is totally bounded and therefore compact.

(b) Recall that the (covering) dimension of a metric space Y is the least number n such that for all sufficiently small $\epsilon > 0$ the space Y admits a covering by ϵ-balls so that the *multiplicity* of this covering is $\le n+1$.

To prove the dimension bound we first review the concept of *Hausdorff dimension* for metric spaces. Let K be a metric space and $\alpha > 0$. The α-Hausdorff measure $\mu_\alpha(K)$ is defined as

$$\lim_{r \to 0} \inf \sum_{i=1}^{N} r_i^\alpha,$$

where the infimum is taken over all finite coverings of K by balls $B(x_i, r_i)$, $r_i \le r$ $(i = 1, ..., N)$. Then the *Hausdorff dimension* of K is defined as:

$$\dim_H(K) := \inf\{\alpha : \mu_\alpha(K) = 0\}.$$

EXAMPLE 55. Verify that the Hausdorff dimension of the Euclidean space $\mathbb{R}^n$ is n.

We will need the following theorem which will be proven in the next section:

THEOREM 5.32 (Sznirelman, Hurewicz-Wallman, **[57]**). $\dim(K) \le \dim_H(K)$*, where* dim *stands for the covering dimension.*

Thus it suffices to show finiteness of the Hausdorff dimension of X_ω. We first verify that the Hausdorff dimension of $B(e_\omega, 1/2)$ is at most $d+1$. Pick $\alpha > d+1$; for each i consider the covering of $B(e_\omega, 1/2)$ by the balls

$$B(x_{j\omega}, 2/i), j = 1, ..., t \le i^{d+1}.$$

Therefore we get:

$$\sum_{j=1}^{t} (2/i)^\alpha \le 2^\alpha i^{d+1}/i^\alpha = 2^\alpha i^{d+1-\alpha}.$$

Since $\alpha > d+1$, $\lim_{i\to\infty} 2^\alpha i^{d+1-\alpha} = 0$. Hence $\mu_\alpha(B(e_\omega, 1/2)) = 0$.

Thus, by homogeneity of X_ω, $\dim_{Haus}(B(x, 1/2)) \le d+1$ for each $x \in X_\omega$. Since the Hausdorff measure is additive, we conclude that for each compact subset $K \subset X_\omega$, $\dim_H(K) \le d+1$.

We now consider the entire space X_ω. Let A_n denote the closed *annulus*

$$\overline{B(e_\omega, n+1) \setminus B(e_\omega, n)}.$$

Then A_n is compact and hence $\mu_\alpha(A_n) = 0$ for each $\alpha > d+1$. Additivity of μ_α implies that

$$\mu_\alpha(X_\omega) \le \sum_{n=1}^{\infty} \mu_\alpha(A_n) = 0.$$

Therefore $\dim(X_\omega) \le \dim_H(X_\omega) \le d+1$. □

7. Hausdorff and topological dimension

The main goal of this section is to prove Theorem 5.32 relating Hausdorff and topological dimensions $\dim_H$ and $\dim$ for (compact) metric spaces. Even though properness of X is not essential in this theorem, we will impose this restriction in order to simplify the arguments. Besides, we will apply the theorem to a proper metric space.

We begin with establishing certain basic facts about metric spaces which should be familiar to the reader in the context of differentiable manifolds: Partition of unity and approximation of continuous maps by (locally) Lipschitz maps. Once these facts are established, the proof of the theorem is rather easy and straightforward.

From now on, we assume that X is a proper metric space. A map from X to a metric space Y is called *locally Lipschitz* if its restriction to each compact is Lipschitz. We let $Lip(X;Y)$ denote the space of locally Lipschitz maps $X \to Y$. We set $Lip(X) := Lip(X;\mathbb{R})$. Let $C(X;Y)$ denote the space of all continuous, maps $X \to Y$; set $C(X) := C(X;\mathbb{R})$. We give the space $C(X;Y)$ the topology of uniform convergence.

Proposition 5.33. *$Lip(X)$ is a dense subset in $C(X)$.*

Proof. Given a $x_0 \in X$ we have define the 1-Lipschitz function d_{x_0} by $d_{x_0}(x) := d(x, x_0)$.

Lemma 5.34 (Lipschitz bump-function). *Let $0 < R < \infty$. Then there exists a $\frac{1}{R}$–Lipschitz function $\varphi = \varphi_{x_0,R}$ such that*

1. *φ is positive on $B(x_0, R)$ and zero on $X \setminus B(x_0, R)$.*
2. *$\varphi(x_0) = 1$.*
3. *$0 \le \varphi \le 1$ on X.*

Proof. We first define the $\zeta : \mathbb{R}_+ \to [0,1]$ which vanishes on the interval $[R, \infty)$, is linear on $[0, R]$ and equals 1 at 0. Then ζ is $\frac{1}{R}$–Lipschitz. Now take $\varphi := \zeta \circ d_{x_0}$. □

Lemma 5.35 (Lipschitz partition of unity). *Suppose that we are given a locally finite covering of X by a countable set of the open R_i-balls $B_i := B_{R_i}(x_i)$, $i \in I \subset \mathbb{N}$. Then there exists a collection of Lipschitz functions $\eta_i, i = 1, ..., N$ so that:*

1. *$\sum_i \eta_i \equiv 1$.*
2. *$0 \le \eta_i \le 1, \quad \forall i \in I$.*
3. *$Supp(\eta_i) \subset \overline{B_{R_i}(x_i)}, \quad \forall i \in I$.*

Proof. For each i define the bump-function using Lemma 5.34

$$\varphi_i := \varphi_{x_i, R_i}.$$

Then the function

$$\varphi(x) := \sum_{i \in I} \varphi_i$$

is positive on X. Finally, define

$$\eta_i := \frac{\varphi_i}{\varphi}.$$

It is clear that the functions η_i satisfy all the required properties. □

Remark 5.36. Since the collection of balls $\{B_i\}$ is locally finite, it is clear that the function

$$L(x) := \sup_{i \in I, \eta_i(x) \neq 0} Lip(\eta_i)$$

is bounded on compacts in X, however, in general, it is unbounded on X.

We are now ready to prove Proposition 5.33. Fix a base-point $o \in X$ and let A_n denote the annulus

$$\{x \in X : n - 1 \leq d(x, o) \leq n\}, n \in \mathbb{N}.$$

Let f be a continuous function on X. Pick $\epsilon > 0$. Our goal is to find a locally Lipschitz function g on X so that $|f(x) - g(x)| < \epsilon$ for all $x \in X$. Since f is uniformly continuous on compacts, for each $n \in \mathbb{N}$ there exists $\delta = \delta(n, \epsilon)$ such that

$$\forall x, x' \in A_n, \quad d(x, x') < \delta \Rightarrow |f(x) - f(x')| < \epsilon$$

Therefore for each n find a finite subset

$$X_n := \{x_{n,1}, ..., x_{n,m_n}\} \subset A_n$$

so that for $r := \delta(n, \epsilon)/4, R := 2r$, the open balls $B_{n,j} := B_r(x_{n,j}), x_j \in X_n$ cover A_n. We reindex the set of points $\{x_{n,j}\}$ and the balls $B_{n,j}$ with a countable set I. Thus we obtain an open locally finite covering of X by the balls $B_j, j \in I$. Let $\{\eta_j, j \in I\}$ denote the corresponding Lipschitz partition of unity. It is then clear that

$$g(x) := \sum_{i \in I} \eta_i(x) f(x_i)$$

is a locally Lipschitz function. For $x \in B_i$ let $J \subset I$ be such that

$$x \notin B_{R_j}(x_j), \quad \forall j \notin J.$$

Then $|f(x) - f(x_j)| < \epsilon$ for all $j \in J$. Therefore

$$|g(x) - f(x)| \leq \sum_{j \in J} \eta_j(x)|f(x_j) - f(x)| < \epsilon \sum_{j \in J} \eta_j(x) = \epsilon \sum_{i \in I} \eta_j(x) = \epsilon.$$

It follows that $|f(x) - g(x)| < \epsilon$ for all $x \in X$. □

Our next goal is to prove a relative version of Proposition 5.33.

Proposition 5.37. *Let $A \subset X$ be a closed subset contained in a subset U which is open in X. Then for every $\epsilon > 0$ and every continuous function $f \in C(X)$ there exists a function $g \in C(X)$ so that:*

1. *g is locally Lipschitz on $X \setminus U$.*
2. *$\|f - g\| < \epsilon$.*
3. *$g|A = f|A$.*

Proof. We will need a lemma which is standard (see for instance [**78**, ??]), but we include the proof for the convenience of the reader.

Lemma 5.38. *Let A, V be disjoint closed subsets of a metric space X. Then there exists a continuous function $\rho = \rho_{A,V} : X \to [0,1]$ "separating" A and V, i.e.*

1. $\rho|A \equiv 0$
2. $\rho|V \equiv 1$.

Proof. Both functions d_A, d_V, which assign to $x \in X$ its minimal distance to A and to B respectively, are clearly continuous. Therefore the ratio

$$\sigma(x) := \frac{d_A(x)}{d_V(x)}, \quad \sigma : X \to [0, \infty]$$

is continuous as well. Let $\tau : [0,\infty] \to [0,1]$ be a continuous monotone function such that $\tau(0) = 0, \tau(\infty) = 1$, e.g.

$$\tau(y) = \frac{2}{\pi} \arctan(y), \quad y \neq \infty, \quad \tau(\infty) := 1.$$

Then the composition $\rho := \tau \circ \sigma$ satisfies the required properties. □

We now can finish the proof of the proposition. For the closed set $V := X \setminus U$ pick a continuous function $\rho = \rho_{A,V}$ as in the previous lemma. According to Proposition 5.33, there exits $h \in Lip(X)$ such that $\|f - h\| < \epsilon$. Then take

$$g(x) := \rho(x)h(x) + (1 - \rho(x))f(x).$$

We leave it to the reader to verify that g satisfies all the requirements of the proposition. □

Let $A \subset X$ be a closed subset. Let $B^n := \bar{B}_1(0) \subset \mathbb{R}^n$ denote the closed unit ball in $\mathbb{R}^n$. Define

$$C(X, A; B^n) := \{f : X \to B^n : f(A) \subset S^{n-1} = \partial B^n\}.$$

As an immediate corollary of Proposition 5.37 we obtain

Corollary 5.39. *For every function $f \in C(X, A; B^n)$ and an open set $U \subset X$ containing A, there exists a sequence of functions $g_i \in C(X, A; B^n)$ so that for all $i \in \mathbb{N}$:*

1. $g_i|A = f|A$.
2. $g_i \in Lip(X \setminus U; \mathbb{R}^n)$.

For a continuous map $f : X \to B^n$ define $A = A_f$ as

$$A := f^{-1}(S^{n-1}).$$

Definition 5.40. The map f is *essential* if it is homotopic rel. A to a map $f' : X \to S^{n-1}$. An *inessential map* is the one which is not essential.

We will be using the following characterization of the topological dimension due to Alexandrov:

Theorem 5.41. *(P. S. Alexandrov, see* [**79**, Theorem III.5]*)* $\dim(X) < n$ *if and only if every continuous map $f : X \to B^n$ is inessential.*

We are now ready to prove Theorem 5.32. Suppose that $\dim_H(X) < n$. We will prove that $\dim(X) < n$ as well. We need to show that every continuous map $f : X \to B^n$ is inessential. Let D denote the annulus $\{x \in \mathbb{R}^n : 1 \le |x| < 1/2\}$. Set $A := f^{-1}(S^{n-1})$ and $U := f^{-1}(D)$.

Take the sequence g_i given by Corollary 5.39. Since each g_i is homotopic to f rel. A, it suffices to show that some g_i is inessential. Since $f = \lim_i g_i$, it follows that for all sufficiently large i,

$$g_i(U) \cap B_{1/3}(0) = \emptyset.$$

We claim that the image of every such g_i misses a point in $B_{1/3}(0)$. Indeed, since $\dim_H(X) < n$, the n-dimensional Hausdorff measure of X is zero. However, $g_i|X \setminus U$ is locally Lipschitz. Therefore $g_i(X \setminus U)$ has zero n-dimensional Hausdorff (and hence Lebesgue) measure. It follows that $g_i(X)$ misses a point y in $B_{1/3}(0)$. Composing g_i with the retraction $B^n \setminus \{y\} \to S^{n-1}$ we get a map $f' : X \to S^{n-1}$ which is homotopic to f rel. A. Thus f is inessential and therefore $\dim(X) < n$. □

8. Proof of Gromov's theorem

The proof is by induction on the degree of polynomial growth. If $\beta_G(R) \prec R^0 = 1$ then G is finite and there is nothing to prove. Suppose that each group of growth at most R^{d-1} is virtually nilpotent. Let G be a (finitely generated group) of growth $\prec R^d$. Find a sequence λ_n satisfying the conclusion of the regular growth theorem and construct the asymptotic cone X_ω of the Cayley graph of G via rescaling by the sequence λ_n. Then X_ω is connected, locally connected, finite-dimensional and proper. Recall that according to Proposition 3.9, we have a homomorphism

$$\alpha : G_\omega \to L := \mathrm{Isom}(X_\omega)$$

such that $\alpha(G_\omega)$ acts on X_ω transitively. We also get a homomorphism

$$\ell : G \to L, \ell = \iota \circ \alpha,$$

where $\iota : G \hookrightarrow G_\omega$ is the diagonal embedding. Since the isometric action $L \curvearrowright X_\omega$ is effective and transitive, according to Theorem 5.27, the group L is a Lie group with finitely many components.

Remark 5.42. Observe that the point-stabilizer L_y for $y \in X_\omega$ is a compact subgroup in L. Therefore $X_\omega = L/L_x$ can be given a left-invariant Riemannian metric ds^2. Hence, since X_ω is connected, by using the exponential map with respect to ds^2 we see that if $g \in L$ fixes an open ball in X_ω pointwise, then $g = id$.

We have the following cases:

(a) The image of ℓ is not virtually solvable. Then by Tits' alternative, $\ell(G)$ contains a free nonabelian subgroup; it follows that G contains a free nonabelian subgroup as well which contradicts the assumption that G) has polynomial growth.

(b) The image of ℓ is virtually solvable and infinite. Then, after passing to a finite index subgroup in G, we get a homomorphism ϕ from G onto $\mathbb{Z}$. According to Proposition 5.8, $K = Ker(\phi)$ is a finitely generated group of growth $\prec R^{d-1}$. Thus, by the induction hypothesis, K is a virtually nilpotent group. Since G has polynomial growth, Theorem 5.7 implies that the group G is virtually nilpotent as well.

(c) $\ell(G)$ is finite.

To see that the latter case can occur consider an abelian group G. Then the homomorphism ℓ is actually trivial. How to describe the kernel of ℓ? For each $g \in G$ define the *displacement function* $\delta(g, r) := \max\{d(gx, x) : x \in B(e, r)\}$. Then

$$K := \{g \in G : g|B(e_\omega, 1) = id\} = \{g \in G : \omega\text{-}\lim \delta(g, \lambda_n)/\lambda_n = 0\}.$$

Here e_ω is the point in X_ω corresponding to the constant sequence (e) in G. On the other hand, by the above remark,

$$Ker(\ell) = K.$$

Let $G' \subset G$ be a finitely-generated subgroup with a fixed set of generators $g_1, ..., g_m$. Define

$$D(G', r) := \max_{j=1,\dots,m} \delta(g_j, r).$$

(This is an abuse of notation, the above function of course depends not only on G' but also on the choice of the generating set.)

Given a point in the Cayley graph, $p \in \Gamma_G$, we define another function

$$D(G', p, r) := \max\{d(g_j x, x), x \in B(p, r), j = 1, ..., m, \}.$$

Clearly $D(G', e, r) = D(G', r)$ and for $p \in G \subset \Gamma_G$,

$$D(G', p, r) = D(p^{-1}G'p, r),$$

where we take the generators $p^{-1}g_j p, j = 1, ..., m$ for the group $p^{-1}G'p$. The function $D(G', p, r)$ is 2-Lipschitz as a function of p.

Lemma 5.43. *Suppose that $D(G', r)$ is bounded as a function of r. Then G' is virtually abelian.*

PROOF. Suppose that $d(g_j x, x) \le C$ for all $x \in G$. Then

$$d(x^{-1}g_j x, 1) \le C,$$

and therefore the conjugacy class of g_j in G has cardinality $\le \beta_G(C) = N$. Hence the centralizer $Z_G(g_j)$ of g_j in G has finite index in G: Indeed, if $x_0, ..., x_N \in G$ then there are $0 \le i \ne k \le N$ such that

$$x_i^{-1} g_j x_i = x_k^{-1} g_j x_k \Rightarrow [x_k x_i^{-1}, g_j] = 1 \Rightarrow x_k x_i^{-1} \in Z_G(g_j).$$

Thus the intersection

$$A := \bigcap_{j=1}^{m} Z_G(g_j)$$

has finite index in G; it follows that $A \cap G'$ is an abelian subgroup of finite index in G'. $\square$

We now assume that $\ell(G)$ is finite and consider the subgroup of finite index $G' := Ker(\ell) \subset G$. Let $g_1, ..., g_m$ be generators of G'. By the previous lemma it suffices to consider the case when the function $D(G', r)$ is unbounded; then, since $G' = Ker(\ell)$, the function $D(G', r)$ has "sublinear growth", i.e.

$$\omega\text{-}\lim \delta(g_j, \lambda_n)/\lambda_n = 0, j = 1, ..., m.$$

If the subgroup G' is virtually abelian, we are done. Therefore we assume that this is not the case. In particular, the function $D(G', p, r)$ is unbounded as the function of $p \in G$.

LEMMA 5.44. *Let ϵ such that $0 < \epsilon \le 1$. Then there exists $x_n \in G$ such that*

$$\omega\text{-}lim \frac{D(x_n^{-1} G' x_n, \lambda_n)}{\lambda_n} = \epsilon.$$

PROOF. For ω-all $n \in \mathbb{N}$ we have $D(G', \lambda_n) \le \epsilon\lambda_n/2$. Fix n. Since $D(G', p, \lambda_n)$ is unbounded, there exists $q_n \in G$ such that

$$D(G', q_n, \lambda_n) > 2\lambda_n.$$

Hence, because Γ_G is connected and the function $D(G', p, \lambda_n)$ is continuous, there exists $y_n \in \Gamma_G$ such that

$$D(G', y_n, \lambda_n) = \epsilon\lambda_n.$$

The point y_n is not necessarily in the vertex set of the Cayley graph Γ_G. Pick a point $x_n \in G$ within the distance $\frac{1}{2}$ from y_n. Then, since the function $D(G', \cdot, \lambda_n)$ is 2-Lipschitz,

$$|D(G', x_n, \lambda_n) - \epsilon\lambda_n| \le 1.$$

It follows that $|D(x_n^{-1} G' x_n, \lambda_n) - \epsilon\lambda_n| \le 1$ and therefore

$$\omega\text{-}\lim \frac{D(x_n^{-1} G' x_n, \lambda_n)}{\lambda_n} = \epsilon. \quad \square$$

Now, given $0 < \epsilon \le 1$ and $g \in G'$ we define a sequence

$$[g] := [x_n^{-1} g x_n] \in G^*.$$

Note that since $D(x_n^{-1} G x_n, \lambda_n) = O(\epsilon)$, the elements $\ell_\epsilon(g_j)$ belong to G_ω. Therefore we obtain a homomorphism $\ell_\epsilon : G' \to G_\omega, \ell_\epsilon : g \mapsto [g]$.

We topologize the group L via the compact-open topology with respect to its action on X_ω, thus ϵ-neighborhood of the identity in L contains all isometries $h \in L$ such that

$$\delta(h, 1) \le \epsilon,$$

where δ is the displacement function of h on the unit ball $B(e_\omega, 1)$. By our choice of x_n, there exists a generator $h = g_j$ of G' such that $\delta(\ell_\epsilon(h), 1) = \epsilon$. If there is an $N \in \mathbb{N}$ such that the order $|\ell_\epsilon(h)|$ of $\ell_\epsilon(h)$ is at most N for all ϵ, then L contains arbitrarily small finite cyclic subgroups $\langle \ell_\epsilon(h) \rangle$, which is impossible since L is a Lie group. Therefore

$$\lim_{\epsilon \to 0} |\ell_\epsilon(h)| = \infty$$

If for some $\epsilon > 0$, $\ell_\epsilon(G')$ is infinite we are done as above. Hence we assume that $\ell_\epsilon(G')$ is finite for all $\epsilon > 0$.

According to Jordan's theorem 1.10, there exists $q = q(L)$ so that each finite group $\ell_\epsilon(G')$ contains an abelian subgroup of index $\leq q$. For each ϵ consider the preimage G'_ϵ in G' of the abelian subgroup in $\ell_\epsilon(G')$ which is given by Jordan's theorem. The index of G'_ϵ in G' is at most q. Let G'' be the intersection of all the subgroups $G'_\epsilon, \epsilon > 0$. Then G'' has finite index in G and G'' admits homomorphisms onto finite abelian groups of arbitrarily large order. Since all such homomorphisms have to factor through the abelianization $(G'')^{ab}$, the group $(G'')^{ab}$ has to be infinite. Since $(G'')^{ab}$ is finitely generated it follows that it has nontrivial free part, hence G'' again admits an epimorphism to $\mathbb{Z}$. Thus we are done by the induction. $\square$

CHAPTER 6

Quasiconformal mappings

DEFINITION 6.1. Suppose that D, D' are domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a homeomorphism. The mapping f is called **quasiconformal** if the function

$$H_f(x) = \limsup_{r \to 0} \frac{\sup\{d(f(z), f(x)) : d(x, z) = r\}}{\inf\{d(f(z), f(x)) : d(x, z) = r\}}$$

is bounded from above in X. A quasiconformal mapping is called K-quasiconformal if the function H_f is bounded from above by K a.e. in X.

The notion of quasiconformality does not work well in the case when the domain and range are 1-dimensional. It is replaced by

DEFINITION 6.2. Let $C \subset \mathbb{S}^1$ be a closed subset. A homeomorphism $f : C \to f(C) \subset \mathbb{S}^1$ is called **quasimoebius** if there exists a constant K so that for any quadruple of mutually distinct points $x, y, z, w \in \mathbb{S}^1$ their cross-ratio satisfies the inequality

$$K^{-1} \leq \frac{\lambda(|f(x) : f(y) : f(z) : f(w)|)}{\lambda(|x : y : z : w|)} \leq K \tag{56}$$

where $\lambda(t) = |\log(t)| + 1$.

Note that if f is K-quasimoebius then for any pair of Moebius transformations α, β the composition $\alpha \circ f \circ \beta$ is again K-quasimoebius.

Recall that a mapping $f : S^n \to S^n$ is *Moebius* if it is a composition of inversions. Equivalently, f is Moebius iff it is the extension of an isometry $\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$. Yet another equivalent definition: Moebius mappings are the homeomorphisms of S^n which preserve the cross-ratio.

Here is another (analytical) description of quasiconformal mappings. A homeomorphism $f : D \to D'$ is called *quasiconformal* if it has distributional partial derivatives in $L^n_{loc}(D)$ and the ratio

$$R_f(x) := \|f'(x)\|/|J_f(x)|^{1/n}$$

is uniformly bounded from above a.e. in D. Here $\|f'(x)\|$ is the operator norm of the derivative $f'(x)$ of f at x. The essential supremum of $R_f(x)$ in D is denoted by $K_O(f)$ and is called the *outer dilatation* of f. Let us compare $H_f(x)$ and $R_f(x)$. Clearly it is enough to consider positive-definite diagonal matrices $f'(x)$. Let Λ be the maximal eigenvalue of $f'(x)$ and λ be the minimal eigenvalue. Then $\|f'(x)\| = \Lambda$, $H_f(x) = \Lambda/\lambda$ and

$$R_f(x) \leq H_f(x) \leq R_f(x)^n.$$

Two definitions of quasiconformality (using H_f and R_f) coincide (see for instance [**89**], [**108**], [**103**]) and we have:

$$K_O(f) \leq K(f) \leq K_O(f)^n.$$

In particular, quasiconformal mappings are differentiable a.e. and their derivative is a.e. invertible.

Note that quasiconformality of mappings and the coefficients of quasiconformality $K(f)$, $K_O(f)$ do not change if instead of the Euclidean metric we consider a conformally-Euclidean metric in D. This allows us to define quasiconformal mappings on domains in S^n, via the stereographic projection.

Examples:

1. If $f : D \to D'$ is a conformal homeomorphism the f is quasiconformal. Indeed, conformality of f means that $f'(x)$ is a similarity matrix for each x, hence $R_f(x) = 1$ for each x. In particular, Moebius transformations are quasiconformal.

2. Suppose that the homeomorphism f extends to a diffeomorphism $\overline{D} \to \overline{D}'$ and the closure $\overline{D}$ is compact. Then f is quasiconformal.

3. Compositions and inverses of quasiconformal mappings are quasiconformal. Moreover, $K_O(f \circ g) \le K_O(f)K_O(g)$, $K_O(f) = K_O(f^{-1})$.

THEOREM 6.3. *(Liouville's theorem for quasiconformal mappings, see* **[89]**, **[77]***.) Suppose that $f : S^n \to S^n$, $n \ge 2$, is a quasiconformal mapping which is conformal a.e., i.e. for a.e. $x \in S^n$, $R_f(x) = 1$. Then f is Moebius.*

Conformality of f at x means that the derivative $f'(x)$ exists and is a similarity matrix (i.e. is the product of a scalar and an orthogonal matrix).

Historical remark. Quasiconformal mappings for $n = 2$ were introduced in 1920-s by Groetch as a generalization of conformal mappings. Quasiconformal mappings in higher dimensions were introduced by Lavrentiev in 1930-s for the purposes of application to hydrodynamics. The discovery of relation between quasi-isometries of hyperbolic spaces and quasiconformal mappings was made by Efremovich and Tihomirova **[32]** and Mostow **[77]** in 1960-s.

THEOREM 6.4. *Suppose that $f : \mathbb{H}^n \to \mathbb{H}^n$ is a (k, c)-quasi-isometry. Then the homeomorphic extension $h = f_\infty$ of f to $\partial_\infty \mathbb{H}^n$ constructed in Theorem 3.12 is a quasiconformal homeomorphism (if $n \ge 3$) and quasimoebius (if $n = 2$).*

PROOF. I will verify quasiconformality of h for $n \ge 3$ and will leave the case $n = 2$ to the reader. According to the definition, it is enough to verify quasiconformality at each particular point x with uniform estimates on the function $H_h(x)$. Thus, after composing h with Moebius transformations, we can take $x = 0 = h(x)$, $h(\infty) = \infty$, where we consider the upper half-space model of $\mathbb{H}^n$.

Take a Euclidean sphere $S_r(0)$ in $\mathbb{R}^{n-1}$ with the center at the origin. This sphere is the ideal boundary of a hyperplane $P_r \subset \mathbb{H}^n$ which is orthogonal to the vertical geodesic $L \subset \mathbb{H}^n$, connecting 0 and ∞. Let $x_r = L \cap P_r$. Let $\pi_L : \mathbb{H}^n \to L$ be the nearest point projection. The hyperplane P_r can be characterized by the following equivalent properties:

$$P_r = \{w \in \mathbb{H}^n : \pi_L(w) = x_r\}$$
$$P_r = \{w \in \mathbb{H}^n : d(w, x_r) = d(w, L)\}.$$

Since quasi-isometric images of geodesics in $\mathbb{H}^n$ are uniformly close to geodesics, we conclude that

$$\mathrm{diam}[\pi_L(f(P_r))] \le Const$$

where $Const$ depends only on the quasi-isometry constants of f. The projection π_L extends naturally to $\partial_\infty \mathbb{H}^n$. We conclude:

$$\mathrm{diam}[\pi_L(h(S_r(0)))] \le Const.$$

Thus $h(S_r(0))$ is contained in a spherical shell

$$\{z \in \mathbb{R}^{n-1} : \rho_1 \le |z| \le \rho_2\}$$

where $\log[\rho_1/\rho_2] \le Const$. This implies that the function $H_h(0)$ is bounded from above by $K := \exp(Const)$. We conclude that the mapping h is K-quasiconformal. $\square$

CHAPTER 7

Quasi-isometries of nonuniform lattices in $\mathbb{H}^n$.

Recall that a *lattice* in a Lie group G (with finitely many components) is a discrete subgroup Γ such that the quotient $\Gamma\backslash G$ has finite volume. Here the left-invariant volume form on G is defined by taking a Riemannian metric on G which is left-invariant under G and right-invariant under K, the maximal compact subgroup of G. Thus, if $X := G/K$, then this quotient manifold has a Riemannian metric which is (left) invariant under G. Hence, Γ is a lattice iff Γ acts on X properly discontinuously so that $vol(\Gamma \setminus X)$ is finite. Note that the action of Γ on X is a priori not free. A lattice γ is called *uniform* if $\Gamma \setminus X$ is compact and Γ is called *nonuniform* otherwise.

Note that each lattice is finitely-generated (this is not at all obvious), in the case of the hyperbolic spaces finite generation follows from the thick-thin decomposition above. Thus, if Γ is a lattice, then it contains a torsion-free subgroup of finite index (Selberg lemma). In particular, if Γ is a nonuniform lattice in $\mathbb{H}^2$ then Γ is virtually free of rank ≥ 2.

EXAMPLE 57. Consider the subgroups $\Gamma_1 := SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$, $\Gamma_2 := SL(2,\mathbb{Z}[i]) \subset SL(2,\mathbb{C})$. Then Γ_1, Γ_2 are nonuniform lattices. Here $\mathbb{Z}[i]$ is the ring of Gaussian integers, i.e. elements of $\mathbb{Z} \oplus i\mathbb{Z}$. The discreteness of Γ_1, Γ_2 is clear, but finiteness of volume requires a proof.

Let's show that Γ_i, $i = 1, 2$, are not uniform. We will give the proof in the case of Γ_1, the case of Γ_2 is similar.

Note that the symmetric space $SL(2,\mathbb{R})/SO(2)$ is the hyperbolic plane. I will use the upper half-plane model of $\mathbb{H}^2$. The group Γ_1 contains the upper triangular matrix

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This matrix acts on $\mathbb{H}^2$ by the parabolic translation $\gamma : z \mapsto z + 1$ (of infinite order). Consider the points $z := (0, y) \in \mathbb{H}^2$ with $y \to \infty$. Then the length of the geodesic segment $\overline{z\gamma(z)}$ tends to zero as y diverges to infinity. Hence the quotient $S := \Gamma_1 \setminus \mathbb{H}^2$ has injectivity radius unbounded from below (from zero), hence S is not compact.

More generally, lattices in a Lie group can be constructed as follows: let $h : G \to GL(N,\mathbb{R})$ be a homomorphism with finite kernel. Let $\Gamma := h^{-1}(GL(N,\mathbb{Z}))$. Then Γ is an *arithmetic lattice* in G.

Recall that a *horoball* in $\mathbb{H}^n$ (in the unit ball model) is a domain bounded by a round Euclidean ball $B \subset \mathbb{H}^n$, whose boundary is tangent to the boundary of $\mathbb{H}^n$ in a single point (called the *center* or *footpoint* of the horoball). The boundary of a horoball in $\mathbb{H}^n$ is called a *horosphere*. In the upper half-space model, the horospheres with the footpoint ∞ are horizontal hyperplanes

$$\{(x_1, ..., x_{n-1}, t) : (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}\},$$

where t is a positive constant.

THEOREM 7.1. *(Thick-thin decomposition) Suppose that Γ is a nonuniform lattice in $Isom(\mathbb{H}^n)$. Then there exists an (infinite) collection C of pairwise disjoint horoballs $C := \{B_j, j \in J\}$, which is invariant under Γ, so that $(\mathbb{H}^n \setminus \cup_j B_j)/\Gamma$ is compact.*

The quotient $(\mathbb{H}^n \setminus \cup_j B_j)/\Gamma$ is called the *thick part* of $M = \mathbb{H}^n/\Gamma$ and its (noncompact) complement in M is called *thin part* of M.

The complement $\Omega := \mathbb{H}^n \setminus \cup_j B_j$ is called a *truncated hyperbolic space.* Note that the stabilizer Γ_j of each horosphere ∂B_j acts on this horosphere cocompactly with the quotient $T_j := \partial B_j/\Gamma_j$. The quotient B_j/Γ_j is naturally homeomorphic to $T_j \times \mathbb{R}_+$, this product decomposition is inherited

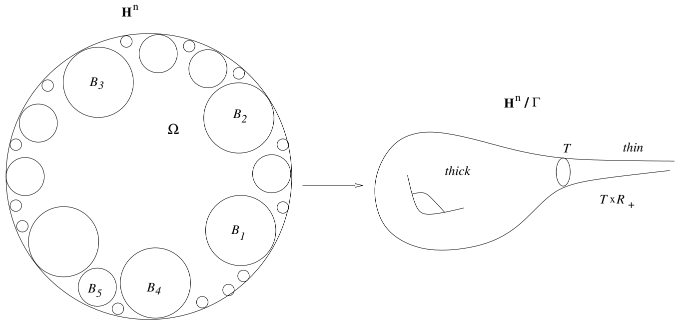

FIGURE 1. *Truncated hyperbolic space and thick-thin decomposition.*

from the foliation of B_j by the horospheres with the common footpoint ξ_j and the geodesic rays asymptotic to ξ_j. In the case Γ is torsion-free, orientation preserving and $n = 3$, the quotients T_j are 2-tori.

DEFINITION 7.2. Let $\Gamma \subset G$ be a subgroup. The *commensurator* of Γ in G, denoted $Comm(\Gamma)$ consists of all $g \in G$ such that the groups $g\Gamma g^{-1}$ and Γ are *commensurable*, i.e. their intersection has finite index in the both groups.

Here is an example of the commensurator: let $\Gamma := SL(2, \mathbb{Z}[i]) \subset SL(2, \mathbb{C})$. Then the commensurator of Γ is the group $SL(2, \mathbb{Q}(i))$. In particular, the group $Comm(\Gamma)$ is nondiscrete in this case. There is a theorem of Margulis, which states that a lattice in G is *arithmetic* if and only if its commensurator is discrete. We note that each element $g \in Comm(\Gamma)$ determines a quasi-isometry $f : \Gamma \to \Gamma$. Indeed, the Hausdorff distance between Γ and $g\Gamma g^{-1}$ is finite. Hence the quasi-isometry f is given by composing $g : \Gamma \to g\Gamma g^{-1}$ with the nearest-point projection to Γ.

The main goal of the remainder of the course is to prove the following

THEOREM 7.3. *(R. Schwartz* **[93]***.) Let $\Gamma \subset Isom(\mathbb{H}^n)$ is a nonuniform lattice, $n \geq 3$. Then:*

(a) For each quasi-isometry $f : \Gamma \to \Gamma$ there exists $\gamma \in Comm(\Gamma)$ which is within finite distance from f. The distance between these maps depends only on Γ and on the quasi-isometry constants of f.

(b) Suppose that Γ, Γ' are non-uniform lattices which are quasi-isometric to each other. Then there exists an isometry $g \in Isom(\mathbb{H}^n)$ such that the groups Γ' and $g\Gamma g^{-1}$ are commensurable.

(c) Suppose that Γ' is a finitely-generated group which is quasi-isometric to a nonuniform lattice Γ above. Then the groups Γ, Γ' are virtually isomorphic, i.e. there exists a finite normal subgroup $F \subset \Gamma'$ such that the groups $\Gamma, \Gamma'/F$ contain isomorphic subgroups of finite index.

The above theorem fails in the case of the hyperbolic plane (except for the last part).

1. Coarse topology of truncated hyperbolic spaces

On each truncated hyperbolic space Ω we put the path-metric which is induced by the restriction of the Riemannian metric of $\mathbb{H}^n$ to Ω. This metric is invariant under Γ and since the quotient Ω/Γ is compact, Ω is quasi-isometric to the group Γ. Note that the restriction of this metric to each peripheral horosphere Σ is a flat metric.

The following lemma is the key for distinguishing the case of the hyperbolic plane from the higher-dimensional hyperbolic spaces (of dimension ≥ 3):

LEMMA 7.4. *Let Ω is a truncated hyperbolic space of dimension ≥ 3. Then each peripheral horosphere $\Sigma \subset \Omega$ does not coarsely separate Ω.*

PROOF. Let $R < \infty$ and let B be the horoball bounded by Σ. Then the union of $N_R(\Sigma) \cup B$ is a horoball B' in $\mathbb{H}^n$ (where the metric neighborhood is taken in $\mathbb{H}^n$). The horoball B' does not separate $\mathbb{H}^n$. Therefore, for each pair of points $x, y \in \Omega \setminus B'$, there exists a PL path p connecting them within $\mathbb{H}^n \setminus B'$. If the path p is entirely contained in Ω, we are done. Otherwise, it can be subdivided into finitely many subpaths, each of which is either contained in Ω or connects a pair of points on the boundary of a complementary horoball $B_j \subset \mathbb{H}^n \setminus \Sigma$. The intersection of $N_R^{(\Omega)}(B')$ with $\Sigma_j = \partial B_j$ is a metric ball in the Euclidean space Σ_j (here $N_R^{(\Omega)}$ is the metric neighborhood taken within Ω). Note that a metric ball does not separate $\mathbb{R}^{n-1}$, provided that $n-1 \geq 2$. Thus we can replace $p_j = p \cap B_j$ with a new path p_j' which connects the end-points of p_j within the complement $\Sigma_j \setminus N_R^{(\Omega)}(B')$. By making these replacements for each j we get a path connecting x to y within $\Omega \setminus N_R^{(\Omega)}(\Sigma)$. $\square$

Let now Ω, Ω' be truncated hyperbolic spaces (of the same dimension), $f : \Omega \to \Omega'$ be a quasi-isometry. Let Σ be a peripheral horosphere of Ω, consider its image $f(\Sigma)$ in Ω'.

PROPOSITION 7.5. *There exists a peripheral horosphere $\Sigma' \subset \partial\Omega'$ which is within finite Hausdorff distance from $f(\Sigma)$.*

PROOF. Note that Ω, being isometric to $\mathbb{R}^{n-1}$, has bounded geometry and is uniformly contractible. Therefore, according to Theorem 2.23, $f(\Sigma)$ coarsely separates $\mathbb{H}^n$; however it cannot coarsely separate Ω', since f is a quasi-isometry and Σ does not coarsely separate Ω. Let $R < \infty$ be such that $N_R(f(\Sigma))$ separates $\mathbb{H}^n$ into (two) deep components X_1, X_2. Suppose that for each complementary horoball B_j' of Ω' (bounded by the horosphere Σ_j'),

$$N_{-R}(B_j') := B_j' \setminus N_R(\Sigma_j') \subset X_1.$$

Then the entire Ω' is contained in $N_R(f(\Sigma))$. It follows that $f(\Sigma)$ does not coarsely separate $\mathbb{H}^n$, a contradiction. Thus there are complementary horoballs B_1', B_2' for Ω' such that $N_{-R}(B_1') \subset X_1, N_{-R}(B_2') \subset X_2$. If either Σ_1 or Σ_2 is not contained in $N_r(f(\Sigma))$ for some r then $f(\Sigma)$ coarsely separates Ω'. Thus we found a horosphere $\Sigma' := \Sigma_1'$ such that

$$\Sigma' \subset N_r(f(\Sigma)).$$

Our goal is to show that $f(\Sigma) \subset N_\rho(\Sigma')$ for some $\rho < \infty$. The nearest-point projection $\Sigma' \to f(\Sigma)$ defines a quasi-isometric embedding $h : \Sigma' \to \Sigma$. However Lemma 3.5 proves that a quasi-isometric embedding between two Euclidean spaces of the same dimension is a quasi-isometry. Thus there exists $\rho < \infty$ such that $f(\Sigma) \subset N_\rho(\Sigma')$. $\square$

LEMMA 7.6. *$d_{Haus}(f(\Sigma), \Sigma')) \leq r$, where r is independent of Σ.*

PROOF. The proof is by inspection of the arguments in the proof of the previous proposition. First of all, the constant R depends only on the quasi-isometry constants of the mapping f and the uniform geometry/uniform contractibility bounds for $\mathbb{R}^{n-1}$ and $\mathbb{H}^n$. The inradii of the shallow complementary components of $N_R(f(\Sigma))$ again depend only on the above data. Therefore there exists a uniform constant r such that Σ_1 of Σ_2 is contained in $N_r(f(\Sigma))$. Finally, the upper bound on ρ such that $N_\rho(Image(h)) = \Sigma'$ (coming from Lemma 3.5) again depends only on the quasi-isometry constants of the projection $h : \Sigma' \to \Sigma$.

2. Hyperbolic extension

The main result of this section is

THEOREM 7.7. *f admits a quasi-isometric extension $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$. Moreover, the extension $\tilde{f}$ satisfies the following equivariance property:*

Suppose that $f : X \to X'$ is equivariant with respect to an isomorphism

$$\rho : G \to G'$$

where $G, G' \subset \mathrm{Isom}(\mathbb{H}^n)$ are subgroups preserving X and X' respectively. Then the extension $\tilde{f}$ is also ρ-equivariant.

PROOF. We will construct the extension $\tilde{f}$ into each complementary horoball $B \subset \mathbb{H}^n \setminus \Omega$. Without loss of generality we can use the upper half-space model of $\mathbb{H}^n$ so that the horoballs B and B' are both given by

$$\{(x_1, ..., x_{n-1}, 1) : (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

We will also assume that $f(\Sigma) \subset \Sigma'$. For each vertical geodesic ray $\rho(t), t \in \mathbb{R}_+$, in B_j we define the geodesic ray $\rho'(t)$ to be the vertical geodesic ray in B' with the initial point $f(\rho(0))$. This gives the extension of f into B:

$$\tilde{f}(\rho(t)) = \rho'(t).$$

Let's verify that this extension is coarsely Lipschitz. Let x and $y \in B$ be points within the (hyperbolic) distance ≤ 1. By the triangle inequality it suffices to consider the case when x, y belong to the same horosphere H_t (of the Euclidean height t) with the footpoint at ∞ (if x and y belong to the same vertical ray we clearly get $d(\tilde{f}(x), \tilde{f}(y)) = d(x, y)$). Note that the distance from x to y along the horosphere H does not exceed ϵ, which is independent of t. Let $\bar{x}, \bar{y}$ denote the points in Σ such that x, y belong to the vertical rays in B_j with the initial points $\bar{x}, \bar{y}$ respectively. Then

$$d_\Sigma(\bar{x}, \bar{y}) = t d_{H_t}(x, y) \leq \epsilon t.$$

Hence, since f is (L, A)-coarse Lipschitz,

$$d_\Sigma(f(\bar{x}), f(\bar{y})) \leq L\epsilon t + A.$$

It follows that

$$d_\Sigma(\tilde{f}(x), \tilde{f}(y)) \leq L\epsilon + A/t \leq L\epsilon + A.$$

This proves that the extension $\tilde{f}$ is coarse Lipschitz in the horoball B. Since the coarse Lipschitz is a local property, the mapping $\tilde{f}$ is coarse Lipschitz on $\mathbb{H}^n$. The same argument applies to the hyperbolic extension $\tilde{f}'$ of the coarse inverse f' to the mapping f. It is clear that the mapping $\tilde{f} \circ \tilde{f}'$ and $\tilde{f}' \circ \tilde{f}$ have bounded displacement. Thus $\tilde{f}$ is a quasi-isometry. The equivariance property of f is clear from the construction. ☐

Since $\tilde{f}$ is a quasi-isometry of $\mathbb{H}^n$, it admits a quasiconformal extension $h : \partial_\infty \mathbb{H}^n \to \partial_\infty \mathbb{H}^n$. Let Λ, Λ' denote the sets of the footpoints of the peripheral horospheres of Ω, Ω' respectively. It is clear that $h(\Lambda) = \Lambda'$.

3. Zooming in

Our main goal is to show that the mapping h constructed in the previous section is Moebius. By the Liouville's theorem for quasiconformal mappings, h is Moebius iff for a.e. point $\xi \in S^{n-1}$, the derivative of h at ξ is a similarity. We will be working with the upper half-space of the hyperbolic space $\mathbb{H}^n$.

PROPOSITION 7.8. *Suppose that h is not Moebius. Then there exists a quasi-isometry $F : \Omega \to \Omega'$ whose extension to the sphere at infinity is a linear map which is not a similarity.*

PROOF. Since h is differentiable a.e. and is not Moebius, there exists a point $\xi \in S^{n-1} \setminus \Lambda$ such that $Dh(\xi)$ exists, is invertible but is not a similarity. By pre- and post-composing f with isometries of $\mathbb{H}^n$ we can assume that $\xi = 0 = h(\xi)$. Let $L \subset \mathbb{H}^n$ denote the vertical geodesic through ξ. Since ξ is not a footpoint of a complementary horoball to Ω, there exists a sequence of points $x_j \in L \cap \Omega$ which converges to ξ. For each $t \in \mathbb{R}_+$ define $\alpha_t : z \mapsto tz$, a hyperbolic translation along L. Let t_j be such that $\alpha_{t_j}(x_1) = x_j$. Set

$$\tilde{f}_j := \alpha_{t_j}^{-1} \circ \tilde{f} \circ \alpha_{t_j},$$

the quasiconformal extensions of these mappings to $\partial_\infty \mathbb{H}^n$ are given by

$$h_j(z) = \frac{h(t_j z)}{t_j}.$$

By the definition of differentiability,

$$\lim_{j \to \infty} h_j = A = Dh(0),$$

where the convergence is uniform on compacts in $\mathbb{R}^{n-1}$. Let's verify that the sequence of quasi-isometries $\tilde{f}_j$ subconverges to a quasi-isometry of $\mathbb{H}^n$. Indeed, since the quasi-isometry constants of all $\tilde{f}_j$ are the same, it suffices to show that $\{\tilde{f}_j(x_1)\}$ is a bounded sequence in $\mathbb{H}^n$. Let L_1, L_2 denote a pair of distinct geodesics in $\mathbb{H}^n$ through x_1, so that the point ∞ does not belong to $L_1 \cup L_2$. Then the quasi-geodesics $\tilde{f}_j(L_i)$ are within distance $\le C$ from geodesics L^*_{1j}, L^*_{2j} in $\mathbb{H}^n$. Note that the geodesics L^*_{1j}, L^*_{2j} subconverge to geodesics in $\mathbb{H}^n$ with distinct end-points (since the mapping A is 1-1). The point $\tilde{f}_j(x_1)$ is within distance $\le C$ from L^*_{1j}, L^*_{2j}. If the sequence $\tilde{f}_j(x_1)$ is unbounded, we get that L^*_{1j}, L^*_{2j} subconverge to geodesics with a common end-point at infinity. Contradiction.

We thus pass to a subsequence such that $\tilde{f}_j$ converges to a quasi-isometry $f_\infty : \mathbb{H}^n \to \mathbb{H}^n$. Note however that f_∞ in general does not send Ω to Ω'. Recall that $\Omega/\Gamma, \Omega'/\Gamma'$ are compact. Therefore there exist sequences $\gamma_j \in \Gamma, \gamma'_j \in \Gamma'$ such that $\gamma_j(x_j)$, $\gamma'_j(\tilde{f}(x_j))$ belong to a compact subset of $\mathbb{H}^n$. Hence the sequences $\beta_j := \alpha_{t_j}^{-1} \circ \gamma_j^{-1}, \beta'_j := \alpha_{t_j}^{-1} \circ \gamma_j'^{-1}$ is precompact in $Isom(\mathbb{H}^n)$ and therefore they subconverge to isometries $\beta_\infty, \beta'_\infty \in Isom(\mathbb{H}^n)$. Set

$$\Omega_j := \alpha_{t_j}^{-1}\Omega = \alpha_{t_j}^{-1} \circ \gamma_j^{-1}\Omega = \beta_j\Omega,$$

$$\Omega'_j := \alpha_{t_j}^{-1}\Omega' = \beta'_j\Omega',$$

then $\tilde{f}_j : \Omega_j \to \Omega'_j$. On the other hand, the sets Ω_j, Ω'_j subconverge to the sets $\beta_\infty\Omega, \beta'_\infty\Omega'$ and $\tilde{f}_\infty$ is a quasi-isometry between $\beta_\infty\Omega$ and $\beta'_\infty\Omega'$. Since $\beta_\infty\Omega$ and $\beta'_\infty\Omega'$ are isometric copies of Ω and Ω' the assertion follows. $\square$

The situation when we have a linear mapping (which is not a similarity) mapping Λ to Λ' seems at the first glance impossible. Here however is an example:

EXAMPLE 58. Let $\Gamma := SL(2, \mathbb{Z}[i]), \Gamma' := SL(2, \mathbb{Z}[\sqrt{-2}])$. Then $\Lambda = \mathbb{Q}(i), \Lambda' = \mathbb{Q}(\sqrt{-2})$.

Define a real linear mapping $A : \mathbb{C} \to \mathbb{C}$ by sending 1 to 1 and i to $\sqrt{-2}$. Then A is not a similarity, however $A(\Lambda) = \Lambda'$).

Thus to get a contradiction we have to exploit the fact that the linear map in question is quasi-conformal extension of an isometry between truncated hyperbolic spaces. This is done using a trick which replaces A with an *inverted linear map*, such maps are defined in the next section.

4. Inverted linear mappings

Let $A : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be an (invertible) linear mapping and I be the inversion in the unit sphere about the origin, i.e.

$$I(x) = \frac{x}{|x|^2}.$$

DEFINITION 7.9. An inverted linear map is the conjugate of A by the inversion in the unit sphere centered at the origin, i.e. the composition

$$h := I \circ A \circ I,$$

which means that

$$h(x) = \frac{|x|^2}{|Ax|^2}A(x).$$

LEMMA 7.10. *The function $\phi(x) = \frac{|x|^2}{|Ax|^2}$ is* asymptotically constant*, i.e. the gradient of ϕ converges to zero as $|x| \to \infty$.*

PROOF. The function ϕ is a rational function of degree zero, hence its gradient is a rational vector-function of degree -1. $\square$

Note however that ϕ is not a constant mapping unless A is a similarity. Hence h is linear iff A is a similarity.

COROLLARY 7.11. *Let R be a fixed positive real number, $x_j \in \mathbb{R}^{n-1}$, $|x_j| \to \infty$. Then the function $h(x - x_j) - h(x_j)$ converges (uniformly on the R-ball $B(0, R)$) to a linear function, as $j \to \infty$.*

We would like to strengthen the assertion that ϕ is not constant (unless A is a similarity). Let G be a discrete group of Euclidean isometries acting cocompactly on $\mathbb{R}^{n-1}$. Fix a G-orbit Gx, for some $x \in \mathbb{R}^{n-1}$.

Lemma 7.12. *There exists a number R and a sequence of points $x_j \in Gx$ diverging to infinity such that the restrictions ϕ to $B(x_j, R) \cap Gx$ are not constant for all j.*

Proof. Let P be a compact fundamental domain for G, containing x. Let ρ denote diam(P). Pick any $R \geq 4\rho$. Then $B(x, R)$ contains all images of P under G which are adjacent to P. Suppose that the sequence x_j as above does not exist. This means that there exists $r < \infty$ such that the restriction of ϕ to $B(x_j, R)$ is constant for each $x_j \in Gx \setminus B(0, r)$. It follows that the function ϕ is actually constant on $Gx \setminus B(0, r)$. Note that the set

$$\{y/|y|, y \in Gx \setminus B(0, r)\}$$

is dense in the unit sphere. Since $\phi(y/|y|) = \phi(y)$ it follows that ϕ is a constant function. $\square$

We now return to the discussion of quasi-isometries.

Let A be an invertible linear mapping (which is not a similarity) constructed in the previous section, by composing A with Euclidean translations we can assume that $0 = A(0)$ belongs to both Λ and Λ':

Indeed, let $p \in \Lambda \setminus \infty, q := A(p)$, define P, Q to be the translations by p, q. Consider $A_2 := Q^{-1} \circ A \circ P$, $\Lambda_1 := \Lambda - p$, $\Lambda_1' := \Lambda' - q$, $\Omega_1 := \Omega - p$, $\Omega_1' := \Omega' - q$. Then $A_2(\Lambda_1) = \Lambda_1'$, $A_2(0) = 0$, $0 \in \Lambda_1 \cap \Lambda_1'$.

We retain the notation $A, \Lambda, \Lambda', \Omega, \Omega'$ for the linear map and the new sets of footpoints of horoballs and truncated hyperbolic spaces.

Then $\infty = I(0)$ belongs to both $I(\Lambda)$ and $I(\Lambda')$. To simplify the notation we replace $\Lambda, \Omega, \Lambda', \Omega'$ with $I(\Lambda), I(\Omega), I(\Lambda'), I(\Omega')$ respectively. Then the truncated hyperbolic spaces Ω, Ω' have complementary horoballs B_∞, B'_∞.

Given $x \in \mathbb{R}^{n-1}$ define $h_*(x) := h(\Gamma_\infty x)$. Let Γ_∞, Γ'_∞ be the stabilizers of ∞ in Γ, Γ' respectively. Without loss of generality we can assume that $\infty \in \Lambda, \Lambda'$, hence Γ_∞, Γ'_∞ act cocompactly (by Euclidean isometries) on $\mathbb{R}^{n-1}$.

Lemma 7.13 (Scattering lemma). *Suppose that A is not a similarity. Then for each $x \in \mathbb{R}^{n-1}$, $h_*(x)$ is not contained in finitely many Γ'_∞-orbits.*

Proof. Let $x_j = \gamma_j x \in \Gamma_\infty x$ and $R < \infty$ be as in Lemma 7.12, where $G = \Gamma_\infty$. We have a sequence of maps $\gamma'_j \in \Gamma'_\infty$ such that $\gamma'_j h(x_j)$ is relatively compact in $\mathbb{R}^{n-1}$. Then the mapping $h|B(x_j, R) \cap \Gamma_\infty x$ is not linear for each j (Lemma 7.12). However the sequence of maps

$$\gamma'_j \circ h \circ \gamma_j := h_j$$

converges to an affine mapping h_∞ on $B(x, R)$ (since h is asymptotically linear). We conclude that the union

$$\bigcup_{j=1} h_j(\Gamma_\infty x \cap B(x, R))$$

is an infinite set. $\square$

Theorem 7.14. *Suppose that h is an inverted linear map which is not a similarity. Then h admits no quasi-isometric extension $\Omega \to \Omega'$.*

Proof. Let x be a footpoint of a complementary horoball B to Ω, $B \neq B_\infty$. Then, by the scattering lemma, $h_*(x)$ is not contained in a finite union of Γ'_∞-orbits. Let $\gamma_j \in \Gamma_\infty$ be a sequence such that the Γ'_∞-orbits of the points $x'_j := h\gamma_j(x)$ are all distinct. Let B'_j denote the complementary horoball to Ω' whose footpoint is x'_j. It follows that the Euclidean diameters of the complementary horoballs B'_j converge to zero. Let B_j be the complementary horoball to Ω whose footpoint is $\gamma_j x$. Then

$$dist(B_j, B_\infty) = dist(B_1, B_\infty) = -\log(\text{diam}(B_1)) = D,$$

$$dist(B'_j, B'_\infty) = -\log(\mathrm{diam}(B'_j)) \to \infty.$$

If $f : \Omega \to \Omega'$ is an (L, A) quasi-isometry whose quasiconformal extension is h then

$$dist(B'_j, B'_\infty) \le L(D + Const) + A.$$

Contradiction. □

Therefore we have proven

Theorem 7.15. *Suppose that $f : \Omega \to \Omega'$ is a quasi-isometry of truncated hyperbolic spaces. Then f admits an (unique) extension to S^{n-1} which is Moebius.*

5. Schwartz Rigidity Theorem

Proof of Theorem 7.3.

(a) For each quasi-isometry $f : \Gamma \to \Gamma$ there exists $\gamma \in Comm(\Gamma)$ which is within finite distance from f.

Proof. The quasi-isometry f extends to a quasi-isometry of the hyperbolic space $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$. The latter quasi-isometry extends to a quasiconformal mapping $h : \partial_\infty \mathbb{H}^n \to \partial_\infty \mathbb{H}^n$. This quasiconformal mapping has to be Moebius according to Theorem 7.15. Therefore $\tilde{f}$ is within finite distance from an γ isometry of $\mathbb{H}^n$ (which is an isometric extension of h to $\mathbb{H}^n$). It remains to verify that γ belongs to $Comm(\Gamma)$. We note that γ sends the peripheral horospheres of Ω within (uniformly) bounded distance of peripheral horospheres of Ω. The same is of course true for all mappings of the group $\Gamma' := \gamma\Gamma\gamma^{-1}$. Thus, if $\gamma' \in \Gamma'$ fixes a point in Λ (a footpoint of a peripheral horosphere Σ), then it has to preserve Σ: Otherwise by iterating γ' we would get a contradiction. The same applies if $\gamma'(\xi_1) = \xi_2$, where $\xi_1, \xi_2 \in \Lambda$ are in the same Γ-orbit: $\gamma'(\Sigma_1) = \Sigma_2$, where ξ_i is the footpoint of the peripheral horosphere Σ_i. Therefore we modify Ω as follows: Pick peripheral horospheres $\Sigma_1, ..., \Sigma_m$ with disjoint Γ-orbits and for each $\gamma' \in \Gamma'$ such that $\gamma'(\Sigma_i)$ is not contained in Ω, we replace the peripheral horosphere parallel to $\gamma'(\Sigma_i)$ with the horosphere $\gamma'(\Sigma_i)$. As the result we get a new truncated hyperbolic space Ω' which is invariant under both Γ and Γ'. Observe now that the group Γ'' generated by Γ, Γ' acts on Ω properly discontinuously and cocompactly: Otherwise the nontrivial connected component of the closure of Γ'' would preserve Ω' and hence the countable Λ, which is impossible.

Therefore the projections

$$\Omega/\Gamma \to \Omega/\Gamma'', \Omega/\Gamma' \to \Omega/\Gamma''$$

are finite-to-one maps. It follows that $|\Gamma'' : \Gamma|$ and $|\Gamma'' : \Gamma'|$ are both finite. Therefore the groups Γ, Γ' are commensurable and $\gamma \in Comm(\Gamma)$.

To prove a uniform bound on the distance $d(f, g|\Sigma)$ we notice that f and g have the same extension to the sphere at infinity. Therefore, by 3.13, the distance $d(f, g)$ is uniformly bounded in terms of the quasi-isometry constants of f.

(b) Suppose that Γ, Γ' are non-uniform lattices which are quasi-isometric to each other. Then there exists an isometry $g \in Isom(\mathbb{H}^n)$ such that the groups Γ' and $g\Gamma g^{-1}$ are commensurable.

Proof. The proof is analogous to (a): The quasi-isometry f is within finite distance from an isometry g. Then the elements of the group $g\Gamma g^{-1}$ have the property that they map the truncated hyperbolic space Ω' of Γ' within (uniformly) bounded distance from Ω. Therefore we can modify Ω' to get a truncated hyperbolic space Ω'' which is invariant under both Γ' and $g\Gamma g^{-1}$. The rest of the argument is the same as for (a).

(c) Suppose that Γ' is a finitely-generated group which is quasi-isometric to a nonuniform lattice Γ above. Then the groups Γ, Γ' are virtually isomorphic, i.e. there exists a finite normal subgroup $K \subset \Gamma'$ such that the groups $\Gamma, \Gamma'/K$ contain isomorphic subgroups of finite index.

Proof. Let $f : \Gamma \to \Gamma'$ be a quasi-isometry and let $f' : \Gamma' \to \Gamma$ be its quasi-inverse. We define the set of uniform quasi-isometries

$$\Gamma'_f := f' \circ \Gamma' \circ f$$

of the truncated hyperbolic space Ω of the groups Γ. Each quasi-isometry $g \in \Gamma'_f$ is within a (uniformly) bounded distance from a quasi-isometry of Ω induced by an element g^* of $Comm(\Gamma)$. We get a map

$$\psi : \gamma' \mapsto f' \circ \gamma' \circ f \mapsto (f' \circ \gamma' \circ f)^* \in Comm(\Gamma).$$

I claim that this map is a homomorphism with finite kernel. Let's first check that this map is a homomorphism:

$$d(f' \circ \gamma'_1\gamma'_2 \circ f, f' \circ \gamma'_1 \circ f \circ f' \circ \gamma'_2 \circ f) < \infty,$$

hence the above quasi-isometries have the same Moebius extension to the sphere at infinity. Suppose that $\gamma' \in Ker(\psi)$. Then the quasi-isometry $f' \circ \gamma' \circ f$ has a bounded displacement on Ω. Since the family of quasi-isometries

$$\{f' \circ \gamma' \circ f, \gamma' \in K\}$$

has uniformly bounded quasi-isometry constants, it follows that they have uniformly bounded displacement. Hence the elements $\gamma' \in K$ have uniformly bounded displacement as well. Therefore the normal subgroup K is finite. The rest of the argument is the same as for (a) and (b): The groups $\Gamma, \Gamma'' := \psi(\Gamma') \subset Comm(\Gamma)$ act on a truncated hyperbolic space Ω' which is within finite distance from Ω. Therefore the groups Γ'', Γ are commensurable. $\square$

6. Mostow Rigidity Theorem

Theorem 7.16. *Suppose that $n \geq 3$ and $\Gamma, \Gamma' \subset \mathrm{Isom}(\mathbb{H}^n)$ are lattices and $\rho : \Gamma \to \Gamma'$ is an isomorphism. Then ρ is induced by an isometry, i.e. there exists an isometry $\alpha \in \mathrm{Isom}(\mathbb{H}^n)$ such that*

$$\alpha \circ \gamma = \rho(\gamma) \circ \alpha$$

for all $\gamma \in \Gamma$.

Proof. **Step 1.**

Proposition 7.17. *There exists a ρ-equivariant quasi-isometry $f : \mathbb{H}^n \to \mathbb{H}^n$.*

Proof. As in the proof of Theorem 7.3 we choose truncated hyperbolic spaces $X \subset \mathbb{H}^n, X' \subset \mathbb{H}^n$ which are invariant under Γ and Γ' respectively. (In case if Γ acts cocompactly on $\mathbb{H}^n$ we would take of course $X = X' = \mathbb{H}^n$.)

Then Lemma 1.47 implies that there exists a ρ-equivariant quasi-isometry

$$f : X \to X'.$$

Note that thick-thin decomposition theorem implies that M is noncompact if and only if $\pi_1(M)$ contains $\mathbb{Z}^{n-1}$. Hence, since ρ is an isomorphism, M is compact if and only if M' is compact.

In the following discussion we assume that M is noncompact. Since ρ sends maximal abelian subgroups to maximal abelian subgroups. Therefore f sends the boundary components of X uniformly close to the boundary components of X' (as they are uniformly close to orbits of maximal parabolic subgroups in Γ, Γ' respectively).

Remark 7.18. Here the argument is much easier than in the proof of Schwartz's theorem.

Therefore, according to Theorem 7.7, f extends to a ρ-equivariant quasi-isometry $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$.

Step 2. Let h denote the ρ-equivariant quasi-conformal homeomorphism $S^{n-1} \to S^{n-1}$ which is the extension of f guaranteed by Theorem 6.4. Our goal is to show that h is Moebius. We argue as in the proof of Theorem 7.15. We will identify S^{n-1} with the extended Euclidean space $\hat{\mathbb{R}}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$. Accordingly, we will identify $\mathbb{H}^n$ with the upper half-space. The key for the proof is the fact that h is differentiable almost everywhere on $\mathbb{R}^{n-1}$ so that its Jacobian is nonzero for almost every $z \in \mathbb{R}^{n-1}$. (The latter fails for quasi-Moebius homeomorphisms of the circle: Although they are differentiable almost everywhere, their derivatives vanish almost everywhere as well.)

Suppose that $z \in S^{n-1}$ is a point of differentiability of h so that $J_z(h) \neq 0$. Since only countable number points in S^{n-1} are fixed points of parabolic elements, we can choose z to be a conical limit point of Γ. By applying a conformal change of coordinates we can assume that $z = h(z) = 0 \in \mathbb{R}^{n-1}$ and that $h(\infty) = \infty$.

Let $L \subset \mathbb{H}^n$ be the "vertical" geodesic emanating from 0, pick a base-point $y_0 \in L$. Since z is a conical limit point, there is a sequence of elements $\gamma_i \in \Gamma$ so that

$$\lim_{i\to\infty} \gamma_i(y_0) \to z$$

and

$$d(\gamma_i(y_0), L) \leq C$$

for each i. Let y_i denote the nearest-point projection of $\gamma_i(y_0)$ to L. Take the sequence of hyperbolic translations $T_i : x \mapsto \lambda_i x$ with the axis L, so that $T_i(y_0) = y_i$. Then the sequence $k_i := \gamma_i^{-1} T_i$ is relatively compact in $\mathrm{Isom}(\mathbb{H}^n)$ and lies in a compact $K \subset \mathrm{Isom}(\mathbb{H}^n)$. Now we form the sequence of quasiconformal homeomorphisms

$$h_i(x) := \lambda_i^{-1} h(\lambda_i x) = T_i^{-1} \circ h \circ T_i(x).$$

Note that $\lambda_i \to 0$ as $i \to \infty$. Since the function h is assumed to be differentiable at zero, there is a linear transformation $A \in GL(n-1, \mathbb{R})$ so that

$$\lim_{i\to\infty} h_i(x) = Ax$$

for all $x \in \mathbb{R}^{n-1}$. Since $h(\infty) = \infty$, it follows that

$$\lim_{i\to\infty} h_i = A$$

pointwise on S^{n-1}.

By construction, h_i conjugates the group $\Gamma_i := T_i^{-1} \Gamma T_i \subset \mathrm{Isom}(\mathbb{H}^n)$ into the group of Moebius transformations. We have

$$\Gamma_i = T_i^{-1} \Gamma T_i = (k_i^{-1}\gamma_i)\Gamma(k_i^{-1}\gamma_i)^{-1} = k_i^{-1}\Gamma k_i.$$

After passing to a subsequence we can assume that

$$\lim_{i\to\infty} k_i = k \in \mathrm{Isom}(\mathbb{H}^n)\,.$$

Therefore the *sequence of sets* Γ_i converges to $\Gamma_\infty := k^{-1}\Gamma k$ (in the Chabauty topology on $\mathrm{Isom}(\mathbb{H}^n)$). For each sequence $\beta_i \in \Gamma_i$ which converges to some $\beta \in \mathrm{Isom}(\mathbb{H}^n)$ we have

$$\lim_{i\to\infty} h_i \beta_i h_i^{-1} = A\beta A^{-1}.$$

Since $h_i\beta_i h_i^{-1} \in \mathrm{Isom}(\mathbb{H}^n)$ for each i, it follows that $A\beta A^{-1} \in \mathrm{Isom}(\mathbb{H}^n)$ for each $\beta \in \Gamma_\infty$. Thus

$$A\Gamma_\infty A^{-1} \subset \mathrm{Isom}(\mathbb{H}^n)\,.$$

Since the group Γ_∞ is nonelementary, the orbit $\Gamma_\infty \cdot (\infty)$ is infinite. Hence Γ_∞ contains an element γ such that $\gamma(\infty) \notin \{\infty, 0\}$.

LEMMA 7.19. *Suppose that $\gamma \in \mathrm{Isom}(\mathbb{H}^n)$ is such that $\gamma(\infty) \neq \infty, 0$, $A \in GL(n-1,\mathbb{R})$ is an element which conjugates γ to $A\gamma A^{-1} \in \mathrm{Isom}(\mathbb{H}^n)$. Then A is a Euclidean similarity, i.e it belongs to $\mathbb{R}_+ \times O(n-1)$.*

PROOF. Suppose that A is not a similarity. Let P be a hyperplane in $\mathbb{R}^{n-1}$ which contains the origin 0 but does not contain $A\gamma^{-1}(\infty)$. Then $\gamma \circ A^{-1}(P)$ does not contain ∞ and hence is a round sphere Σ in $\mathbb{R}^{n-1}$.

Since A is not a similarity, the image $A(\Sigma)$ is an ellipsoid which is not a round sphere. Hence the composition $A\gamma A^{-1}$ does not send planes to round spheres and therefore it is not Moebius. Contradiction. $\square$

We therefore conclude that the derivative of h at 0 is Moebius transformation $A \in \mathbb{R}_+ \times O(n-1)$.

Step 3. We will be using the notation from Step 2.

Let $G := \mathrm{Isom}(\mathbb{H}^{\mathrm{n}})$, regarded as the group of Moebius transformations of S^{n-1}. Consider the quotient

$$Q = G \backslash Homeo(S^{n-1})$$

consisting of the cosets $[f] = \{g \circ f : g \in G\}$. Give this quotient the quotient topology, where we endow $Homeo(S^{n-1})$ with the topology of pointwise convergence. Since G is a closed subgroup in $Homeo(S^{n-1})$, it follows that every point in Q is closed. (Actually, Q is Hausdorff, but we will not need it.) The group $Homeo(S^{n-1})$ acts on Q by the formula

$$[f] \mapsto [f \circ g], g \in Homeo(S^{n-1}).$$

It is clear from the definition of the quotient topology on Q that this action is continuous, i.e. the map

$$Q \times Homeo(S^{n-1}) \to Q$$

is continuous.

Since h is a ρ-equivariant homeomorphism, we have

$$[h] \circ \gamma = [h], \quad \forall \gamma \in \Gamma.$$

Recall that we have a sequence of dilations T_i (fixing the origin), a sequence $\gamma_i \in \Gamma$ and a sequence $k_i \in G$ which converges to $k \in G$, so that

$$T_i = \gamma_i \circ k_i,$$

and

$$\lim_i h_i = A \in \mathbb{R}_+ \times O(n-1) \subset G,$$

where

$$h_i = T_i^{-1} \circ h \circ T_i.$$

Therefore

$$[h_i] = [h\gamma_i k_i] = [h] \circ k_i,$$
$$[1] = [A] = \lim_i [h_i] = \lim_i ([h_i] \circ k_i) = [h] \circ \lim_i k_i = [h] \circ k.$$

(Recall that every point in Q is closed.) Thus $[h] = [1] \circ k^{-1} = [1]$, which implies that h is the restriction of an element $\alpha \in \mathrm{Isom}(\mathbb{H}^{\mathrm{n}})$. Since

$$\rho(\gamma) \circ \alpha(x) = \alpha \circ \gamma(x), \quad \forall x \in S^{n-1},$$

it follows that $\alpha : \mathbb{H}^n \to \mathbb{H}^n$ is ρ-equivariant. $\square$

CHAPTER 8

A survey of quasi-isometric rigidity

Given a group G one defines the *abstract commensurator* $Comm(G)$ as follows. The elements of $Comm(G)$ are equivalence classes of isomorphisms between finite index subgroups of G. Two such isomorphisms $\psi : G_1 \to G_2, \phi : G_1' \to G_2'$ are equivalent if their restrictions to further finite index subgroups $G_1'' \to G_2''$ are equal. The composition and the inverse are defined in the obvious way, making $Comm(G)$ a group.

Let X be a metric space or a group G. Call X *strongly QI rigid* if each (L, A)-quasi-isometry $f : X \to X$ is within finite distance from an isometry $\phi : X \to X$ or an element ϕ of $Comm(G)$ and moreover $d(f, \phi) \le C(L, A)$.

Call a group G *QI rigid* if any group G' which is quasi-isometric to G is virtually isomorphic to G.

Call a class of groups $\mathcal{G}$ *QI rigid* if each group G which is quasi-isometric to a member of $\mathcal{G}$ is virtually isomorphic to a member of $\mathcal{G}$.

Theorem 8.1. *(P. Pansu, [**84**]) Let X be a quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{H}}$ $(n \ge 2)$ or the hyperbolic Cayley plane $\mathbb{H}^2_{Ca}$. Then X is strongly QI rigid.*

Theorem 8.2. *(P. Tukia, [**101**] for the real-hyperbolic spaces $\mathbb{H}^n, n \ge 3$ and Chow [**22**] for complex-hyperbolic spaces $\mathbb{H}^n_{\mathbb{C}}, n \ge 2$). Let X be a symmetric space of negative curvature which is not the hyperbolic plane $\mathbb{H}^2$. Then the class of uniform lattices in X is QI rigid.*

Theorem 8.3. *(Combination of the work by D. Gabai [**40**], A. Casson and D. Jungreis [**20**] and P. Tukia [**102**]) The fundamental groups of closed hyperbolic surfaces are QI rigid.*

Theorem 8.4. *(J. Stallings, [**97**]) Each nonabelian free group is QI rigid. Thus each nonuniform lattice in $\mathbb{H}^2$ is QI rigid.*

Theorem 8.5. *(B. Kleiner, B. Leeb, [**67**]) Let X be a symmetric space of nonpositive curvature such that each deRham factor of X is a symmetric space of rank ≥ 2. Then X is strongly QI rigid.*

Theorem 8.6. *(B. Kleiner, B. Leeb, [**67**]) Let X be a Euclidean building such that each deRham factor of X is a Euclidean building of rank ≥ 2. Then X is strongly QI rigid.*

Theorem 8.7. *(B. Kleiner, B. Leeb, [**67**]) Let X be a symmetric space of nonpositive curvature without Euclidean deRham factors. Then the class of uniform lattices in X is QI rigid.*

Theorem 8.8. *(A. Eskin, [**33**]) Let X be an irreducible symmetric space of nonpositive curvature of rank ≥ 2. Then each nonuniform lattice in X is strongly QI rigid and QI rigid.*

Theorem 8.9. *(B. Kleiner, B. Leeb, [**68**]) Suppose that Γ is a finitely-generated groups which is quasi-isometric to a Lie group G with center C and semisimple quotient $G/C = H$. Then Γ fits into a short exact sequence*

$$1 \to K \to \Gamma \to Q \to 1,$$

where K is quasi-isometric to C and Q is virtually isomorphic to a uniform lattice in H.

Problem 8.10. Prove an analogue of the above theorem for all Lie groups G (without assuming that the sol-radical of G is central).

Theorem 8.11. *(M. Bourdon, H. Pajot [**12**]) Let X be a thick hyperbolic building of rank 2 with right-angled fundamental polygon and whose links are complete bipartite graphs. Then X is strongly QI rigid.*

PROBLEM 8.12. Construct an example of a hyperbolic group with Menger curve boundary, which is QI rigid.

PROBLEM 8.13. Let G be a *random* k-generated group, $k \geq 2$. Is G QI rigid?

Randomness can be defined for instance as follows. Consider the set $B(n)$ of presentations

$$\langle x_1, ..., x_k | R_1, ..., R_l \rangle$$

where the total length of the words $R_1, ..., R_l$ is $\leq n$. Then a class C of k-generated groups is said to consist of random groups if

$$\lim_{n\to\infty} \frac{|B(n) \cap C|}{|B(n)|} = 1.$$

Here is another notion of randomness: fix the number l of relators, assume that all relators have the same length n; this defines a class of presentations $S(k, l, n)$. Then require

$$\lim_{n\to\infty} \frac{|S(k, l, n) \cap C|}{|S(k, l, n)|} = 1.$$

See **[58]** for a comparison of various notions of randomness for groups.

THEOREM 8.14. *(M. Kapovich, B. Kleiner,* **[60]***) There is a 3-dimensional hyperbolic group which is strongly QI rigid.*

THEOREM 8.15. *Each finitely generated abelian group is QI rigid.*

THEOREM 8.16. *(B. Farb, L. Mosher,* **[36]***) Each solvable Baumslag-Solitar group*

$$BS(1, q) = \langle x, y : xyx^{-1} = y^q \rangle$$

is QI rigid.

THEOREM 8.17. *(K. Whyte,* **[110]***) All non-solvable Baumslag-Solitar groups*

$$BS(p, q) = \langle x, y : xy^px^{-1} = y^q \rangle,$$

$|p| \neq 1, |q| \neq 1$ *are QI to each other.*

THEOREM 8.18. *(B. Farb, L. Mosher,* **[37]***) The class of non-polycyclic abelian-by-cyclic groups, i.e. groups* Γ *which fit into an exact sequence*

$$1 \to A \to \Gamma \to \mathbb{Z} \to 1$$

with A *is an abelian group, is QI rigid.*

THEOREM 8.19. *(A. Dyubina,* **[30]***) The class of finitely generated solvable groups is not QI rigid.*

PROBLEM 8.20. Is the class of finitely generated polycyclic groups QI rigid?

The recent papers of Eskin, Fisher and Whyte **[34, 35]** made a major progress in the direction of this conjecture. In particular, they proved QI rigidity for 3-dimensional Sol groups.

EXAMPLE 59. Let S be a closed hyperbolic surface, M is the unit tangent bundle over S. Then we have an exact sequence

$$1 \to \mathbb{Z} \to G = \pi_1(M) \to Q := \pi_1(S) \to 1.$$

This sequence does not split even after passage to a finite index subgroup in G, hence G is not virtually isomorphic to $Q \times \mathbb{Z}$. However, since Q is hyperbolic, the group G is quasi-isometric to $Q \times \mathbb{Z}$. See Theorem 1.65.

EXAMPLE 60. There are uniform lattices in $\mathbb{H}^n, n \geq 3$, which are not virtually isomorphic. For instance, take an arithmetic and a nonarithmetic lattice in $\mathbb{H}^n$.

EXAMPLE 61. The product of free groups $G = F_n \times F_m$, $(n, m \geq 2)$ is not QI rigid.

Proof. The group G acts discretely, cocompactly, isometrically on the product of simplicial trees $X := T \times T'$. However there are examples [**111**], [**17**], of groups G' acting discretely, cocompactly, isometrically on X so that G' contains no proper finite index subgroups. Then G is quasi-isometric to G' but these groups are clearly not virtually isomorphic. □

Problem 8.21. Suppose that G is (a) a Mapping Class group, (b) $Out(F_n)$, (c) an Artin group, (d) a Coxeter group, (e) the fundamental group of one of the negatively curved manifolds constructed in [**51**], (f) $\pi_1(N)$, where N is a finite covering of the product of a hyperbolic surface by itself $S \times S$, ramified over the diagonal $\Delta(S \times S)$. Is G QI rigid?

One has to exclude, of course, Artin and Coxeter groups which are commensurable with the direct products of free groups.

Remark 8.22. The Mapping Class group case was recently settled by Behrstock, Kleiner, Minsky and Mosher [**7**].

Theorem 8.23. *(M. Kapovich, B. Leeb,* [**65**]*) The class of fundamental groups G of closed 3-dimensional Haken 3-manifolds, which are not Sol-manifolds*[1]*, is QI rigid.*

In view of rigidity for Sol groups proven by Eskin, Fisher and Whyte, the above rigidity theorem also holds for the fundamental groups of arbitrary closed 3-manifolds.

Theorem 8.24. *(P. Papasoglu,* [**85**]*) The class of finitely-presented groups which split over $\mathbb{Z}$ is QI rigid. Moreover, quasi-isometries of 1-ended groups G preserve the JSJ decomposition of G*

Theorem 8.25. *(M. Kapovich, B. Kleiner, B. Leeb,* [**64**]*) Quasi-isometries preserve deRham decomposition of the universal covers of closed nonpositively curved Riemannian manifolds.*

Problem 8.26. Are there finitely generated (amenable) groups which are quasi-isometric but not bi-Lipschitz homeomorphic? Solved in 2009 by T. Dymarz [**29**] who constructed lamplighter group examples. Her examples, however, are commensurable. Hence, one can ask the following:

Generate an equivalence relation CLIP on groups by combining commensurability and bi-Lipschitz homeomorphisms. Is CLIP equivalent to the quasi-isometry equivalence relation?

Cf. Theorem 2.27 and following discussion.

Theorem 8.27. *(S. Gersten,* [**43**]*) The cohomological dimension (over an arbitrary ring R) is a QI invariant within the class of finitely-presented groups of type FP (over R).*

I refer to [**16**] for the definitions of cohomological dimension and the type FP.

Theorem 8.28. *(Y. Shalom,* [**96**]*) The cohomological dimension (over $\mathbb{Q}$) of amenable groups is a QI invariant.*

Problem 8.29. Is the cohomological dimension of a group (over $\mathbb{Q}$) a QI invariant? Solved by R. Sauer [**92**].

Recall that a group G has *property (T)* if each isometric affine action of G on a Hilbert space has a global fixed point, see [**26**] for more thorough discussion. In particular, such groups cannot map onto $\mathbb{Z}$.

Theorem 8.30. *The property (T) is not a QI invariant.*

Proof. This theorem should be probably attributed to S. Gersten and M. Ramachandran; the example below is a variation on the *Raghunathan's example* discussed in [**42**].

Let Γ be a hyperbolic group which satisfies property (T) and such that $H^2(\Gamma, \mathbb{Z}) \neq 0$. To construct such a group, start for instance with an infinite hyperbolic group F satisfying Property (T) which has an aspherical presentation complex (see for instance [**4**] for the existence of such groups). Then $H^1(F, \mathbb{Z}) = 0$ (since F satisfies (T)), if $H^2(F, \mathbb{Z}) = 0$, add enough random relations to F, keeping the resulting groups F' hyperbolic, infinite, 2-dimensional. Then $H^1(F', \mathbb{Z}) = 0$ since F' also satisfies

[1] I.e. excluding G which are polycyclic but not nilpotent.

(T). For large number of relators we get a group $\Gamma = F'$ such that $\chi(\Gamma) > 0$ (the number of relator is larger than the number of generators), hence $H^2(\Gamma, \mathbb{Z}) \neq 0$. Now, pick a nontrivial element $\omega \in H^2(\Gamma, \mathbb{Z})$ and consider a central extension

$$1 \to \mathbb{Z} \to G \to \Gamma \to 1$$

with the extension class ω. The cohomology class ω is *bounded* since Γ is hyperbolic; hence the groups G and $G' := \mathbb{Z} \times \Gamma$ are quasi-isometric, see Theorem 1.65 or [**42**]. The group G' does not satisfy (T), since it surjects to $\mathbb{Z}$. On the other hand, the group G satisfies (T), see [**26**, 2.c, Theorem 12]. $\square$

However the following question is still open:

PROBLEM 8.31. A group G is said to be *a-T-menable* (see [**21**]) if it admits a proper isometric affine action on a Hilbert space. Is a-T-menability a QI invariant?

Below is a (potentially) interesting variation of Property (T) and a-T-menability whose QI invariance also deserves attention:

Instead of isometric affine actions on Hilbert spaces, consider affine actions on Hilbert spaces $G \curvearrowright H$ where the linear part $l(G)$ of the action consists of uniformly bounded operators, i.e., $\|l(g)\| \leq C < \infty$ for all $g \in G$. Can one promote such actions to isometric actions on Hilbert spaces?

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