Calculus I, Review of Functions

Course web page: http://math.hunter.cuny.edu/olgak/calculus1fall.html
The classkey is: hunter 7757 8224
MATH 150 Fall 2012

(Olga Kharlampovich)

A chalkboard or blackboard is a reusable writing surface on which text or drawings are made with sticks of calcium sulfate or calcium carbonate, known, when used for this purpose, as chalk. Chalkboards were originally made of smooth, thin sheets of black or dark grey slate stone. Modern versions are often green because the colour is considered easier on the eyes [wiki].
The blackboard was invented by James Pillans, headmaster of the Royal High School , Edinburgh, Scotland (1128). He used it with colored chalk to teach geography. The chalkboard was in use in Indian schools in the 11th century.
The term "blackboard" dates from around 1815 to 1825 while the newer and predominantly American term, "chalkboard" dates from 1935 to 1940.
The chalkboard was introduced into the US education system in 1801.

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The graph of f is the set of pairs

$$Graph(f) = \{(x, f(x)) \mid x \in Dom(f)\}$$

which can be viewed as a subset of the real plane $\mathbb{R}^2$.

Definition of function

Example

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Fact (Vertical Line Test)

A set of points S in $\mathbb{R}^2$ is the graph of a function if and only if no vertical line passes through two distinct points in S.

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(b) $f(x) = \sqrt{x^2 - x}$.

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- $|a + b| \leqslant |a| + |b|$.

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Example

$$f(x) = \begin{cases} 1 - x, & \text{if } x \leqslant 1, \\ \sqrt{x - 1}, & \text{if } x > 1. \end{cases}$$

Some properties of functions

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Power functions

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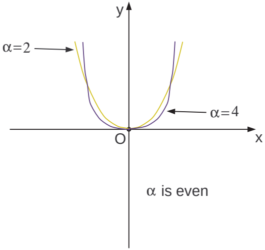

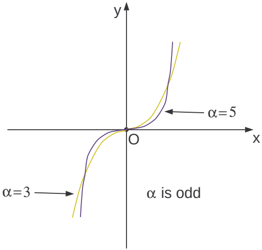

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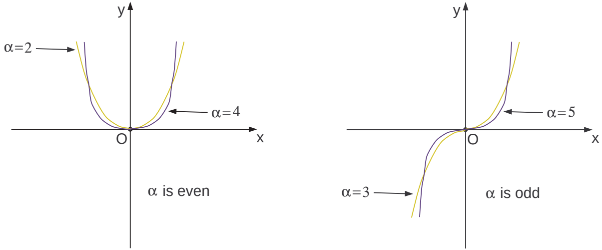

The domain of x^α in this case is the whole real line.

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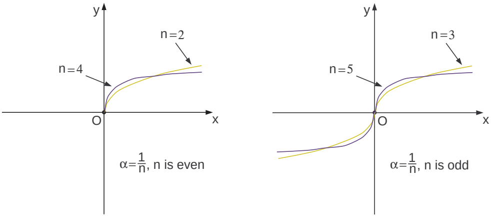

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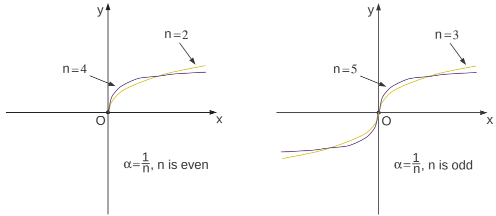

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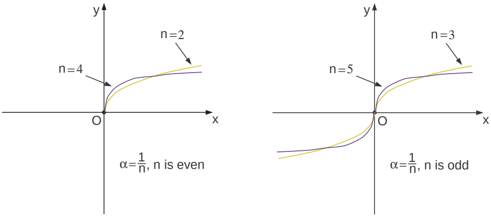

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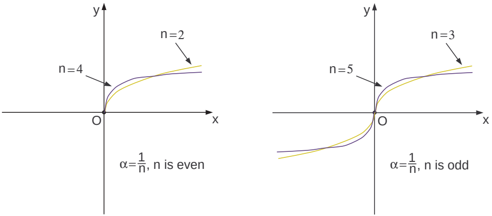

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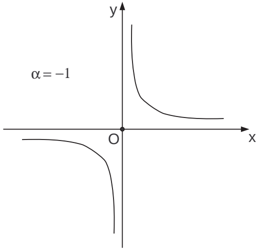

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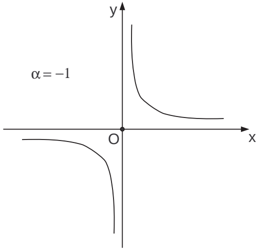

The domain of $f(x) = x^{-1}$ is all real numbers except for $x = 0$.

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$$Dom(f) = \{x \in \mathbb{R} \mid Q(x) \neq 0\}.$$

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- Trigonometric functions

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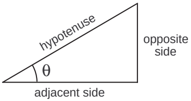

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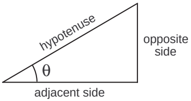

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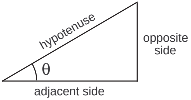

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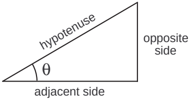

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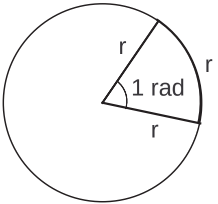

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(a) Find the radian measure of 60°.

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(b) Express $\frac{5\pi}{4}$ rad in degrees.

Example

(a) Find the radian measure of 60°.

(b) Express $\frac{5\pi}{4}$ rad in degrees.

Remark

In Calculus we use radians to measure angles except when otherwise indicated.

Angles

The standard position of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis.

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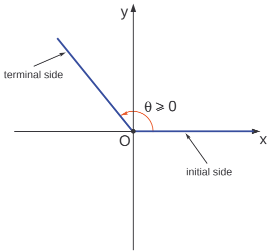

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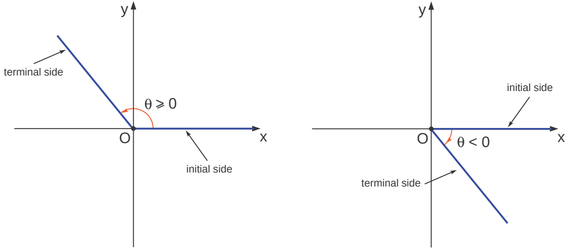

Notice that different angles θ_1 and θ_2 can have the same terminal side. It happens when $\theta_1 - \theta_2 = 2\pi n, \; n \in \mathbb{N}$.

Trigonometric functions

For a general angle θ in standard position let $P(x, y)$ be any point on the terminal side of θ and let r be the distance $|OP|$.

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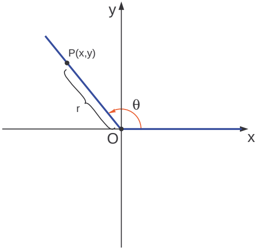

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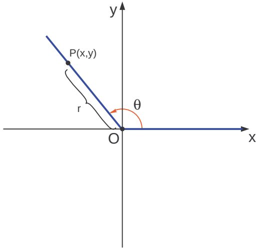

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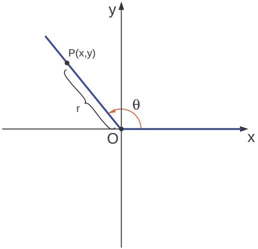

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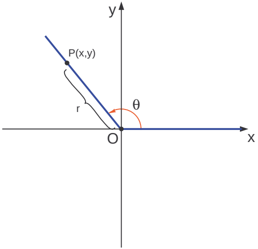

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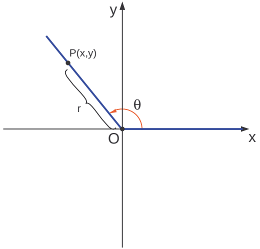

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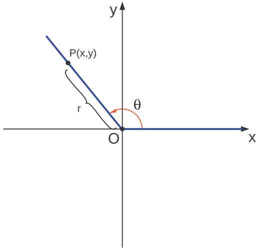

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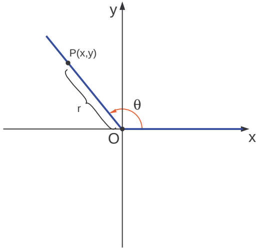

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(b) $\theta = -\frac{2\pi}{3}$.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \sin\frac{\pi}{6} = \frac{1}{2}, \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

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Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$, find the other five trigonometric functions of θ.

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can be obtained from the addition formulas by replacing ϕ by $-\phi$ in the addition identities.

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$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Example

Prove the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Equations

Example

(a) Find all values of θ which satisfy the equation $2\cos\theta - 1 = 0$.

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(b) Find all values of θ in the interval $[0, \pi]$ which satisfy the equation $2\cos\theta + \sin 2\theta = 0$.

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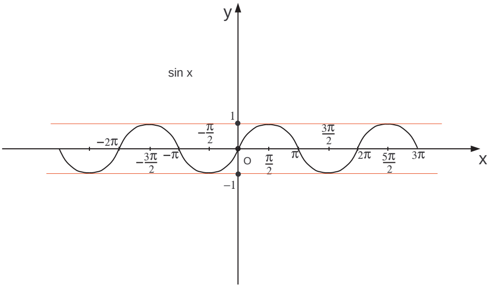

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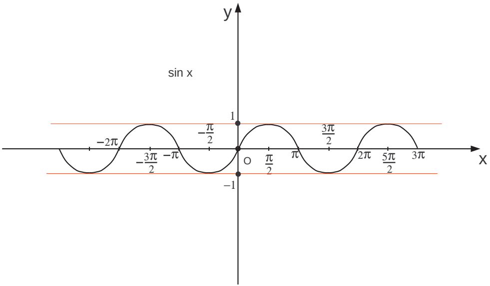

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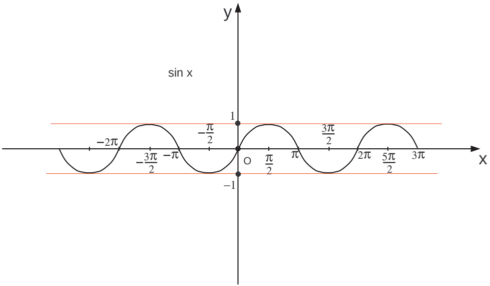

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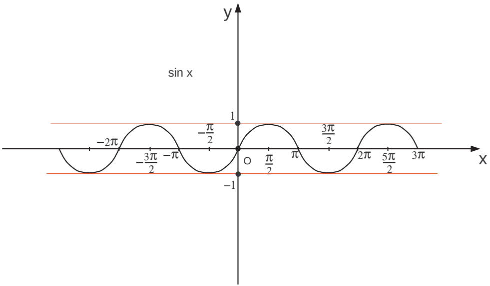

Observe that $|\sin x| \leqslant 1$ and $\sin x = 0$ only when $x = n\pi, \ n \in \mathbb{Z}$. The domain of $\sin x$ is the whole real line.

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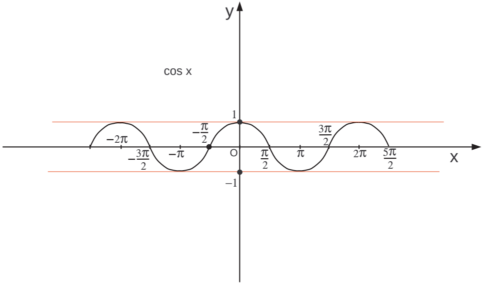

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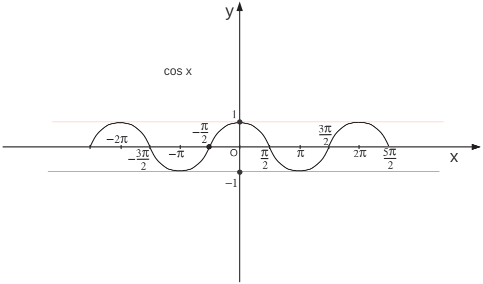

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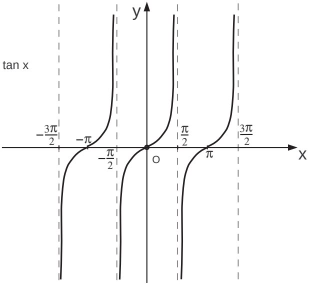

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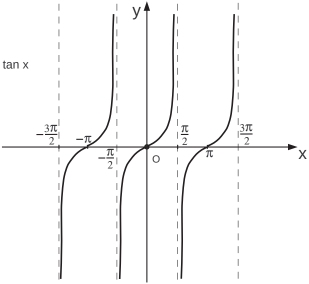

This function is unbounded and its domain excludes points $n\pi + \frac{\pi}{2}, \; n \in \mathbb{Z}$.

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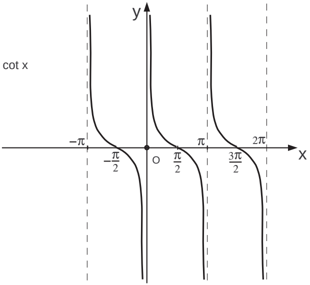

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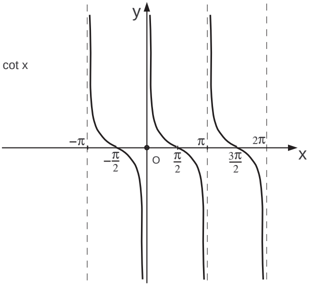

The function is also unbounded and with the domain excluding points $n\pi, \;\; n \in \mathbb{Z}$.

Sumerian astronomers introduced angle measure. The first trigonometric table was apparently compiled by Hipparchus, who is known as "the father of trigonometry". Driven by the demands of navigation and the growing need for accurate maps of large areas, trigonometry grew to be a major branch of mathematics.

The Canadarm2 robotic manipulator on the International Space Station is operated by controlling the angles of its joints. Calculating the final position of the astronaut at the end of the arm requires repeated use of trigonometric functions of those angles.

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Transformations

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These transformations are called *translations*.

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Transformations

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Example

Using the laws of exponents simplify the expression

$$\left(\frac{x^2 y^2 z^5 x^{-3}}{x^3 y^2 z^{-3}}\right)^{-1}$$

Exponential functions

An *exponential function* is a function of the form

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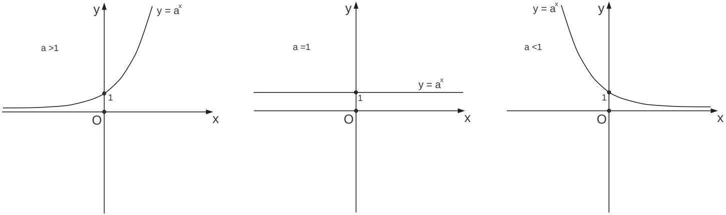

Exponential functions

Example

Sketch the graph of the function $g(x) = 3 - 2^{1-x}$.

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If the graph of a function f is given then it is possible to find out if f is one-to-one using the *Horizontal Line Test*: f is one-to-one if and only if no horizontal line intersects its graph more than once.

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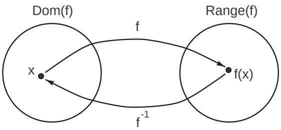

Fact

The graphs of f and f^{-1} are symmetric in the straight line $y = x$.

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Example

Given $f(x) = x^3$, graph $f^{-1}(x)$.

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Example

Find the inverse function of $f(x) = \frac{4x-1}{2x-3}$.

Logarithms

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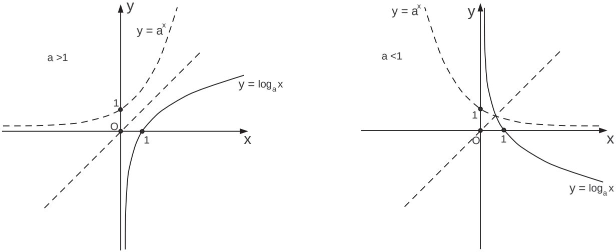

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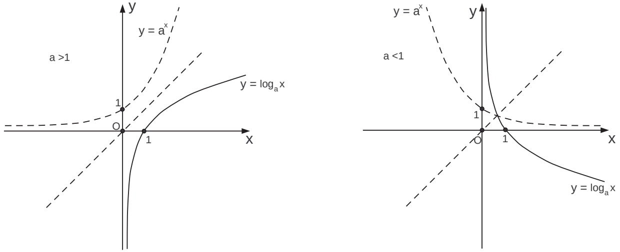

By definition, the domain of $\log_a x$ is the interval $(0, \infty)$ for every value of a.

Example

Sketch the graph of $f(x) = -\log_2(2 - x)$.

Logarithms

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Example

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Logarithms

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In particular, $\ln e = 1$.

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Find all values of x satisfying the given equation

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Logarithms

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$$\log_a x = \frac{\log_b x}{\log_b a}.$$