Model theory and algebraic geometry in groups, non-standard actions and algorithms

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Outline

- Tarski's problems
- Malcev's problems
- Open problems

First-order language of groups

The language L of group theory consists of multiplication $\cdot$, inversion $^{-1}$, and a constant symbol 1 for the identity in the group.

For a given group G one may include all elements of G as constants to the language L thus obtaining a language L_G.

If G is generated by a finite set A it suffices to include only elements of A into L obtaining L_A.

First-order formulas in groups

By $\phi(p_1, \ldots, p_n)$ we denote a first-order formula in the language L (or L_G) whose free variables are contained in the set $\{p_1, \ldots, p_n\}$.

We also use the tuple notation for variables referring to ϕ above as to $\phi(P)$ where $P = (p_1, \ldots, p_n)$.

One may consider only first-order formulas of the type

$$\phi(P) = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \ \phi_0(P, X, Y),$$

where $\phi_0(P, X, Y)$ has no quantifiers.

A formula without free variables is called a sentence.

First-order Theory

The first order theory $Th(G)$ of a group G is the set of all first-order sentences in L (or in L_G) that are true in G.

$Th(G)$ is all the information about G describable in first-order logic.

Examples

Let F be a free group.

The $\forall$-formula

$$F \models \forall x(x^n = 1 \to x = 1)$$

says that there is no n-torsion in the group.

The $\forall\exists$-formula

$$\forall x, y \exists z(xy = yx \to (x = z^2 \vee y = z^2 \vee xy = z^2))$$

holds in F and states that the abelian subgroup generated by x and y is cyclic.

Alfred Tarski

Tarski's type problems for a given group (or a ring, or a structure) G:

- First-order classification: Describe groups H such that $Th(G) = Th(H)$.
- Decidability: Is the theory $Th(G)$ decidable?

Here $Th(G)$ is decidable if there exists an algorithm which for any given sentence Φ decides if $\Phi \in Th(G)$ or not.

There are a few comments in order here.

First-order classification: Describe groups H such that $Th(G) = Th(H)$.

Sometimes this question is too general, so replace an arbitrary H with a finitely generated one, or with a group of a particular kind.

It helps if one can find a natural set of axioms of $Th(G)$ which have a clear algebraic meaning.

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Elimination of quantifiers

A theory $Th(G)$ has quantifier elimination to some set of formulas E, called the eliminating set, if for every formula $\phi(P)$ of L (or L_G) there exists a formula $\phi^*(P)$, which is a boolean combination of formulas from E, such that ϕ is equivalent to ϕ^* in G.

A quantifier elimination is computable if the function $\phi \to \phi^*$ is computable.

If formulas from the set E are simple enough a quantifier elimination to E gives a powerful tool to study $Th(G)$.
There is another equivalent look at quantifier elimination.

Elimination of quantifiers

A subset $S \subseteq G^n$ is definable in a group G if there exists a first-order formula $\phi(P)$ in L_G such that S is precisely the set of tuples in G^n where $\phi(P)$ holds.

$Th(G)$ has an elimination of quantifiers to E iff every set definable in G is a boolean combination of sets defined by formulas from E.

For the field of complex numbers $\mathbb{C}$ the following holds:

- $Th(\mathbb{C}) = Th(F)$ if and only if F is an algebraically closed field of characteristic 0;
- $Th(\mathbb{C})$ is decidable;
- The axioms of the theory state that F is a field of characteristic 0, and every polynomial equation over F has a solution in F.
- This theory admits elimination of quantifiers to a set of quantifier-free formulas.
- Definable sets are precisely the constructible sets (boolean combinations of algebraic sets).

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Tarski's approach to real numbers $\mathbb{R}$

[Tarski 1930-48] For the field of real numbers $\mathbb{C}$ the following holds:

- $Th(\mathbb{R}) = Th(F)$ if and only if F is a real closed field, i.e., F is a formally real field of characteristic 0, where every odd degree polynomial with coefficients in F has a root in F and for every element $a \in F$ either a or $-a$ is a square;
- $Th(\mathbb{R})$ is decidable;
- There is quantifier elimination to quantifier free formulas and formulas of the type $\exists t(x = t^2)$.
- There is quantifier elimination to quantifier free formulas if the order $\leq$ is added to the language.
- Definable sets are boolean combinations of semi-algebraic sets (solution sets of systems of polynomial equations and inequalities);

Around 1945 Tarski put forward two problems:

Let F_n be a free group of rank n.

- Is it true that $Th(F_2) = Th(F_3)$?
- Is $Th(F_n)$ decidable for $n \geq 2$?

It took a while and a joint effort of many mathematicians to develop techniques to get to the point when one could put her/his hands on this problem.

Abelian groups

in 1954 W. Szmielew solved Tarski's problems for an arbitrary abelian group A:
She defined some invariants $\alpha_{p,k}(A)$, $\beta_{p,k}(A)$, $\gamma_{p,k}(A)$, $\delta_{p,k}(A)$
For a given abelian group she did described precisely:

- described all groups B with $Th(A) = Th(B)$;
- described when $Th(A)$ is decidable;
- proved quantifier elimination for $Th(A)$ to a very particular set of existential formulas (pp-formulas). In modern terms, every subset of A^n definable in A is a boolean combination of cosets of pp-definable subgroups of A^n.

Intuitively: there are a lot of definable subgroups in A^n and they completely determine $Th(A)$.

Solvable groups

[Malcev]: Let N_m and S_m be respectively free nilpotent and free solvable groups of finite rank m. Then

- $Th(N_m) = Th(N_k)$ iff $m = k$.
- $Th(S_m) = Th(S_k)$ iff $m = k$.

[Ershov, Romanovskii, Noskov]: A finitely generated soolvable group G has a decidable $Th(G)$ if and only if G is virtually abelian.

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Malcev's problems

Malcev: Let F be a free non-abelian group.

1) Describe definable sets in F;
2) Describe definable subgroups in F;
3) Is the commutant $[F, F]$ of F definable in F?

Theorem [Kharlampovich-Myasnikov, Sela, 2006]

- $Th(F_n) = Th(F_m), m, n > 1.$
- Described all finitely generated groups H with $Th(F_n) = Th(H)$.

Theorem [Kharlampovich and Myasnikov], 2006

The elementary theory $Th(F)$ of a free group F even with constants from F in the language is decidable.

(there were some small inaccuracies noticed by Sela, they are clarified in our note "On Tarski decidability problem" in the arxive, they are indeed of minor nature)

Techniques

Solutions to Tarski's problems in free groups is technically hard and is based on several recent achievements of modern group theory:

- Bass-Serre theory,
- group actions on $\mathbb{R}$-trees and $\mathbb{Z}^n$-trees,
- JSJ decompositions of groups,
- theory of equations in groups,
- algebraic geometry over groups.
- theory of fully residually free (i.e., limit) groups,

However, the whole machinery works nicely for other groups as well, not only for free groups.

In the rest of talk I will discuss our results on Tarski's and Malcev's problems for hyperbolic groups.

Let G be a group generated by a finite set A,

$F(X)$ be a free group with basis $X = \{x_1, x_2, \ldots x_n\}$,

Put $G[X] = G * F(X)$.

If $S \subset G[X]$ then the expression $S(X, A) = 1$ is called *a system of equations* over G.

A solution of the system $S = 1$ over G can be described as a G-homomorphism $\phi : G[X] \longrightarrow G$ such that $\phi(S) = 1$.

Solutions of $S(X, A) = 1$ in G correspond to G-homomorphisms $G[X]/ncl(S) \to G$, and vice versa.

The solution set of $S(X, A) = 1$ in G is the algebraic set $V_G(S)$ defined by S.

The normal subgroup

$$R(S) = \{T(X) \in G[X] \mid \forall Y \in G^n(S(Y, A) = 1 \to T(Y) = 1\}.$$

is the radical of S.

$$G_{R(S)} = G[X]/R(S)$$

is the coordinate group of the system $S(X, A) = 1$.

Zariski topology

One can define Zariski topology on G^n by taking algebraic sets in G^n as a sub-basis for the closed sets of this topology.

Zariski topology is Noetherian if and only if the group G is equationally Noetherian, i.e., every system of equations $S(X, A) = 1$ over G is equivalent to a finite subsystem.

It is known that free and hyperbolic groups are equationally Noetherian.

In a Noetherian topology every closed set is a finite union of its irreducible components.

Γ-limit groups

Let Γ be a non-elementary torsion-free hyperbolic group. A group G is fully residually Γ if for any finite number of non-trivial elements in G there is a homomorphism $G \to \Gamma$ such that the images of these elements are non-trivial.
A finitely generated fully residually Γ group is the same as a Γ-limit group.
Warning: Nor all Γ-limit groups are finitely presented!

Γ-limit groups

Let Γ be an equationally Noetherian group and G a finitely generated group containing Γ. Then the following conditions are equivalent:

1) G is fully residually Γ;
2) G is universally equivalent to Γ (in the language with constants);
3) G is the coordinate group of an irreducible algebraic set over Γ;
4) G is a Γ- limit group;
5) G embeds into an ultrapower of Γ.

Triangular quasi-quadratic (TQ) system is a finite system that has the following form

$$S_1(X_1, X_2, \ldots, X_n, A) = 1,$$
$$S_2(X_2, \ldots, X_n, A) = 1,$$
$$\ldots$$
$$S_n(X_n, A) = 1$$

where either $S_i = 1$ is quadratic in variables X_i, or $S_i = 1$ is a system $[x_j, x_k] = 1$ and, in addition, equations $[x, u] = 1$ for all $x, x_j, x_k \in X_i$ and some $u \in F_{R(S_{i+1},\ldots,S_n)}$ or S_i is empty.
A TQ system above is non-degenerate (NTQ) if for every i, $S_i(X_i, \ldots, X_n, A) = 1$ has a solution in the coordinate group $G_i = F_{R(S_{i+1},\ldots,S_n)}$, where $G_n = F$ (or $G_n = \Gamma$).

Description of solution sets of equations

Theorem

For a system of equations

$$S(X, A) = 1$$

over F one can find finitely many NTQ systems

$$U_1(Y_1, A) = 1, \ldots, U_m(Y_m, A) = 1$$

such that

$$V_F(S) = P_1(V(U_1)) \cup \ldots \cup P_m(V(U_m))$$

for some word mappings $P_1, \ldots, P_m$. (P_i maps a tuple $\bar{Y}_i \in V(U_i)$ to a tuple $\bar{X} \in V_F(S)$.

Similarly one can effectively describe the solution set of a system over a torsion-free hyperbolic group Γ.

First-order classification for free groups

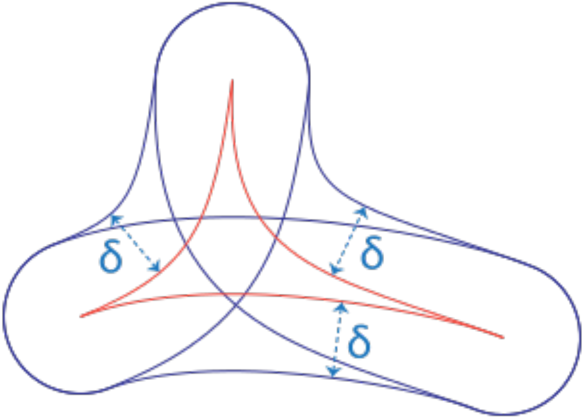
Now we can explain which finitely generated groups have the same theory $Th(F)$ as a free non-abelian group.

Theorem [KM, Sela, 2006]

Hyperbolic NTQ groups are exactly the finitely generated groups H such that $Th(F) = Th(H)$.

Hyperbolic groups

A finitely generated group G is δ-hyperbolic if all geodesic triangles in the Cayley graph of G are δ-thin.



Quantifier elimination [Sela, 2008]

Every formula in the theory of F is equivalent to a boolean combination of AE-formulas. The same is true for a torsion free hyperbolic group.

Note, the quantifier elimination is not efficient here.

Effective quantifier elimination

Theorem [Kharlampovich and M.], 2013

Let Γ be a torsion free hyperbolic group. Then $Th(\Gamma)$ with constants from Γ in the language admits a computable quantifier elimination to $\forall\exists$-formulas.

Thus, i.e., there exists an algorithm which given a first-order formula ϕ with constants from Γ finds a boolean combination of $\forall\exists$-formulas that is equivalent to ϕ in Γ.

Note, the elimination set in this case, as well as in free groups, is much more complex then in $\mathbb{C}, \mathbb{R}$, or in abelian groups.

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Decidability

Let Γ be a torsion free hyperbolic group.

Theorem [Kharlampovich and M.], 2013

The $\forall\exists$-theory of a Γ torsion-free hyperbolic group with constants from Γ in the language is decidable.

Combining this with effective quantifier elimination one gets

Theorem [Kharlampovich and M.], 2013

The elementary theory $Th(\Gamma)$ of a torsion free hyperbolic group Γ with constants from Γ in the language is decidable.

Theorem (KM, 2011)

Let Γ be a torsion free hyperbolic group. Then the group Γ and cyclic subgroups of Γ are the only definable subgroups of Γ.

In particular, this solves the Malcev's problems 2 and 3 for free groups. There are other interesting corollaries.

The proof is based on our description of sets definable in torsion-free hyperbolic groups (solution to Malcev's problem 1) and a beautiful idea due to Bestvina and Feighn on negligible sets.

I will try to explain some ideas of the description of the definable sets.

Algebraic sets: let $W(P, A) = 1$ be an equation (with constants) in a group G. Then the algebraic set

$$V_G(W) = \{g \in G^n \mid W(g, A) = 1\}$$

is definable in G.

Examples of definable subgroups

The following subgroups are definable in any group G:

- The centralizer of a finite subset $M = \{g_1, \ldots, g_m\} \subseteq G$:

$$C_G(M) = \{x \in G \mid \wedge_{i=1}^{m} [x, g_i] = 1\};$$

- In particular, maximal cyclic subgroups are definable in a free group or a torsion-free hyperbolic group, as well as their subgroups.

As we know from quantifier elimination every definable set in Γ is a boolean combination of $\forall\exists$-definable sets.

Hence we need to describe only sets $\forall\exists$-definable in Γ.

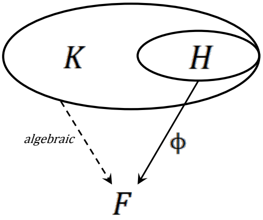
We first improve the quantifier elimination.

More on quantifier elimination

Quantifier elimination [Kh,M]

Every formula in the theory of F (or Γ) in the language L_A is equivalent to some boolean combination of formulas $\exists X \forall Y (U(P,X) = 1 \wedge V(P,X,Y) \neq 1)$, where X, Y, P are tuples of variables. Moreover, there is an algorithm that for a given formula $\phi(P)$ finds an equivalent boolean combination of the type above.

What do these formulas define?



$U(P, X) = 1$ is an irreducible equation such that the limit group $K = F_{R(U)}$ has "not sufficiently many" H-automorphisms, $H =< P >$, $V(P, X, Y) \neq 1$ demonstrates that each $\phi : H \to F$ can be extended to at least n "algebraic" $K \to F$.

Parametrization theorem for definable sets

Theorem [Kharlampovich, M., 2011]

Let Γ be a torsion free hyperbolic group. Then for every subset $P \subseteq \Gamma^m$ definable in Γ, either P or its complement $\neg P$ is a finite union of sets admitting parametrization and one can find these parametrizations effectively.

The formula defines the set P "implicitly", while the parametrization defines it (or its complement) "explicitly".

This theorem is the main tool when proving that some sets are non-definable.

Now I will explain what is parametrization in the case of free groups, the hyperbolic case is similar but more technical.

Parametrization

A piece of a word $u \in F$ is a non-trivial subword v that appears in u in at least two different ways (maybe the second time as v^{-1}, maybe with overlapping).

A proper subset P of F admits parametrization if every words $p \in P$ satisfies a system of equations (with coefficients) without cancellations of the following type

$$p \doteq w_1(y_1, \ldots, y_n), \ldots, p \doteq w_k(y_1, \ldots, y_n),$$

where each y_i is non-empty, appears at least twice in the system, and each y_j that appears in w_1 is a piece of p.

Parametrizations of subsets of F^k are defined similarly.

Recall that a RAAG is a group G given by a presentation of the form $< a_1, \ldots, a_r | R >$, where R is a subset of the set $\{[a_i, a_j] | i, j = 1, \ldots, r\}$.

Tarski's problems for RAAGs

Let A be a right angled Artin group.

- Describe RAAGs B such that $Th(A) = Th(B)$.
- Is the first order theory $Th(A)$ decidable?

Notice that Casals-Ruiz and Kazachkov obtained an algorithmic description of the solution set of a system of equations in RAAGs. Note that Diekert and Muscholl showed that the universal theory of any RAAG is decidable.