Notes on Agol's Virtual Haken Theorem and Wise's Malnormal Special Quotient Theorem and Hierarhy Theorem

Olga Kharlampovich

October 31, 2013

1 The virtual Haken conjecture, introduction

Agol proves that cubulated hyperbolic groups are virtually special.

Theorem 1. *(Agol [A])Let G be a hyperbolic group which acts properly and cocompactly on a CAT(0) cube complex X (we say that G is cubulated). Then G has a finite index subgroup G' so that X/G' is a special cube complex (we say that such G is virtually special).*

[cocompactly means that X/G is a compact, *properly* means that the map $G \times X \to X \times X$, defined as $(g, x) \to (gx, x)$ is proper (pre-image of a compact is compact)]

Work of Haglund and Wise [HW] on special cube complexes (GAFA) implies that virtually special hyperbolic groups are linear groups, and quasi-convex subgroups are separable (both terms to be defined). A consequence is that

Theorem 2. *Closed hyperbolic 3-manifolds have finite-sheeted covers which are Haken manifolds.*

This resolves the Waldhausen virtual Haken conjecture from 1968.

Conjecture 1. *(Waldhausen) Every aspherical (this means that $\pi_i(M) = 0, i \geq 2$) closed 3-manifold M has a finite cover which is Haken.*

The conjecture was reduced to the case of closed hyperbolic manifolds by Perelman's hyperbolization theorem.

This also resolves Thurstons virtual fibering question

Theorem 3. *Let M be a closed hyperbolic 3-manifold. Then M has a finite cover which fibers over the circle.*

The results depend on the following results:

1) Recent result of Wise, the malnormal special quotient theorem [W1];

2) The cubulation of closed hyperbolic 3-manifolds by Bergeron-Wise'2009, using the existence of nearly geodesic surfaces by Kahn-Markovic: (**Projects**)) Closed hyperbolic 3-manifolds are cubulated (=homotopy equivalent to compact locally CAT(0) cube complexes (equivalently, NPC cube complexes) = have $\pi_1(M)$ isomorphic to the fundamental group of such a complex).

3) A generalization of previous work of Agol with Groves and Manning to the case of torsion (which is joint with Groves and Manning).

It follows from Agols theorem that given a hyperbolic 3-manifold M with fundamental group $\pi(M)$ that

(1) $\pi_1(M)$ admits a finite index subgroup which is a quasi-convex subgroup of a right angled Artin group (this follows from Haglund-Wise)

(2)M is virtually fibered (in fact $\pi_1(M)$ is virtually RFRS (to be defined) and the statement of Agol's virtual fibering theorem holds)

(3) π_1 is LERF (also known as subgroup separable), this uses work of Haglund and the proof of the tameness conjecture. In fact a stronger statement holds: any geometrically finite subgroup of pi is a virtual retract.

(4) $\pi_1(M)$ is large, i.e. pi admits a finite index subgroup which maps onto a non-cyclic free group, in particular the Betti numbers of finite covers can become arbitrarily large

(5) $\pi_1(M)$ is linear over $\mathbb{Z}$, i.e. $\pi_1(M)$ admits a faithful representation to $GL(n, \mathbb{Z})$ for some n.

(6) $\pi_1(M)$ is conjugacy separable (this uses work of Minasyan)

(7) $\pi_1(M)$ is virtually biorderable.

Almost every nice property of fundamental groups which one can possibly ask for either holds for $\pi(M)$ or a finite index subgroup of $\pi(M)$.

As Calegari states "this marks the end of an era in 3-manifold topology, since the proof ties up just about every loose end left over on the list of problems in 3-manifold topology from Thurstons famous Bulletin article (with the exception of problem 23 to show that volumes of closed hyperbolic 3-manifolds are not rationally related which is very close to some famous open problems in number theory). "

The purpose of this course is to say what the Virtual Haken Conjecture is, formulate and explain the results of Wise's paper "Groups with a quasi-convex hierarchy" and give some of the background that goes into Agol's and Wise's arguments.

We make the assumption in what follows that all manifolds that we discuss are connected and orientable (which can be achieved by passing to double covers, if necessary).

Definition 1. *A compact 3-manifold M, possibly with boundary, is said to be irreducible if every embedded 2-sphere in M bounds a 3-ball in M.*

Notice that the only orientable prime 3-manifold that is not irreducible is $S^1 \times S^2$. A connected 3-manifold M is called prime if $M = P\#Q \to P = S^3$or $Q = S^3$ (the fundamental group of a prime manifold cannot be decomposed as a non-trivial free product).

Definition 2. *A compact 3-manifold is said to be Haken (the terminology sufficiently large is also standard) if it is irreducible, and if it contains a closed, 2-sided π_1- embedded surface S (other than a 2-sphere)*

[By other words, the surface is incompressible and boundary incompressible; incompressibility means that if an embedded disk intersects S only in an embedded loop, then this loop is (homotopically) inessential in S (bounds a disc in S), and boundary incompressibility means that if an embedded disk intersects S only in a proper arc (with the rest of the boundary of the disk on the boundary of M) then the arc on S is (homotopically) inessential in S. Incompressibility and boundary incompressibility mean roughly that the surface cant be simplified by a local move. Such a surface is also said to be essential.]

Remark 1. *(to be explained later) Haken used such surfaces to solve the homeomorphism problem for the 3-manifolds that contain them. (Hence the word problem in the fundamental group of a Haken manifold is solvable). This includes as a very important special case manifolds obtained as complements of (open tubular neighborhoods of) knots in the 3-sphere; hence Hakens methods solve the knot recognition problem,- the problem of deciding when two different diagrams determine the same knot type. The key point of Hakens approach is that once one has an essential surface, one can cut along it to produce a simpler 3-manifold with boundary. Every irreducible 3-manifold with nonempty boundary is Haken, so if this decomposition process can be begun, it can be continued.*

2 Cube Complexes

The technical content of Agol's preprint is the proof of a conjecture in Wise's paper about groups acting on CAT(0) cube complexes (Theorem 1 above). So lets start with the definition of a CAT(0) cube complex.

Definition 3. *A cube complex is a space obtained by gluing Euclidean cubes of edge length 1 along subcubes. It is CAT(0) if it is CAT(0) as a geodesic metric space.*

Definition 4. *A geodesic metric space X is CAT(0) id its triangles are at least as thin as triangles in Euclidean space. That is, if abc are three points in X and $\bar{a}\bar{b}\bar{c}$ are three points in the Euclidean plane with $d_X(a,b) = d_{\mathbf{E}^2}(\bar{a},\bar{b})$, $d_X(c,b) = d_{\mathbf{E}^2}(\bar{c},\bar{b})$, $d_X(a,c) = d_{\mathbf{E}^2}(\bar{a},\bar{c})$, then for any point p on the geodesic $[b,c]$ and corresponding point $\bar{p} \in [\bar{b},\bar{c}]$ (in the sense that $\bar{p}$ is at the same distance from $\bar{b}$ as p from b), we have the inequality*

$$d_X(a,p) \leq d_{\mathbf{E}^2}(\bar{a},\bar{p}).$$

Gromov showed (see also [BH] that a cube complex X is CAT(0) if and only if it is simply connected, ($\pi_1(X) = 1$) and satisfies the flag condition, namely the link of every vertex is a flag complex i.e.

Definition 5. *A complex is a flag complex if edges of every complete subgraph of the 1-skeleton are the edges of a simplex in the complex.*

Definition 6. *A cube complex satisfying the flag condition without necessarily being simply connected is said to be nonpositively curved, or NPC.*

Flag complex=NPC cube complex = locally CAT(0) cube complex.

Project: A theorem of Sageev '95 associates a cocompact action of $\pi_1(M)$ on a (globally) CAT (0) cube complex if M contains an immersed essential surface. Globally CAT (0)=simply-connected and locally CAT(0)= simply connected and NPC.

The universal cover of an NPC complex is a CAT(0) cube complex.

Theorem 4. *(Bergeron-Wise'2009 using Kahn-Markovich (***Projects***)) Closed hyperbolic 3-manifolds are cubulated (=homotopy equivalent to compact locally CAT(0) cube complexes=have $\pi_1(M)$ isomorphic to the fundamental group of such a complex).*

Definition 7. *The edges in a cube complex fall into equivalence classes, where two edges are equivalent if they are on opposite sides of a square in the complex. Equivalence classes are called* **walls***. Dual to an equivalence class is a* **hyperplane***, which can be thought of as a collection of codimension 1 faces in each cube dual to the edges in that cube in the given equivalence class. For each edge a denote by $W(a)$ the unique wall containing a.*

Haglund and Wise singled out a particular class of NPC cube complexes, which they call special cube complexes.

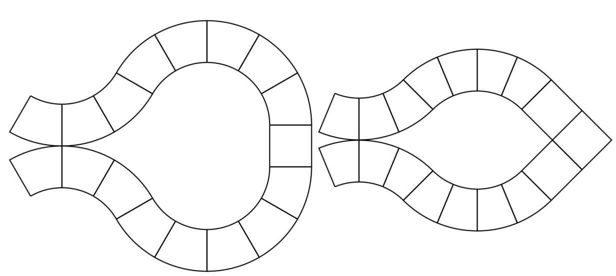
Definition 8. *A* special cube complex *is an NPC cube complex C satisfying the following conditions:*

(i) hyperplanes are embedded;

(ii) hyperplanes are 2-sided (this can be arranged by passing to a 2-fold cover if necessary, given (i))

(iii) there are no self-osculating hyperplanes; and

(iv) there is no interosculation (see pictures below or in [C]).



An example of a one-sided hyperplane (Mobius band).

Interosculation rules out the condition that two hyperplanes osculate somewhere and are transverse elsewhere:

Haglund-Wise show (and this is the main reason to introduce the class of special NPC complexes)

Theorem 5 (HW'07)**.** *The fundamental group $\pi_1(X)$ of a special NPC cube complex X embeds into a right-angled Artin group (RAAG) with one generator for each hyperplane , with the relation that generators commute whenever hyperplanes cross, and no other relations (denote it $A_(\Gamma(X))$, where $\Gamma(X)$ is the "crossing graph" with vertices the hyperplanes). Moreover, quasi-convex subgroup of $\pi_1(X)$ are separable.*

There is a natural map from X to the Salvetti complex $S_{A_{(\Gamma(X))}}$ with is locally isometric immersion when X is special. This gives $\pi_1(X) < A_{(\Gamma(X))}$.

We will consider the paper [HW] in details.

3 Some group-theoretic properties

Group theoretic properties of 3-manifold fundamental groups.

Definition 9. *A group G is residually finite if $\{1\} = \cap H_{[G:H]<\infty}$.*

Alternatively, $\{1\} = \cap_{[G:H]<\infty} H$.

Examples:

f.g. linear groups (Malcev' 40)

3-manifold groups (Hempel' 84+Geometrization)

mapping class groups of surfaces (i.e. $Out(\pi_1\Sigma)$) (Grossman "74)

Lecture 2

Definition 10. *A subgroup $H < G$ is separable if $\forall g \in G - H\ \exists \phi : G \to K$, where K is finite and $\phi(g) \notin \phi(H)$.*

Alternatively, $h = \cap_{H \leq L \leq G, [G:L]<\infty} L$

Residual finiteness means that $1 < G$ is separable.

Definition 11. *A group G is LERF if every finitely generated subgroup of G is separable.*

Examples:

Free groups are LERF, $\mathbb{Z}^n$, surface groups are LERF (to be proved).

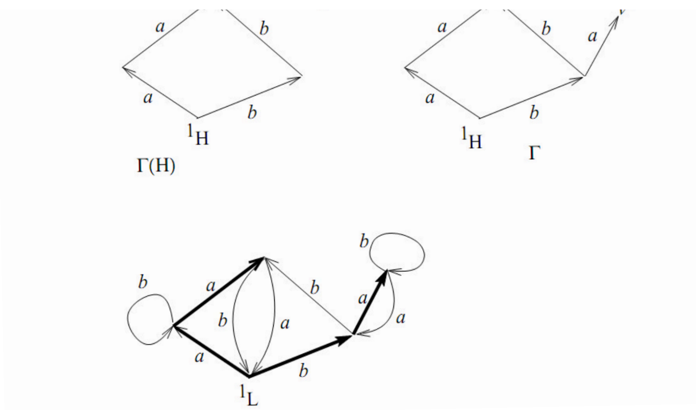
Theorem 6 (Marshall Hall's Theorem)**.** *Let H be a finitely generated subgroup of $F(X)$. Let $g \in F(X)$ be such that $g \notin H$.*

Then there exists a finitely generated subgroup K of $F(X)$ such that

1. *$L = \langle H, K \rangle = H * K$;*
2. *L has finite index in $F(X)$;*
3. *$g \notin L$.*

Proof. **Idea of the proof** The Stallings graph $\Gamma(H)$ for a subgroup of a free group is the minimal subgraph of the graph of right cosets of H that contains all the loops beginning at the coset H (denote this vertex 1_H). Notice that H has finite index in $F(X)$ if and only if $\Gamma(X)$ is complete, namely each vertex for each $x, x \in X^{\pm 1}$ has an edge emanating from this vertex and labeled x.

Write g as a reduced word in X. Add a path with label g beginning at 1_H to $\Gamma(H)$. Fold the obtained graph (denote the result by Γ). Add edges to Γ to obtain a finite folded complete graph Γ'. $L = L(\Gamma', 1_H)$. Consider example:



$F(X) = F(a, b),\ H =< a^2b^{-2} >,\ g = ba.$

□

Example If $\sigma^2 \to M^3 \to^{\eta} S^1$ is a fibration (to be defined later), then there is a short exact sequence $\pi_1(\Sigma) \to \pi_1(M) \to \mathbb{Z}$. Then $\pi_1(\Sigma)$ is separable in $\pi_1(M)$, since

$$\pi_1(\Sigma) = \cap_n \eta^{-1}(n\mathbb{Z}).$$

Not all 3-manifold groups are LERF. For example the fundamental group of the complement of the olympic 5-ring emblem is not LERF (Niblo, Wise' 01)

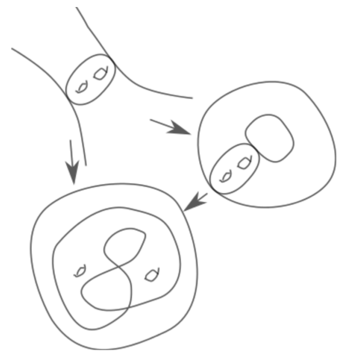
Importance of LERF property in topology. If M is a hyperbolic 3-manifold, and $\pi_1(M)$ is LERF, then this allows to lift an immersed π_1-embedded surface $f : \Sigma \to M$ to an embedding in the finite cover, see Fig. 3

4 Finitely generated groups viewed as metric spaces

Let G be a group given as a quotient

$$\pi : F(S) \to G,$$

of the free group on a set S. Therefore $G = \langle S|R \rangle$. The word length $|g|$ of an element $g \in G$ is the smallest integer n for which there exists a sequence



$s_1.\cdots s_n$ of elements in $S \cup S^{-1}$ such that

$$g = \pi\left(s_1.\cdots s_n\right).$$

The word metric $d_S\left(g_1, g_1\right)$ is defined on G by

$$d_S\left(g_1, g_2\right) = \left|g_1^{-1}g_2\right|.$$

G acts on itself from the left by isometries.
(Cayley graph)

Definition 12. *Note, that if S and $\overline{S}$ are two finite generating sets of G then d_S and $d_{\overline{S}}$ are bi-Lipschitz equivalent, namely $\exists\ C\ \forall\ g_1, g_2 \in G$,*

$$\frac{1}{C} d_S\left(g_1, g_2\right) \leq d_{\overline{S}}\left(g_1, g_2\right) \leq C\ d_S\left(g_1, g_2\right).$$

Definition 13. *A ball of radius n is $Cay\left(G, S\right)$ is*

$$B_n = \left\{g \in G|\ \ |g| \leq n\right\}.$$

5 Hyperbolic spaces and groups

See corresponding section in [W].

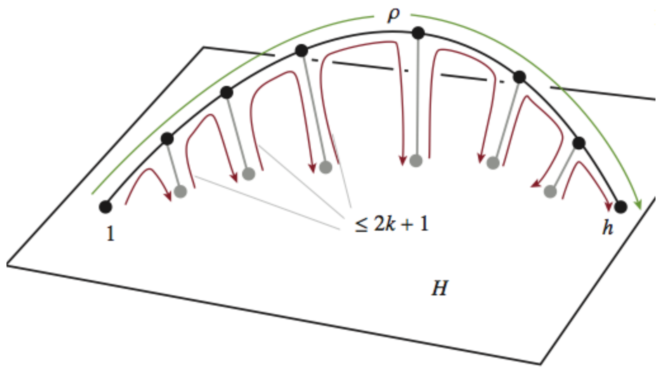
Definition 14. *A geodesic metric space is called $\delta-$hyperbolic if for every geodesic triangle, each edge is contained in the δ neighborhood of the union of the other two edges.*

Definition 15. *If $\delta = 0$ the space is called a real tree, or $\mathbb{R}-$ tree.*

Example 1. *A group G is hyperbolic $Cay\left(G, X\right)$ is hyperbolic (= It looks like a tree)*

Example 2. *Geodesic triangles are $\delta-$thin hyperbolic.*

Example 3. *$F\left(S\right)$ is hyperbolic so is a $0-$ hyperbolic, so $\delta = 0$.*



Definition 16. *Let G be a hyperbolic group, with Cayley graph Γ. A subgroup $H < G$ may be regarded as a subspace of $H \subset V(\Gamma) = G$. A subgroup H is k+-quasiconvex if for every $h \in H$, the geodesic $\rho = [1, h] \subset \mathcal{N}_{k(H)}$.*

We say that H is quasi convex if it is k-quasiconvex for some k.

6 Quasi-isometry

Definition 17. *Let (X, d_X) and (Y, d_Y) be metrics spaces. Given real numbers $k \geq 1$ and $C \geq 0$, a map*

$$f : X \to Y,$$

is called a $(k, C)-$ quasi-isometry if

1. $\frac{1}{k} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2) + C$, $\forall x_1, x_2 \in X$.
2. *the C neighborhood of $f(X)$ is all of Y.*

Examples of quasi-isometries

Example 4. $(\mathbb{R}; d)$ *and* $(\mathbb{Z}; d)$ *are quasi-isometric. The natural embedding of $\mathbb{Z}$ in $\mathbb{R}$ is isometry. It is not surjective, but each point of $\mathbb{R}$ is at most $\frac{1}{2}$ away from $\mathbb{Z}$.*

Example 5. *All regular trees of valence at least 3 are quasi-isometric. We denote by T_k the regular tree of valence k and we show that T_3 is quasi-isometric to T_k for every $k \geq 4$. We define that map*

$$q : T_3 \to T_k,$$

sending all edges drawn in thin lines isometrically onto edges and all paths of length $k-3$ drawn in thick lines onto one vertex. The map q thus defined is surjective and it satisfies the inequality

$$\frac{1}{k-2} dist(x, y) - 1 \leq dist(q(x), q(y)) \leq dist(x, y).$$

Free groups of different rank are quasi-isometric

Example 6. *All non-Abelian free groups of finite rank are quasi-isometric to each other. The Cayley graph of the free group or rank* n *with respect to a set of* n *generators and their inverses is the regular simplicial tree valence* $2n$.

Example 7. *Let* G *be a group with a finite generating set* S, *and let* $Cay\,(G,S)$ *be the corresponding Cayley graph. We can make* $Cay\,(G,S)$ *into a metric space by identifying each edge with a unit interval* $[0,1]$ *in* $\mathbb{R}$ *and defining* $d\,(x,y)$ *to be the length of the shortest path joining* x, *and* y. *This coincides with the path length metric when* x *and* y *are vertices. Since every point of* $Cay\,(G,S)$ *is in the* $\frac{1}{2}-$*neighborhood of some vertex, we see that* G *and* $Cay\,(G,S)$ *are quasi-isometric for this choice of* d.

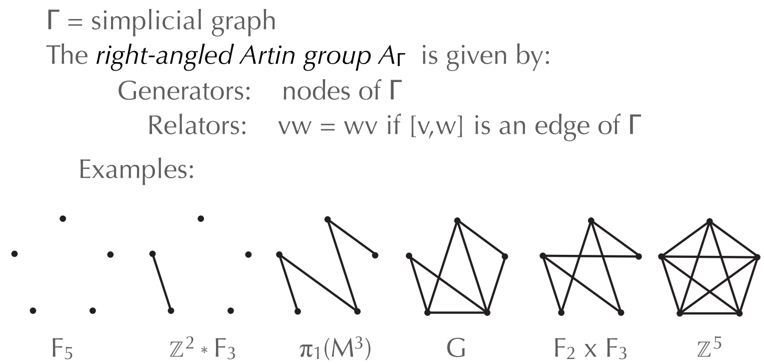
Example 8. *Every bounded metric space is quasi-isometric to a point.*

Example 9. *The main example, which partly justifies the interest in quasi-isometries, is the following. Given* M *a compact Riemannian manifold, let* $\widetilde{M}$ *be its universal covering and let* $\pi_1\,(M)$ *be its fundamental group. The group* $\pi_1\,(M)$ *is finitely generated, in fact even finitely presented. The metric space* $\widetilde{M}$ *with the Riemannian metric is quasi-isometric to* $\pi_1\,(M)$ *with some word metric.*

Example 10. *If* G_1 *is a finite index subgroup of* G, *then* G *and* G_1 *are quasi-isometrically equivalent.*

Connections between group properties and virtual properties of 3-manifolds

7 Right-angled Artin groups (RAAGs) and their cube complexes.



The standard 2-complex X of the presentation extends to a nonpositively curved cube complex $S_{A(X)}$ by adding an n-cube (in the form of an n- torus $S^1 \times \ldots \times S^1$) for each set of n pairwise commuting generators.

Ex. Consider a torus $S^1 \times S^1$ as a cube complex with one 0- cube, two 1-cubes and one 2-cube. The link of the vertex is a circle with 4 vertices.

8 Amalgamated products, HNN extensions and graphs of groups

Definition 18. *A graph of groups $\Gamma(\mathcal{G}, X)$ consists of*

1) a connected graph X;

2) a function $\mathcal{G}$ which for every vertex $v \in V(X)$ assigns a group G_v, and for each edge $e \in E(X)$ assigns a group G_e such that $G_{\bar{e}} = G_e$.

3) For each edge $e \in E(X)$ there exists a monomorphism $\sigma : G_e \longrightarrow G_{\sigma e}$.

Let $\Gamma(\mathcal{G}, X)$ be a graph of groups. Since $G_{\bar{e}} = G_e$ then there exists a monomorphism $G_e \longrightarrow G_{\sigma\bar{e}} = G_{\tau e}$ which we denote by $\tau : G_e \longrightarrow G_{\tau e}$.

Let $\Gamma = \Gamma(\mathcal{G}, X)$ be a graph of groups, and let T be a maximal subtree of X. Suppose the groups G_v are given by presentations $G_v = \langle X_v \,|\, R_v \rangle, v \in V(X)$. We define a *fundamental group* $\pi(\Gamma)$ of the graph of groups Γ by generators and relations :

Generators of $\pi(\Gamma)$:

$$\bigcup_{v \in V(X)} X_v \cup \{t_e \,|\, e \in E(X)\}$$

Relations of $\pi(\Gamma)$:

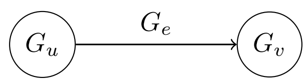
$$\bigcup_{v \in V(X)} R_v \cup \{t_e^{-1} \sigma g t_e = \tau g \,|\, g \in G_e,\, e \in E(X)\}$$

$$\cup \{t_{\bar{e}} = t_e^{-1} \,|\, e \in E(X)\} \cup \{t_e = 1 \,|\, e \in T\}.$$

We assume here that $\sigma(g)$ and $\tau(g)$ are words in generators $X_{\sigma e}$ and $X_{\tau e}$, correspondingly.

Examples.

i) Let Γ be

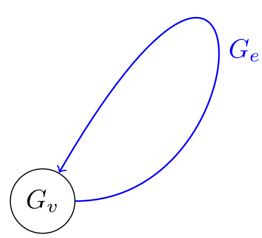


then

$$\pi(\Gamma) = G_u *_{G_e} G_v$$

free product with amalgamation.

ii) If Γ is



then

$$\pi(\Gamma) = G_v *_{G_e}$$

- *HNN extension* of G_v.

- A manifold M **fibers over the circle** if there is a submersion $\eta : M \to S^1$. Each preimage $\eta^{-1}(x)$ is a codimension-one submanifold of M called the **fiber**.
- If M is 3-dimensional and fibers over S^1, then the fiber is a surface Σ, and M is obtained as the mapping torus of a homeomorphism $\phi : \Sigma \to \Sigma$,
$$M \cong T_\phi = \frac{\Sigma \times [0,1]}{\{(x,0) \sim (\phi(x),1)\}}.$$
- M is **virtually fibered** if there exists a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers
- There are known examples of **Seifert-fibered spaces** and **graph manifolds** which are not virtually fibered
- Thurston asked whether every hyperbolic 3-manifold is virtually fibered?

9 Lecture ? will be done later

****

Theorem 7. *(Agol) If M is aspherical and $\pi_1(M)$ is RFRS, then M virtually fibers.*

The **rational derived series of a group** G is defined inductively as follows.
If $G^{(1)} = [G,G]$, then $G_r^{(1)} = \{x \in G | \exists k \neq 0, x^k \in G^{(1)}\}$.
If $G_r^{(n)}$ has been defined, define $G_r^{(n+1)} = (G_r^{(n)})_r^{(1)}$.
The rational derived series gets its name because $G_r^{(1)} = \ker\{G \to \mathbb{Q} \otimes_{\mathbb{Z}} G/G^{(1)}\} = \ker\{G \to H_1(G;\mathbb{Q})\}$.
The quotients $G/G_r^{(n)}$ are solvable.

Definition

A group G is *residually finite $\mathbb{Q}$-solvable* or *RFRS* if there is a sequence of subgroups $G = G_0 > G_1 > G_2 > \cdots$ such that $\cap_i G_i = \{1\}$, $[G : G_i] < \infty$ and $G_{i+1} \geq (G_i)_r^{(1)}$.

By induction, $G_i \geq G_r^{(i)}$, and thus G/G_i is solvable with derived series of length at most i. We remark that if G is RFRS, then any subgroup $H < G$ is as well.
Examples of RFRS groups are free groups, surface groups, RAAGs.
For a 3-manifold M with RFRS fundamental group, the condition is equivalent to there existing a **cofinal tower** of finite-index covers

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

such that M_{i+1} is obtained from M_i by taking a finite-sheeted cyclic cover dual to an embedded non-separating surface in M_i.
Equivalently, $\pi_1(M_{i+1}) = ker\{\pi_1(M_i) \to \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}\}$.
This condition implies that M has virtual infinite β_1, unless $\pi_1(M)$ is virtually abelian.

10 Quasiconvex hierarchies, Baumslag's conjecture, will be discussed later

Definition 19. *A* **hierarchy** *for a group G is a way to repeatedly build it starting with trivial groups by repeatedly takling amalgamated products $A *_C B$ and HNN extensions $A*_{C^t=D}$ whose vertex groups have shorter length hierarchies. The hierarchy is* **quasi convex** *if the amalgamated subgroup C is a finitely generated that embeds by a quasi-isometric embedding, and the hierarchy is* **malnormal** *is C is malnormal in $A *_c B$ or $A*_{C^t=D}$.*

Theorem 8. *(Wise, [W2]) Suppose G is hyperbolic and has a malnormal quasi convex hierarchy. Then G is virtually special (therefore has a finite index subgroup that .is a subgroup of a RAAG).*

A **one related group** has a presentation $< g_1, \ldots, g_n | R >$. One related groups need not be residually finite. For example, the Baumslag-Solitar group

$$< a, b | ab^2 = b^3 a >$$

is not residually finite.
Baumslag (67) conjectured that one-related groups with torsion are residually finite. A one related group has torsion iff a relator is a proper power.

Theorem 9. *(Wise, 2011) $< g_1, \ldots, g_n | R^m >$ is residually finite if $m > 1$.*

These groups have quasi convex malnormal hierarchy.

11 Relatively hyperbolic groups and Hyperbolic Dehn filling, will be discussed later

We will use the following definition of relative hyperbolicity. A f.g. group G with generating set Σ is relatively hyperbolic relative to a collection of finitely generated subgroups $\mathcal{P} = \{P_1, \ldots, P_k\}$ if the Cayley graph $Cayley(G, \Sigma \cup \Pi)$ (where Π is the set of all non-trivialelements of subgroups in $\mathcal{P}$) is a hyperbolic metric space (there exists $\delta > 0$ such that all geodesic triangles are δ-slim, and the pair $\{G, \mathcal{P}\}$ has *Bounded Coset Penetration* property (BCP property for short).

The pair $(G, \{P_1, P_2, ..., P_k\})$ satisfies the *BCP property*, if for any $\lambda 1$, there exists constant $a = a(\lambda)$ such that the following conditions hold. Let p, q be $(\lambda, 0)$-quasi-geodesics without backtracking in $Cayley(G, \Sigma \cup \Pi)$ (do not have a subpath that joins a vertex in a left coset of some P_k to a vertex in the same coset (and is not in P_k)) such that their initial points coincide ($p_- = q_-$), and for the terminal points p_+, q_+ we have $distance(p_+, q_+) \leq 1$.

1) Suppose that for some i, s is a P_i-component of p such that $distance(s_-, s_+) \geq a$; then there exists a P_i-component t of q such that t is connected to s (there exists a path c in $Cayley(G, \Sigma \cup \Pi)$ that connects some vertex of p to some vertex of q and the label of this path is a word consisting of letters from P_i).

2) Suppose that for some i, s and t are connected P_i-components of p and q respectively. Then $distance_\Sigma(s_-, t_-) \leq a$ and $distance_\Sigma(s_+, t_+) \leq a$.

A group G that is hyperbolic relative to a collection $\{P_1, \ldots, P_k\}$ of subgroups is called *toral*, if $P_1, \ldots, P_k$ are all abelian and G is torsion-free. In this section we always assume that Σ contains generators of all subgroups $P_1, \ldots, P_k$.

Theorem 10. *(Groves-Manning'08, Osin '07) Let G be a group which hyperbolic relative to the subgroup P. Then there is a finite set of elements $S \subset P - \{1\}$ so that if $P' \trianglelefteq P$ is finite-index with $S \cap P' = \emptyset$, then the quotient $G/ << P' >>$ is a hyperbolic group. Moreover $P \cap << P' >>= P'$.*

12 The Malnormal Special Quotient Theorem, will be discussed later

Definition 20. *A collection of subgroups $\{H_1, \ldots, H_m\}$ is almost malnormal if $|H_i^g \cap H_j| = \infty \to\ i = j, g \in H_i$.*

Theorem (Wise, 2011, we, probably, won't be able to see the proof in details)

> *Let G be hyperbolic, virtually special, and $\{H_1, \ldots, H_M\} < G$ a almost malnormal collection of quasi convex subgroups. Then there exists $\dot{H}_i \lhd H_i$ such that for any $H_i' < \dot{H}_i$, the quotient group $\overline{G} = G/ << H_1', \ldots, H_m' >>$ is virtually special hyperbolic.*

Remarks on the proof: Using hyperbolic Dehn filling results of Groves-Manning and Osin, one may conclude that $\overline{G}$ is hyperbolic whenever $H_i/\dot{H}_i$ has "large girth" (this is called *hyperbolic Dehn filling*, and is a descendent of small-cancellation theory). The difficult thing is showing that the quotient is cubulated. What Wise actually proves is that there is (virtually) a malnormal quasi convex hierarchy, and then applies his joint work with Haglund and Hsu to conclude that it is virtually special. This represents hundreds of pages of mathematics.

****

13 Lecture 3, Sept. 27, Proof of Theorem 5

We discussed paper [HW], proof of Theorem 1.1 =Theorem 5 in these notes. A special cube complex is called A-special in [HW] (Definition 3.2).

We discussed Section 2, in particular, we proved Lemma 2.5 from Subsection 2.1 that states the relation between a NPC cube complex and its 2-skeleton , studied Definition 2.9 (of immersion and local isometry).

We skipped Coxeter groups.

In Section 3 we discussed why the Salvetti complex of a RAAG is special (Example 3.3 (2)).

Remark 3.4, [HW] X is special iff X^2 (the 2-skeleton) is special.

Definition 21. Definition 3.14, Typing maps. *Let B be a simple square complex in which each hyperplane embeds. Let Γ_B be the simpleton graph whose vertices are hyperplanes in B, and whose edges connect distinct intersecting hyperplanes. We wish to map B to the cube complex $ART(\Gamma_B)$.*

All hyperplanes of B are two-sided, therefore there is a combinatorial map $B^1 \to ART(\Gamma_B)$ sending parallel oriented edges of some wall W to the loop in $ART(\Gamma_B)$ that is labelled by W and with the same orientation. Such a map immediately extends to a combinatorial map $\tau_A : B \to ART(\Gamma_B)$ which we call A-typing of B.

14 Lecture 4, Oct 4

We finished the proof of Theorem 5 (Theorem 1.1. from [HW]), namely, we proved [HW], Lemma 3.7, Corollary 3.9, Lemma 3.15, Lemma 4.1 and Theorem 4.2.

Defined Coxeter groups and their cube complexes COX(Γ) [HW].

15 Lectures 5, 6, Oct 11, 18

Sections 6 and 7 from [HW] where the following result is proved.

Theorem 11. *(Th. 1.3, [HW]) Let X be a compact special cube complex with $\pi_1(X)$ is word-hyperbolic, then every quasi convex subgroup is separable.*

16 Lecture 7, Oct 25, Introduction to non-positive curvature

We used [BH],[Wil] and [S].

1) Defined model spaces M^n_κ.

2) Introduced basic concepts in general metric spaces: geodesics, local geodesics, Alexandroff's angle ([S], Definition 2.1), proved that an angle between two geodesic germs defines a pseudo-metric ([S], Proposition 2.3), showed the existence of a comparison triangle in M^2_κ ([S], Lemma 2.5).

3) Introduced CAT(κ) metric spaces. Proved that a CAT(κ) space X is uniquely geodesic ([S], Proposition 3.2), and that the metric is convex in CAT(0) spaces ([Wil], Lemma 2.3, Lemma 2.4).

[Wil], Lemma 2.5: Let X be a proper (every closed ball is compact) uniquely geodesic metric space. The geodesics vary continuously in the compact-open topology with their endpoints.

[Wil] Proposition 2.6, Any Cat(0) space X is contractible.

4) Proved Alexanfroff's lemma: Suppose the triangles $\Delta_1 = \Delta(A, B, C)$ and $\Delta_2 = \Delta(A, B', C)$ satisfy the CAT (κ) condition, and $c \in [B, B']$. Then $\Delta = \Delta(A, B, B')$ satisfies the CAT(κ) condition (having angles not greater than the angles in a comparison triangle).

(This is [BH], I.2.14 or [W], Lemma 2.20 or [S], Lemma 2.6.)

5) Defined length-metrics ([Wil], Def 2.23, 2.24)

Theorem 12. *(Cartan-Hadamard) ([Wil],2.27) If X is a complete, connected length space of non-positive curvature, then the universal cover $\tilde{X}$, with the induced length metric, is CAT(0).*

We will prove the theorem under the additional hypothesis ot local compactness, foliowing Ballmann (Werner Ballmann. Singular spaces of nonpositive curvature. In Sur les groupes hyperboliques dapr‘es Mikhael Gromov (Bern, 1988), volume 83 of Progr. Math., pages 189201. Birkhauser Boston, Boston, MA, 1990) as in [Wil] notes.

Proved the theorem under the assumption that geodesics in $\tilde{X}$ are unique:

Lemma 1. *If X is proper and non-positively curved, and there is a unique geodesic between each pair of points in X, then X is CAT(0).*

Proof. By [Wil], Lemma 2.5, the hypothesis that geodesics are unique implies that they vary continuously with their endpoints. Consider a triangle $\Delta = \Delta(x, y, z)$, contained in a ball $\bar{B} = \bar{B}_x(R)$. Because $\bar{B}$ is compact, there is an

$\epsilon > 0$ such that $B_p(\epsilon)$ is $CAT(0)$ for every $p \in \bar{B}$. Let $\alpha = [y, z]$ and, for each t, let γ_t be the geodesic from x to $\alpha(t)$. Because geodesics vary continuously with their endpoints, there is a δ such that $d(\alpha_{t_1}(s), \alpha_{t_2}(s)) < \epsilon$, for all s, whenever $|t_1 - t_2| < \delta$. To prove the lemma, we now divide Δ into a patchwork of geodesic triangles, each contained in a ball of radius ϵ. By Alexandrovs Lemma, it follows by induction that Δ satisfies the CAT(0) condition. □

17 Lecture 8, Nov 1, Gromov's Link Condition

We prove [Wil], Theorem 2.30. and then consider Section 3 of [B]. An English translation of Section 3 of [B] is, basically, Section 2.5 in [Wil]. In particular, we prove

Theorem 13. *(Gromov's Linl Condition). Any Euclidean complex X is NPC iff $Lk(x_0)$ is CAT(1) for every $x_0 \in X$.*

Bibliography

[C] Danny Calegari, Notes on Agol's virtual Haken theorem,

[A] I. Agol (with an appendix by D. Groves and J. Manning), The virtual Haken conjecture, preprint,

[B] Werner Ballmann. Singular spaces of nonpositive curvature. In Sur les groupes hyperboliques dapr'es Mikhael Gromov (Bern, 1988), volume 83 of Progr. Math., pages 189201. Birkhauser Boston, Boston, MA, 1990

[H] A. Hatcher, Notes on basic 3-manifold topology,

[W1] D. Wise, The structure of groups with a quasiconvex hierarhy,

[W2] D. Wise, From Riches to RAAGS: 3-Manifolds, right-angled Artin groups, and cubical geometry,

[HW] F. Haglund and D. Wise, Special Cube Complexes, GAFA, vol 17, 2008, 1551-1620.

[Wil] H. Wilton, Non-positively curved cube complexes.

[BH] Martin R. Bridson and Andr Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[S] T. Susse, Introduction to non-positive curvature, notes.