Free groups, Lecture 1, part 2

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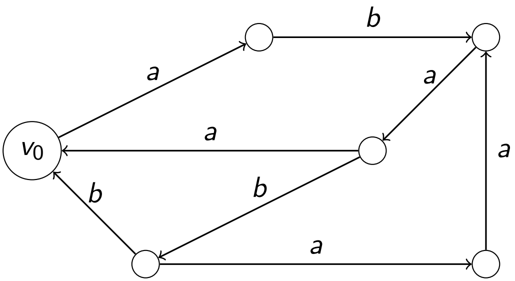
Theorem

Every subgroup $H \leq F$ of a free group F is free, and given generators of H we can compute its basis.

Before proving this theorem we need to develop some machinery that will allow us to associate to any subgroup of a free group an automaton, i.e. a finite oriented labeled graph, that *accepts* only elements of H.

Such an automaton is called $\Gamma(H)$ – the Stallings graph of H. For example, $\Gamma(H)$ for $H = \langle aba^2, a^{-1}b^2, aba^{-2}b \rangle$:



Remark

Traversing an edge $\xrightarrow{a}$ forward, i.e. along its direction, we read a, and traversing it backward, we read a^{-1}.

Definition

One way reading property (OR) – no two edges outgoing from a vertex are labeled by the same symbol.

Definition

A path in $\Gamma(H)$ is a sequence $e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n}$, where e_i are edges and $\epsilon_i = \pm 1$. We say that a path is reduced if it contains no subpaths $e_i e_i^{-1}$ or $e_i^{-1} e_i$.

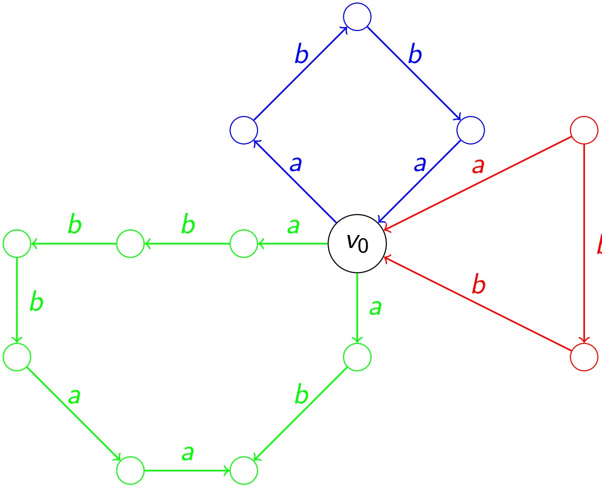
The subgroup H corresponds to labels of loops beginning at v_0 in $\Gamma(H)$:

Fact

Every element of H is a loop from v_0 in $\Gamma(H)$.

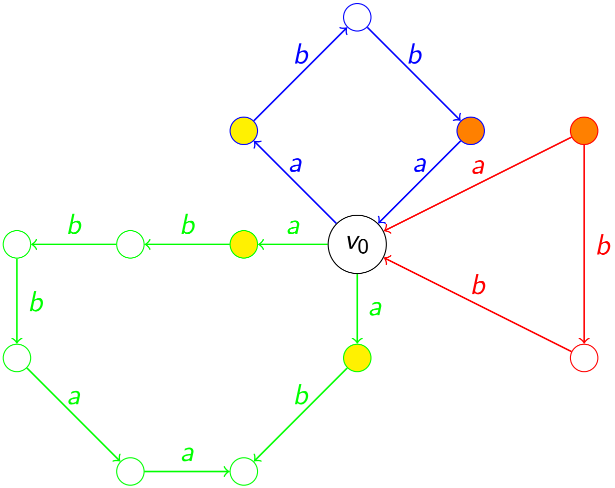
This gives an easy procedure to decide whether $g \in H$ of not. However, how does one contruct $\Gamma(H)$, given $H \leq F$?

For example, suppose $H = \langle ab^2a, a^{-1}b^2, aba^{-2}b^{-3}a^{-1} \rangle$.
Step 1 Initial graph consists of subdivided circles around a common distinguished vertex v_0, each labeled by one of the generators: note that it does not have the OR property!

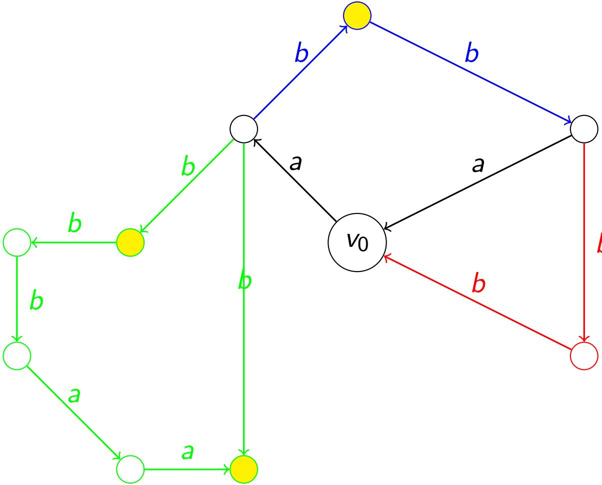


Then, we iteratively identify all edges from the same vertex that have the same label

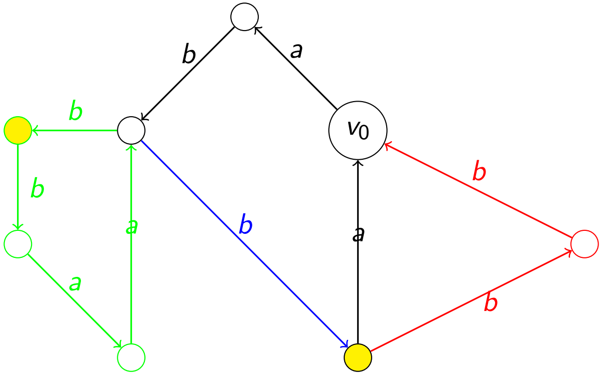
Step 2 Graph before folding. Nodes that are going to be identified are colored.



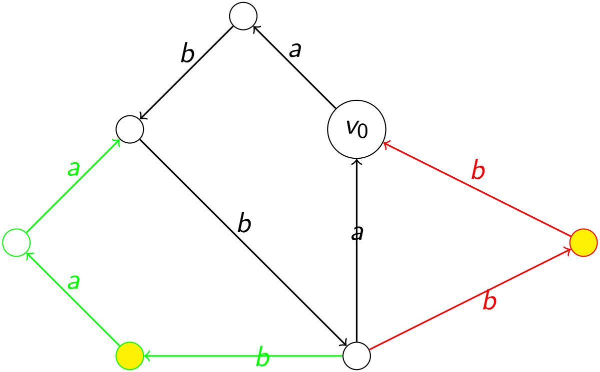
Step 3 Graph after identifying edges labeled by a and a^{-1} coming out of v_0. Nodes to be indetified next are colored again.



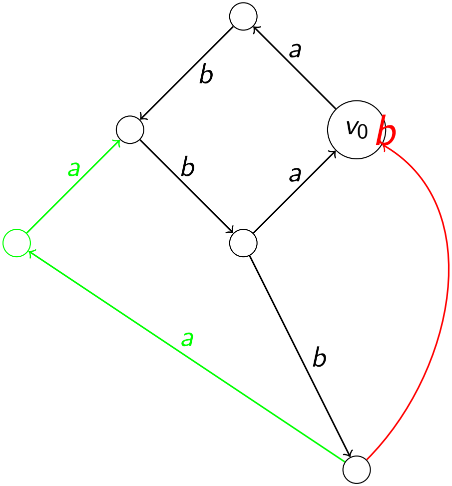
Step 4



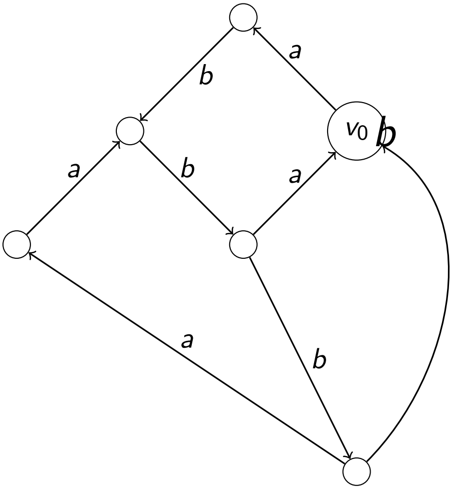
Step 5



Step 6



Step 7 Final result $\Gamma(H)$, the graph has the OR property.



Formal definition of $\Gamma(H)$

Definition

An oriented graph Γ consists of a set of vertices $V(\Gamma)$, a set of edges $E(\Gamma)$, and two functions

$$
\begin{aligned}
E \to V \times V \qquad & e \mapsto (\alpha(e), \tau(e)) && \textit{(endponts of } e\textit{)} \\
E \to E \qquad & e \mapsto \bar{e} := (\tau(e), \alpha(e)) && \textit{(inverse of } e\textit{)}
\end{aligned}
$$

Definition

1. $u, v \in V$ *are adjacent if* $\exists\, (u, v) \in E$.
2. *A path in* Γ *is a sequence of edges* $e_1 e_2 \ldots e_n$ *s.t.* $\tau(e_i) = \alpha(e_{i+1})$ *for* $i = 1, \ldots, n-1$.
3. *A path is simple if all* $\alpha(e_i)$ *are distinct.*
4. *A path is a loop if* $\alpha(e_1) = \tau(e_n)$.
5. *A path is reduced if* $\bar{e}_i \neq e_{i+1},\ \forall i$.
6. *A graph is connected if there is a path between any two vertices.*
7. *A graph is a forest if it does not have simple loops.*
8. *A graph is a tree if it is a connected forest.*

Remark

In a tree T, $\forall u, v \in V(T)$, $\exists!$ reduced path joining u and v.

Proof.

Sketch. If p_1, p_2 are two different paths, $p_1 \bar{p_2}$ is a loop. Removing pairs $e\bar{e}$ from it makes it reduced. Contradiction.

Definition

This unique path is called a geodesic between u and v.

By Zorn's Lemma, every connected graph Γ has a maximal subtree T^*. If Γ is finite, there is an algorithm to contruct T^*:

1. Take an edge.
2. Add edges without forming a simple loop.
3. Stop when no more edges can be added.

Labeled graphs with orientation

Definition

Let $G = (V, E)$ be a graph.

1. *An orientation is $E_+ \subset E$ such that*

$$E_+ \cap \bar{E_+} = \emptyset \qquad E_+ \cup \bar{E_+} = E, \text{if } e = (u, v), \text{then } \bar{e} = (v, u),$$

2. *Given an orientation E_+, and an alphabet $S = \{a, b, \ldots\}$. We set $S^{-1} := \{a^{-1}, b^{-1}, \ldots\}$. A labeling is a function*

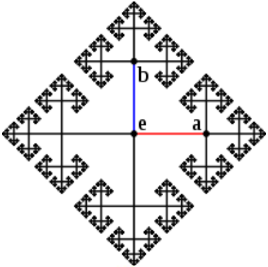
$$\lambda : E_+ \to S$$

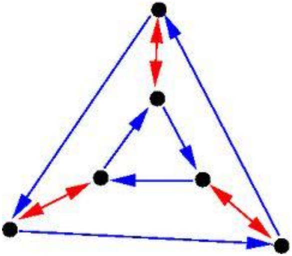
such that $\forall e \in E_+,\ (\lambda(e) = x) \Rightarrow \left(\lambda(\bar{e}) = x^{-1}\right)$.

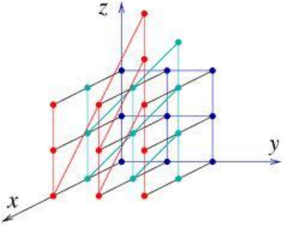
Definition

A Cayley graph of a group $G = \langle S \mid R \rangle$ is the labeled graph $\mathrm{Cayley}(G) = (V, E, \lambda)$, where $V = G$ and

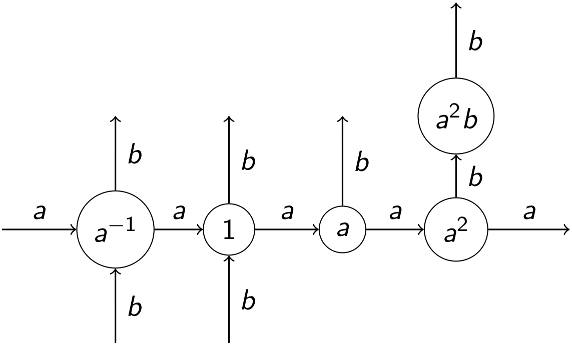
$$(\forall s \in S)\,(\forall g \in G)\,(\exists e \in E_+)\,(e = (g, gs), \lambda(e) = s)$$


b
e
a




z
y
x

For example, let $F = F(a, b)$ be a free group on two elements. Then, Cayley(F) is



Remark

Cayley(G), where $G = \langle S \mid R \rangle$ is a tree iff G is a free group with basis S.

Proof.

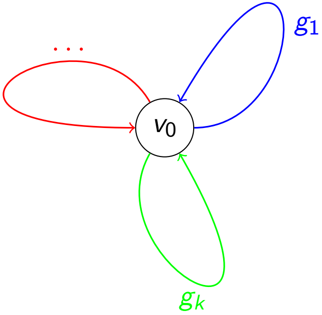
If Cayley(G) is a tree, then every word in $S^{\pm}$ corresponds to a unique vertex in the graph, and this vertex is different from 1. Hence, this word$\neq 1$.

Conversely, if Cayley(G) is not a tree, $\exists$ a simple loop from some $g \in G$:

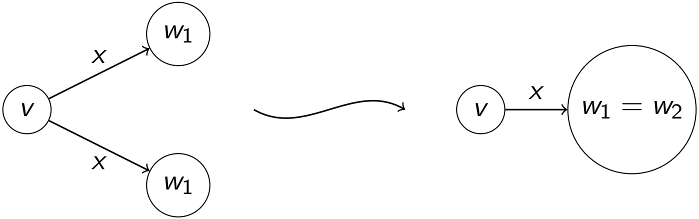
$$e_1 e_2 \dots e_n \qquad \tau(e_n) = \alpha(e_1) = g$$

Thus, $g = g\lambda(e_1)\cdots\lambda(e_n) \Rightarrow \lambda(e_1)\cdots\lambda(e_n) = 1.$ That is there is a reduced word in G that is equal to the identity, hence, G is not free. □

Let $H \leq F = F(S)$ be defined as $H = \langle g_1, \ldots, g_k \rangle$. Construct $\gamma(H)$ by drawing loops corresponding to the generators starting from v_0:



Then we fold (do reductions):



As the generating set of H is finite, this process must stop, so we will end up with $\Gamma(H)$ – the Stalling's graph of H with the OR property.
In $\gamma(H)$, $g \in H$ iff g is a label of a loop at v_0 in $\gamma(H)$. This property is preserved during foldings, thus it is also true for $\Gamma(H)$. This proves the fact above.

Schreir's graph

The graph of right cosets of H, denoted by $\Gamma_0(H)$, is called the Scheir's graph of H.

$$V(\Gamma_0) = G/H = \{Hg \mid g \in \text{set of right coset representatives}\}$$

$$\forall Hg, \forall s \in S, \exists e \in E_+ \qquad e = (Hg, Hgs) \qquad \lambda(e) = s$$

Theorem

If $F = F(S)$, $H \leq F$, then $\Gamma(H)$ is the minimal subgraph of $\Gamma_0(H)$, containing all loops at $v_0 = H$.

Proof is left as an exercise.

Example $\Gamma(H)$ for $H = \langle aba^2, a^{-1}b^2, aba^{-2}b \rangle$.