Graphs of groups, Lecture 6

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Let $\Gamma = (G, X)$ be a graph of groups and $T \subseteq X$ be a maximal subtree. Then the fundamental group $\pi(\Gamma, T)$ was defined by generators and relations as follows:

$$Generators: \bigcup_{\nu \in V(X)} gen(G_\nu) \cup \{e \mid e \in E(X)\}$$

$$Rel: \bigcup_{\nu \in V(X)} rel(G_\nu) \cup \left\{e^{-1}\sigma(g)e = \tau(g), e\bar{e} = 1 \mid e \in E(X),\, g \in G_e\right\} \cup$$

$$\cup \{e = 1 \mid e \in T\}$$

Now we will construct $\pi(\Gamma, T)$ in a different way, which does not depend on a choice of the tree T.
Let

$$G = *_{\nu \in V(X)} G_\nu * F(E(X)) = \langle \cup_{v \in V(X)} gen(G_\nu) \cup E(X) \mid \cup_{v \in V(X)} rel(G_\nu) \rangle$$

Put (here by $ncl_G(X)$ we denote the normal subgroup generated by X):

$$N = ncl_G \langle e^{-1}\sigma(g)e = \tau(g),\ e\bar{e} \ \mid \ e \in E(X),\ g \in G_e \rangle$$

and denote $F(\Gamma) = G/N$. If we set

$$M_T = ncl_{F(\Gamma)}(eN \ \mid \ e \in T),$$

then

$$F(\Gamma)/M_T \simeq \pi(\Gamma, T).$$

So it suffices to show that $F(\Gamma)/M_T$ does not depend on T.

Fix a point $\nu_0 \in V(X)$ and define a subgroup $H \leq G$ which consists of all elements of the free product G that can be presented in the form

$$g_0 e_1 g_1 e_2 g_2 \dots e_n g_n$$

where $g_0 \in G_{\nu_0}, g_i \in G_{\tau e_i}, e_i \in E(X)$ and $e_1 \dots e_n$ is a closed path in X at ν_0.

HW Problem 10a. Check that H is a subgroup of G.

For a vertex $\nu \in V(X)$ denote by p_ν the unique path in T from ν_0 to ν. Let $\phi : G \longrightarrow H$ be the homomorphism which is defined on free factors of G as follows:

$$\begin{cases} \phi(g_\nu) = p_\nu \, g_\nu \, p_\nu^{-1}, \\ \phi(e) = p_{\sigma e} \, e \, p_{\tau e}^{-1} \end{cases}$$

HW Problem 10b. Check that ϕ is a *homomorphism*.

Hint. It suffices to show that $\phi(Rel(G_\nu)) = 1$.

HW Problem 10c. Prove that ϕ is a *retract* on H, i.e. $\phi(G) = H$ and $\phi|_H = id_H$.

Denote by $\lambda : G \to F(\Gamma)$ the canonical epimorphism and put

$$\pi(\Gamma, \nu_0) = \lambda(H) \leq F(\Gamma).$$

Clearly, $\pi(\Gamma, \nu_0)$ does *not* depend on T.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\lambda"'] \arrow[dr, dashed, "\varphi_1"] & H \arrow[d, "\lambda|_H"] \\
G/N = F(\Gamma) \arrow[r, dashed, "\varphi_2"] & \pi(\Gamma, \nu_0)
\end{tikzcd}$$

Figure: Diagram 1

The composition $\phi_1 = \lambda|_H \circ \phi$ gives a homomorphism

Claim 1. $\phi_1(N) = 1$.
Indeed,

$$\phi_1(e\bar{e}) = (p_{\sigma e} e p_{\tau e}^{-1})(p_{\tau e} \bar{e} p_{\sigma e}^{-1}) = p_{\sigma e} e \bar{e} p_{\sigma e}^{-1} = p_{\sigma e} p_{\sigma e}^{-1} = 1.$$

Similarly, $\phi_1\big(e^{-1}\sigma(g)e\tau(g^{-1})\big) = 1$.
Therefore ϕ_1 induces a homomorphism

$$\phi_2 : F(\Gamma) \longrightarrow \pi(\Gamma, \nu_0).$$

Claim 2. $\phi_2(M_T) = 1$.

Indeed, $\phi_2(eN) = \lambda(\phi(e)) = \lambda(p_{\sigma e} e p_{\tau e}^{-1})$. If $e \in T$ then $p_{\sigma e} e p_{\tau e}^{-1} = 1$ hence $\phi_2(eN) = 1$.

Therefore ϕ_2 induces a homomorphism

$$\phi_3 : F(\Gamma)/M_T \longrightarrow \pi(\Gamma, \nu_0)$$

and we have the following commutative diagram where

$$\mu : F(\Gamma) \to F(\Gamma)/M_T$$

is the canonical epimorphism

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\lambda"'] \arrow[dr, dashed, "\varphi_1"] & H \arrow[d, "\lambda|_H"] \\
F(\Gamma) \arrow[r, dashed, "\varphi_2"] \arrow[d, "\mu"'] & \pi(\Gamma, \nu_0) \\
\pi(\Gamma, T) = F(\Gamma)/M_T \arrow[ur, dashed, "\varphi_3"] &
\end{tikzcd}$$

On the other hand the restriction $\bar{\mu}$ of μ on $\pi(\Gamma, \nu_0)$ gives a homomorphism

$$\bar{\mu} : \pi(\Gamma, \nu_0) \longrightarrow \pi(\Gamma, T).$$

We claim that

$$\bar{\mu} \circ \phi_3 = \phi_3 \circ \bar{\mu} = id,$$

i.e. $\bar{\mu}$ and ϕ_3 are *isomorphisms*. Indeed, since $p_\nu = e_1 \dots e_n$, $e_i \in T$, and $\mu(\lambda(e_i)) = 1$ we have the following chain of equalities

$$\phi_3(g_\nu M_T) = \phi_2(g_\nu N) = \lambda(\phi(g_\nu)) = \lambda(p_\nu g_\nu p_\nu^{-1});$$

$$\mu(\lambda(p_\nu g_\nu p_\nu^{-1})) = \mu(\lambda(g_\nu)).$$

We have

$$\phi_3(eM_T) = \phi_2(eN) = \phi(e) = p_{\sigma e} e p_{\tau e}^{-1}.$$

Therefore

$$\bar{\mu}(\phi_3(eM_T)) = \mu(\lambda(p_{\sigma e} e p_{\tau e}^{-1})) = eM_T.$$

$\square$

Graph of spaces

Let G be a fundamental group of a graph of groups with underlying graph Y. For each edge or vertex a of Y choose X_a to be a connected CW-complex with fundamental group G_a. We can do this so that there is an embedding $X_e \hookrightarrow X_v$ realizing the inclusion of groups. We form the topological space $\Gamma(\mathcal{X}, Y)$ by beginning with the disjoint union

$$\coprod_{v \in V} X_v \coprod \coprod_{e \in E} X_e \times I,$$

and (a) identifying $X_e \times I$ with $X_{\bar{e}} \times I$ via $(x, t) \equiv (x, 1 - t)$ and (b) gluing $X_e \times \{0\}$ to $X_{\sigma(e)}$ via the given inclusion. The resulting topological space has fundamental group G.

Lemma Let $p : X \longrightarrow Y$ be a locally surjective morphism of graphs and T be a subtree of Y. Then there exists a morphism of graphs $j : T \longrightarrow X$ such that $p \circ j = id_T$. In particular, j is injective.

Proof. Consider all the pairs (T', j') with $T' \subseteq T$, $j' : T' \longrightarrow X$, $p \circ j' = id_{T'}$. By Zorn's lemma there exists a maximal pair (T_0, j_0) with respect to inclusion. We claim that $T_0 = T$. Suppose not then there exists an edge $e \in T$ with $\sigma e \in T_0$, $\tau e \notin T_0$. Since p is locally surjective there exists $e' \in V(X)$ with $pe' = e$, $\sigma e' = j\sigma e$. Put $T' = T_0 \cup \{e\}$, $j' = j \cup \{(e, e')\}$. Then $j' : T' \longrightarrow X$ is a graph morphism and $p \circ j' = id_{T'}$. But $T' \supsetneq T_0$ - contradiction. $\square$

Suppose now that G acts on X without inversions, $p : X \longrightarrow X/G$ is the canonical projection, and $T \subseteq X/G$ is a maximal subtree of X/G. By the lemma above there exists a morphism $j : T \longrightarrow X$ with $p \circ j = id_T$. In particular, jT is a tree, and $j : T \longrightarrow jT$ is an isomorphisms of graphs.

We call jT a *representative* tree of the action of G on X.

Lemma Let G acts on X without inversions and $p : X \longrightarrow X/G$ be the canonical projection. If Y_+^1 is an orientation of $Y = X/G$ then $X_+^1 = p^{-1} Y_+^1$ is an orientation of X.

Theorem Let G be a group which acts freely and without inversions on a tree X. Then G is free.

Proof. Let T be a representative tree for the action of G on X.Then

$$gT \cap hT = \emptyset, \quad \mathrm{g} \neq \mathrm{h}.$$

Indeed, if for some vertices $v, w \in T$ we have $gv = hw$ then $p(v) = p(w)$ contradicting the fact that $p|_T$ is injective, unless $v = w$. But $gv = hv$ implies $h^{-1}gv = v$ and hence $h^{-1}g = 1$, i.e. $h = g$. Contradiction.

Observe that every vertex of X belongs to gT for some $g \in G$. It follows that X consists of subgraphs gT, $g \in G$, and some other edges. Since X is a tree there exists at most one edge connecting gT and hT for $g \neq h$. Denote by Y the graph which is obtained from X by contracting every subtree gT, $g \in G$, to a single vertex which we denote by gT. Clearly, Y is a tree.

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Put

$$S = \{g \in G \,|\, g \neq 1 \;\exists\, e \in X^1_+ \text{ such that } \sigma e \in T \text{ and } \tau e \in gT\}.$$

We label edges from Y by elements from $S^{\pm 1} = S \cup S^{-1}$ as follows. Let $e \in X^1_+$, e connects gT to hT. Then the edge $g^{-1}e$ connects T to $g^{-1}hT$, and $g^{-1}e \in X^1_+$. Hence $g^{-1}h \in S$. We label the edge e by $g^{-1}h$, and the edge $\bar{e}$ by $h^{-1}g$.

Claim. The labeled graph Y is isomorphic to the Cayley graph $\Gamma(G, S^{\pm 1})$.

We define a map $\phi : \Gamma(G, S^{\pm 1}) \longrightarrow Y$ in the folowing way:
a) $\phi(g) = gT$, $g \in G$.
b) if (g, s) is an edge in $\Gamma(G, S^{\pm 1})$ such that $s \in S$ then there exists an edge e' in Y connecting T to sT. Define $\phi : (g, s) \longrightarrow ge'$. Also, if

$$(g, s) \in E(\Gamma(G, S^{\pm 1}))$$

and $s \notin S$, then $s^{-1} \in S$ and we define

$$\phi : (g, s) \longrightarrow \overline{\phi(gs, s^{-1})}.$$

It is not hard to check that ϕ is a morphism of graphs. Indeed, if $e \in E(\Gamma(G, S^{\pm 1}))$ and $e = (g, s)$ then $\sigma e = g$, $\tau e = gs$. Now $\phi(e) = ge'$, where $\sigma e' = T$, $\tau e' = sT$. Hence

$$\sigma\phi(e) = \sigma(ge') = g\sigma e' = gT = \phi(\sigma e),$$

$$\tau\phi(e) = \tau(ge') = g\tau e' = gsT = \phi(\tau e).$$

It is left to verify that ϕ is bijective on edges.
Problem. Verify that ϕ is bijective on edges. $\square$

The Structure theorems

Let G be a group acting on a connected graph X, $Y = X/G$, $p : X \longrightarrow X/G$ be the canonical projection, $T \subseteq Y$ be a maximal subtree of Y, and $j : T \longrightarrow X$ be a morphism of graphs with $p \circ j = id_T$.

We want to make Y into a *graph of groups* by assigning to its vertices and edges certain groups which are stabilizers of some vertices and edges under the action of G on X. To this end we want to extend $j : T \longrightarrow X$ to a map (not a graph morphism, in general) $j : Y \longrightarrow X$ such that $p \circ j = id_Y$.

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Since $V(T) = V(Y)$ we need to define j on edges from $Y - T$. We will define j on $Y_+^1 - T$ and put $j(e) = \overline{j(\bar{e})}$ for $e \notin A$. Let $e \in Y_+^1 - T$. Since p is locally surjective there exists an edge $e' \in E(X)$ with $p\sigma e' = \sigma e$ and $pe' = e$.
Then we define

$$j(e) = e' \text{ and } j(\bar{e}) = \bar{e}'.$$

Clearly, $p \circ j = id_Y$. Observe that $p(\tau je) = \tau e = p(j\tau e)$ hence $G\tau je = Gj\tau e$ so $\exists\, \gamma_e \in G$ such that $\gamma_e j\tau e = \tau je$. For uniformity, if $e \in Y_+^1 \cap T$ then we define $\gamma_e = 1$, and for arbitrary e we put $\gamma_{\bar{e}} = \gamma_e^{-1}$.

Recall that for $x \in X$ $Stab_G(x) = \{g \in G \mid gx = x\}$. Now we define a graph of groups on Y as follows:

$$y \in Y \implies G_y = Stab_G(jy).$$

It is left to define monomorphisms

$$\sigma : G_e \longrightarrow G_{\sigma e}, \;\; \tau : G_e \longrightarrow G_{\tau e}.$$

It suffices to do it for each $e \in Y^1_+$. Let $e \in Y^1_+$, then $\sigma je = j\sigma e$. Hence $Stab_G(je) \leq Stab_G(j\sigma e)$, so $G_e \leq G_{\sigma e}$, and in this case σ is the inclusion map. Also

$$Stab_G(je) \leq Stab_G(\tau je) = Stab_G(\gamma_e j\tau e) = \gamma_e Stab_G(j\tau e)\gamma_e^{-1}.$$

We define

$$\tau : G_e \longrightarrow G_{\tau e} \text{ as } g \longrightarrow \gamma_e g \gamma_e^{-1}.$$

This defines a graph of groups $\Gamma = (G, Y)$.

It is easy to check that maps $G_v \longrightarrow G_v$, $t_e \longrightarrow \gamma_e$ extend to a homomorphism ϕ

$$\phi : \pi(\Gamma) \longrightarrow G.$$

HW Problem. Check that ϕ is a homomorphism.

Now we would like to reconstruct (as much as possible) the original graph X and the action of G on X from the quotient graph of groups $\Gamma = \Gamma(G, X/G)$. We can do this when X is a tree.

Theorem Let $G = \pi(Y, T)$. Then there exists a tree X, on which G acts without inversion on edges such that the factor graph $G \backslash X$ is isomorphic to Y and stabilizers of the vertices and edges in X are conjugate to the canonical images in G of the groups G_v and $\sigma(G_e)$.

Moreover for the projection $p : X \to Y$ there exists a lift $(\tilde{T}, \tilde{Y})$ of the pair (T, Y) such that p maps $\tilde{T}$ isomorphically onlo T and $\tilde{Y}^1 - \tilde{T}^1$ bijectively onto $Y^1 - T^1$ and each edge from $\tilde{Y}^1 - \tilde{T}^1$ has the initial or terminal vertex in $\tilde{T}$.

Proof **Let** $X^0 = \bigcup_{v \in Y_0} G/G_v$ (union of left cosets) and $X^1_+ = \bigcup_{e \in Y^1_+} G/G_e$**.** Put

$\sigma(gG_e) = gG_{\sigma(e)}, \tau(gG_e) = gt_e G_{\tau(e)}, g \in G, e \in Y^1_+$

and let $\tilde{T}$ be the lift of the tree T in X with $\tilde{T}^0 = \bigcup_{v \in T^0} \{G_v\}$, $\tilde{T}^1_+ = \bigcup_{e \in T^1_+} \{G_e\}$. G acts on X by left multiplication.

Let $\Gamma = \Gamma(\mathcal{X}, Y)$ be the graph of spaces. Take a closed subset $Z = \coprod_e X_e \times 1/2 \subset \Gamma$ which has a collar neighborhood in Γ. Let $\tilde{\Gamma}$ be the universal covering of Γ and $\tilde{Z} \subset \tilde{\Gamma}$ be the preimage of Z. We define the dual tree X to $\tilde{Z} \subset \tilde{\Gamma}$. Its vertices are the components of $\tilde{\Gamma} - \tilde{Z}$. Its unoriented edges are the components of $\tilde{Z}$. The vertices of an edge given by a component $\tilde{Z}_0$ of $\tilde{Z}$ are the two components of $\tilde{\Gamma} - \tilde{Z}$ which have $\tilde{Z}_0$ in their closure. The fact that $\tilde{\Gamma}$ is simply connected (as the universal cover) implies that X is contractible and hence is a tree.
The natural action of G on $\tilde{\Gamma}$ leaves $\tilde{Z}$ invariant, therefore induces action on X.

Observation Given a universal covering space $p : \tilde{X} \to X$ and a connected, locally path-connected subspace $A \subset X$ such that the inclusion is injective on π_1, then each component $\tilde{A}$ of $p^{-1}(A)$ is a universal cover of A .

To see this, note that $p : \tilde{A} \to A$ is a covering space, so we have injective maps $\pi_1(\tilde{A}) \to \pi_1(A) \to \pi_1(X)$ whose composition factors through $\pi_1(\tilde{X}) = 1$, hence $\pi_1(\tilde{A}) = 1$

Lemma Let $\Gamma = \Gamma(\mathcal{X}, Y)$ a graph of spaces. $\tilde{\Gamma}$ is itself a graph of spaces. Also, each vertex space (resp. edge space) is the universal cover of a vertex space of Γ (resp edge space).

Sketch of Proof For any $v \in Y$, let $L_v = X_v \coprod (\coprod_{e \in E} X_e \times I/2)$, where E is the set of edges that are the edges incident to v. It should be noted that X_v is a deformation retract of L_v. Also, recall that the edge maps are π_1-injective. It follows that $\tilde{X}_v \hookrightarrow \tilde{L}_v$. So $\tilde{L}_v$ is built from $\tilde{X}_v$ by attaching covering spaces of edge spaces. Because σ, τ are π_1-injective, these covering spaces of edge spaces really are universal covers. By iteratively gluing together copies of the $\tilde{L}_v$ we can construct a simply connected cover of Γ.

Let Γ be a graph of groups and let $G = \pi_1(\Gamma)$. Fix a base point $u \in X_v$ and a choice of lift $\tilde{u} \in \tilde{X}$. Let $g \in G_v$. The space $\tilde{X}_v$ is a universal cover of X_v, and so the lift of g to the universal cover $\tilde{\Gamma}$ at $\tilde{u}$ is contained in $\tilde{X}_v$. Therefore the pre-images of u that are contained in $\tilde{X}_v$ correspond to the elements of G_v.

Now consider $g \in G \setminus G_v$. If g is lifted at $\tilde{u}$, then the terminus of this lift is not in $\tilde{X}_v$, but in some other component of the preimage of X_v. Call the component where the lift terminates $\tilde{X}_{\tilde{v}_1}$. If g, h are such that both have lifts that terminate in $\tilde{X}_{\tilde{v}_1}$ then $h^{-1}g \in G_v$. We have just proved the following lemma.

Lemma Let Γ be a graph of groups and let X be the underlying graph of the universal cover $\tilde{\Gamma}$. For any vertex $v \in Y$ the set of vertices of X lying above v is in bijection with G/G_v and G acts by left translation.

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