Some problems on moment Generating functions and the Central Limit Theorem

1) Compute the moment generating function of a binomial random variable with parameters $n$ and $p$. Use this result to find the mean, variance, and the third moment. If $X_i$ is binomial with parameters $n_i$ and $p_i = p$ for $i = 1, 2, \ldots, n$, and the $X_i$ are independent, use moment generating functions to show that $\sum_{i=1}^{n} X_i$ is binomial.

2) Use the mgf to show that if $X$ is exponential, so is $cX$.

3) Find the moment generating function of a geometric random variable and use it to compute the mean and variance.

4) Assuming $X \sim N(0, \sigma^2)$, use the mgf to show that the odd moments are zero and the even moments are given by

$$\mu_{2n} = E(X^{2n}) = \frac{(2n)! \sigma^{2n}}{2^n n!}.$$ 

(HINT: use the Taylor series expansion.)

5) Using moment-generating functions, show that as $n \to \infty$, $p \to 0$, and $np \to \lambda$, the binomial distribution with parameters $n$ and $p$ tends to the Poisson distribution.

6) Using moment-generating functions, show that as $\alpha \to \infty$, the gamma distribution $\Gamma(\lambda, \alpha)$, properly standardized, tends to the standard normal distribution.

7) A soft-drink vending machine is set so that the amount of drink dispensed is a random variable with a mean of 200 milliliters, and a standard deviation
of 15 milliliters. What is the probability that the average amount dispensed in a random sample of size 36 is at least 204 milliliters? (Hint: use the central limit theorem.)

8) Consider the position of a particle following a random walk: each minute the particle moves north or south by 50 cm, with probability $p = 1/2$. Use the central limit theorem to estimate the probability that the position of the particle will be within 400 cm of the start after 1 hour.