Some problems on moment Generating functions and the Central Limit Theorem

1) Compute the moment generating function of a binomial random variable with parameters $n$ and $p$. Use this result to find the mean, variance, and the third moment. If $X_i$ is binomial with parameters $n_i$ and $p_i = p$ for $i = 1, 2, \ldots, n$, and the $X_i$ are independent, use moment generating functions to show that $\sum_{i=1}^{n} X_i$ is binomial.

2) Use the mgf to show that if $X$ is exponential, so is $cX$.

3) Find the moment generating function of a geometric random variable and use it to compute the mean and variance.

4) Assuming $X \sim N(0, \sigma^2)$, use the mgf to show that the odd moments are zero and the even moments are given by

$$\mu_{2n} = E(X^{2n}) = \frac{(2n)! \sigma^{2n}}{2^n(n!)}. $$

(HINT: use the Taylor series expansion.)

Solution: The moment generating function of a normal variable with mean 0 and standard deviation $\sigma$ is $M(t) = e^{\sigma^2 t^2 / 2}$. Using the Taylor series expansion for $e^u$ we get that the Taylor series expansion for $M(t)$ is

$$M(t) = 1 + \frac{\sigma^2 t^2}{2} + \left(\frac{\sigma^2 t^2}{2}\right)^2 \frac{1}{2!} + \left(\frac{\sigma^2 t^2}{2}\right)^3 \frac{1}{3!} + \cdots = 1 + \frac{\sigma^2 t^2}{2} + \frac{\sigma^4 t^4}{2^2 \cdot 2!} + \frac{\sigma^6 t^6}{2^3 \cdot 3!} + \cdots $$

from which it follows that all the even power terms are zero.

Since the $n$th term of the Taylor series of the function $M(t)$ is $\frac{M^{(n)}(0) t^n}{n!}$, we get

$$\frac{\sigma^{2n} t^{2n}}{2^n \cdot n!} = \frac{M^{(2n)}(0) t^{2n}}{(2n)!}$$
from which it follows that

\[ E(X^{2n}) = M^{(2n)}(0) = \frac{(2n)!\sigma^{2n}}{2^n(n!)}. \]

5) Using moment-generating functions, show that as \( n \to \infty, p \to 0 \), and \( np \to \lambda \), the binomial distribution with parameters \( n \) and \( p \) tends to the Poisson distribution.

6) Using moment-generating functions, show that as \( \alpha \to \infty \), the gamma distribution \( \Gamma(\lambda, \alpha) \), properly standardized, tends to the standard normal distribution.

7) A soft-drink vending machine is set so that the amount of drink dispensed is a random variable with a mean of 200 milliliters, and a standard deviation of 15 milliliters. What is the probability that the average amount dispensed in a random sample of size 36 is at least 204 milliliters? (Hint: use the central limit theorem.)

8) Consider the position of a particle following a random walk: each minute the particle moves north or south by 50 cm, with probability \( p = 1/2 \). Use the central limit theorem to estimate the probability that the position of the particle will be within 400 cm of the start after 1 hour.

Solution:

Let \( X_n \) be the Bernoulli variable which is +1 if the particle moves north at the \( n \)th step, and −1 if it moves south. Then the position after \( n \) steps is \( S_n = 50(X_1 + X_2 + \cdots + X_n) \). By the Central Limit Theorem, if \( n \) is large then \( S_n \) is approximately normal.

The mean of \( X_i \) is zero, and the variance of \( X_i \) is 1 for every \( i \), so we get \( E(S_n) = 0 \) and \( \text{var}(S_n) = 2500n \). So if \( n = 60 \) we have \( E(S_{60}) = 0 \) and \( \text{var}(S_{60}) = (2500)(60) \), or \( \text{s.d}(S_{60}) = 100\sqrt{15} \). Using the approximation to a
standard normal \( \frac{S_{60}}{100\sqrt{15}} \sim Z \) we have

\[
P(-400 \leq S_{60} \leq 400) = P\left(\frac{-400}{100\sqrt{15}} \leq Z \leq \frac{400}{100\sqrt{15}}\right) = \\
P(-1.033 \leq Z \leq 1.033) = .6984
\]