1.3.1 We have $A \cap B \subset B \Rightarrow \mathbb{P}(A \cap B) \leq \mathbb{P}(B) = \frac{1}{3}$.

We also have from the inclusion-exclusion principle that

\[
\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \\
= \frac{13}{12} - \mathbb{P}(A \cup B) \\
\geq \frac{13}{12} - 1 = \frac{1}{12}
\]

since $\mathbb{P}(A \cup B) \leq 1$.

For examples of attaining each bound, let $\Omega = \{1, 2, \ldots, 12\}$ let $\mathcal{F} = \mathbb{P}(\Omega)$, and let $\mathbb{P}(\omega) = \frac{1}{12}$ for each $\omega \in \Omega$. Further, let $A = \{\omega|\omega \leq 9\}$.

When $B = \{\omega|\omega \geq 9\}$, $\mathbb{P}(A \cap B) = \frac{1}{12}$ and for $B = \{\omega|\omega \leq 4\}$, $\mathbb{P}(A \cap B) = \frac{1}{3}$. In each case, $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$.

For comparable bounds on $\mathbb{P}(A \cup B)$, we have $A \subset A \cup B$ so $\frac{3}{4} = \mathbb{P}(A) \leq \mathbb{P}(A \cup B)$.

Also,

\[
\mathbb{P}(A \cup B) = \frac{13}{12} - \mathbb{P}(A \cap B) \\
\leq \frac{13}{12} - \frac{1}{12} = 1
\]

since $\mathbb{P}(A \cap B) \geq \frac{1}{12}$.

1.3.2 Let $H_n$ be the event ”Heads turns up for the first time on toss $n$”. Then we get $P(H_1) = \frac{1}{2}$, $P(H_2) = \frac{1}{4}$, $P(H_3) = \frac{1}{8}$, and in general $P(H_n) = \frac{1}{2^n}$. If we add up these probabilities we get that the probability of heads turning up after a finite number of tosses is

\[
\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.
\]

Therefore if we also consider the event $H_\infty$ which is ”heads never turns up”, then by countable additivity $P(H_\infty) = 0$.

1.4.3 Let 0, 1, and 2 be the events that the coin has 0, 1, and 2 heads, respectively. Let $H_L$ (resp. $H_U$) be the events that the lower (upper) side is heads. Note that $H_U \cap H_L = 2$ (the event 2) and $\mathbb{P}(H_L) = \mathbb{P}(H_U)$.
\[ P(H_L) = P(H_L|0)P(0) + P(H_L|1)P(1) + P(H_L|2)P(2) \]
\[ = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 1 \cdot \frac{2}{5} \]
\[ = \frac{3}{5} \]

\[ P(H_L|H_U) = \frac{P(H_L \cap H_U)}{P(H_U)} = \frac{P(2)}{P(H_L)} = \frac{2/5}{3/5} = \frac{2}{3} \]

Let \( H_UH_L \) be the event that the first toss is heads up and the second is heads down.

\[ P(H_UH_L|H_U) = \frac{P(H_UH_L \cap H_U)}{P(H_U)} = \frac{P(H_UH_L)}{3/5} = \frac{5}{3} \left( 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 1 \cdot \frac{2}{5} \right) \]
\[ = \frac{5}{6} \]

\[ P(H_UH_L|H_UH_U) = \frac{P(H_UH_L \cap H_UH_U)}{P(H_UH_U)} = \frac{P(2)}{P(H_UH_U)} = \frac{P(2)}{P(H_UH_L)} = \frac{2/5}{1/2} = \frac{4}{5} \]

We do the final question in two parts. First, we condition on what the first coin was. First, suppose the first coin was 1 (it cannot have been 0).
\[
\mathbb{P}(H_UH_UH_U|1 \cap H_UH_U) = \mathbb{P}(H_UH_UH_U|10 \cap H_UH_U) \mathbb{P}(10 \cap H_UH_U) \\
+ \mathbb{P}(H_UH_UH_U|11 \cap H_UH_U) \mathbb{P}(11 \cap H_UH_U) \\
+ \mathbb{P}(H_UH_UH_U|12 \cap H_UH_U) \mathbb{P}(12 \cap H_UH_U) \\
= 0 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \\
= \frac{5}{8}
\]

Next, suppose it was 2.

\[
\mathbb{P}(H_UH_UH_U|1 \cap H_UH_U) = \mathbb{P}(H_UH_UH_U|20 \cap H_UH_U) \mathbb{P}(20 \cap H_UH_U) \\
+ \mathbb{P}(H_UH_UH_U|21 \cap H_UH_U) \mathbb{P}(21 \cap H_UH_U) \\
+ \mathbb{P}(H_UH_UH_U|22 \cap H_UH_U) \mathbb{P}(22 \cap H_UH_U) \\
= 0 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} \\
= \frac{1}{2}
\]

We have \(\mathbb{P}(2|H_UH_U) = \mathbb{P}(H_UH_L|H_UH_U) = \frac{4}{5}\), and since \(\mathbb{P}(0|H_UH_U) = 0\), we must have \(\mathbb{P}(1|H_UH_U) = \frac{1}{5}\).

Finally, combining the results:

\[
\mathbb{P}(H_UH_UH_U|1 \cap H_UH_U) = \mathbb{P}(H_UH_UH_U|1 \cap H_UH_U) \cdot \mathbb{P}(1|H_UH_U) \\
+ \mathbb{P}(H_UH_UH_U|2 \cap H_UH_U) \cdot \mathbb{P}(2|H_UH_U) \\
= \frac{5}{8} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{4}{5} \\
= \frac{21}{40}
\]

1.5.1

\[
\mathbb{P}(A^C) \mathbb{P}(B) = (1 - \mathbb{P}(A)) \mathbb{P}(B) \\
= \mathbb{P}(B) - \mathbb{P}(A) \mathbb{P}(B) \\
= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
= \mathbb{P}(A^C \cap B)
\]

By the exact same logic, replacing \(B\) with \(A^C\) and \(A\) with \(B\) in the equations above, \(A^C\) and \(B^C\) are independent.

1.5.2 For pairwise independence, consider \(A_{ij}\) and \(A_{k\ell}\), possibly with \(i = k\) but definitely with \(j \neq \ell\). If the \(i^{th}\) roll comes up \(x\) and the \(k^{th}\) roll is \(y\) (possibly with \(x = y\)), then \(A_{ij}\) is the event that the \(j^{th}\) roll is \(x\) and \(A_{k\ell}\) is the event that the \(\ell^{th}\) roll is \(y\). Since
the rolls are independent, each of these events has conditional probability $\frac{1}{6}$ and the conditional probability of both is $\frac{1}{36}$, no matter what $x$ and $y$ are. Thus, the probability of each event is $\frac{1}{6}$ and the probability of both is $\frac{1}{36}$.

For independence of all the events, note that having all the pairs be equal is equivalent to having all the dice be equal, and that there are six ways to get this outcome, each having probability $\frac{1}{6^2}$, so we have $P(\bigcap_{1<j} A_{ij}) = \frac{1}{6^{n-1}}$. On the other hand, there are $\frac{n(n-1)}{2}$ pairs $i < j$ so $\prod_{i<j} P(A_{ij}) = \frac{1}{6^{n(n-1)/2}}$. For $n > 2$, these are not equal to each other.

1.5.9 Let $S$ be the event that the sum is 7. There are 36 equally likely outcomes of the roll, and six of them ($\{(n,7-n)|n = 1, 2, \ldots, 6\}$) result in a sum of 7. Therefore, $P(S) = \frac{1}{6}$.

Choose $n$ for $1 \leq n \leq 6$, and let $T$ be the event first die comes up $n$. So $P(T) = \frac{1}{6}$. Now $S \cap T$ is the same as saying the pair come up $(n, 7-n)$ which has a probability of $\frac{1}{36}$. So we have $P(S \cap T) = P(S)P(T)$ or $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$, so the events $S$ and $T$ are independent.

1.8.1 (a) Define the events: $A_1 = 6$ turns up on first die and $A_2 = 6$ turns up on second die, $A = 6$ turns up exactly once.

$$P(A) = P((A_1 \cup A_2) - (A_1 \cap A_2))$$
$$= P(A_1 \cup A_2) - P(A_1 \cap A_2)$$
$$= P(A_1) + P(A_2) - 2P(A_1 \cap A_2)$$
$$= \frac{1}{6} + \frac{1}{6} - 2 \cdot \frac{1}{36}$$
$$= \frac{5}{18}$$

(b) $A_1 =$ first number is odd, $A_2 =$ second number is odd. The rolls are independent, so $A_1$ and $A_2$ are.

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$
$$= \frac{1}{2} \cdot \frac{1}{2}$$
$$= \frac{1}{4}$$

(c) There are three ways for this to happen: $A = \{(1,3), (2,2), (3,1)\}$. Each way is equally as likely as every other of the 36 joint possibilities, so $P(A) = \frac{1}{12}$.

(d) Of the numbers 1 through 6, two each are congruent to 0, 1, and 2 modulo 3. If the first roll is $x$ modulo 3, then regardless of $x$ there is a one-third chance that the second is congruent to $-x$ modulo 3 (i.e. that their sum is 0 modulo 3). Thus the probability that the sum is divisible by 3 is $\frac{1}{3}$.

1.8.4 Let $\mathcal{P}(S)$ denote the power set of a set $S$. When $\mathcal{F} = \mathcal{P}(\Omega)$ it is sufficient to define $P(\omega)$ for each $\omega \in \Omega$. 

4
(a) Let $p$ be the probability of heads on a given flip. Let $h(\omega)$ for $\omega \in \Omega$ be the number of total heads (e.g. $h(HHT) = 2$).

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$\mathbb{P}(\omega) = p^{h(\omega)}(1-p)^{3-h(\omega)}$$

(b) Let $d(\omega)$ be the indicator that the first and second balls have different colors (1 if different, 0 o.w.). The color of the first ball is one half probability either way, and for the second ball to have the same color as the first has probability $\frac{1}{3}$.

$$\Omega = \{UU, UV, VU, VV\}$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$\mathbb{P}(\omega) = \frac{1 + d(\omega)}{6}$$

(c) The probability of heads coming up first on toss $n$ ($\omega = n$) is the probability of $n - 1$ tails followed by a heads, in other words $(1-p)^{n-1}p$. The probability of heads never coming up ($\omega = \infty$, countable intersection of the nested events “tails comes up on each of the first $n$ tosses”) is the limit of the probabilities of those events, $\lim_{n \to \infty} (1-p)^n$, which is 0 if $p > 0$ and 1 otherwise.

$$\Omega = \mathbb{N} \cup \infty$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

and $\mathbb{P}$ as defined above.

1.8.15 Let $D_i$ be the event that the $i^{th}$ die comes up 1.

(a) If $S = 4$ then $N <= 4$ so our sum in the following stops at $N = 4$.

$$\mathbb{P}(N = 2 | S = 4) = \frac{\mathbb{P}(S = 4 | N = 2) \mathbb{P}(N = 2) \sum_{i=1}^{4} \mathbb{P}(S = 4 | N = i) \mathbb{P}(N = i)}{\sum_{i=1}^{4} \mathbb{P}(S = 4 | N = i) \mathbb{P}(N = i)}$$

$$= \frac{1/12 \cdot 1/4}{1/6 \cdot 1/2 + 1/12 \cdot 1/4 + 3/216 \cdot 1/8 + 1/67 \cdot 1/16}$$

$$\approx 0.197$$

(b) Again, we must have $N <= 4$. Let $E$ be the event that $N$ is even.
\[ P(S = 4|E) = \frac{P(S = 4 \cap E)}{P(E)} = \frac{P(S = 4|N = 2) P(N = 2) + P(S = 4|N = 4) P(N = 4)}{\sum_{i=1}^{\infty} P(N = 2i)} \]
\[ = \frac{\frac{1}{12} \cdot \frac{1}{4} + \frac{1}{64} \cdot \frac{1}{16}}{\sum_{i=1}^{\infty} (1/4)^i} \]
\[ = 3 \left( \frac{1}{12} \cdot \frac{1}{4} + \frac{1}{64} \cdot \frac{1}{16} \right) \]
\[ = 0.063 \]

\[ P(N = 2|S = 4 \cap D_1 = 1) = \frac{P(S = 4 \cap D_1 = 1|N = 2) P(N = 2)}{\sum_{i=1}^{4} P(S = 4 \cap D_1 = 1|N = i) P(N = i)} \]
\[ = \frac{\frac{1}{36} \cdot \frac{1}{4}}{0 \cdot \frac{1}{2} + \frac{1}{36} \cdot \frac{1}{4} + \frac{2}{216} \cdot \frac{1}{8} + \frac{1}{64} \cdot \frac{1}{16}} \]
\[ \approx 0.852 \]

(c) Define:
\[ f(r) = P \left( \bigcap_{i=1}^{N} D_i \leq r \right) \]
\[ = \sum_{n=1}^{\infty} \left( \frac{r}{12} \right)^n 2^{-n} \]
\[ = \sum_{n=1}^{\infty} \left( \frac{r}{12} \right)^n \]
\[ = \frac{r/12}{1 - r/12} \]
\[ = \frac{r}{12 - r} \]

Note \( f(6) = 1 \). Then the probability that the highest roll is exactly \( r \) is \( f(r) - f(r - 1) \), or \( \frac{12}{(12-r)(13-r)} \).

1.8.22 Five cherries have stones, 15 don’t. The pig eats 5.

(a) This is the same thing as picking one of the 20 cherries at random - the fact that five of them happen to be in the pig’s stomach is not relevant. So the answer is \( \frac{5}{20} \), or .25.

(b) We’ll compute the conditional probability that the pig didn’t get any stones, and then subtract from one.

Think of 20 slots, with the first 5 being the ones in the pigs stomach, and the 6th being the one we pick.
The condition is that slot 6 has a stone so there are $5 \cdot 19!$ total orderings of the cherries having a stone in place 6.

There are $C(15, 5)$ ways to choose the un-stoned cherries to put in slots 1-5, and $5!$ ways to order those. There are 5 ways to pick the stone to put in slot 6, and that leaves $14!$ ways to order the remaining 14 cherries. So there are a total of $C(15, 5) \cdot 5! \cdot 5 \cdot 14!$ ways to choose and arrange the cherries so that the pig got no stones, and the one we picked got a stone. So the conditional probability that the pig got at least one stone is

$$1 \frac{C(15, 5) \cdot 5! \cdot 5 \cdot 14!}{5 \cdot 19!} = .74174407.$$ 

1.8.30 Inductive proof that the probability no students have the same birthday is $q(m) = \frac{365!}{(365-m)365^m}$. $q(1) = 1$, so the base case works. For the inductive step, assume the formula works for $m-1$. Then there are 365 equally likely choices for the $m$th student who enters the room, of which $365 - (m-1)$ do not intersect with the set of already represented birthdays. Thus the new probability of no common birthdays is $q(m-1) \cdot \frac{365!}{(365-\cdot(m-1))365^{m-1}365} = \frac{365!}{(365-m)365^m} = q(m)$.

Thus the formula works for all $m$, and it follows that the probability of the complement of this event, that at least 2 students have the same birthday, is $1 - q(m)$. In particular, for $m = 23$, $q(m) = .493$ and $1 - q(m) = .507 > \frac{1}{2}$.

1.8.33 There are $\binom{52}{5} = 2,598,960$ total hands of poker, so to compute the probability of a given hand type, we simply divide the number of ways to get that hand type by 2,598,960.

First consider the hands other than flushes and straights. We first choose the duplicated values (i.e. 2,3,...,A), then the unduplicated values, then the suits of each type.

Pair: $\binom{13}{1} \binom{12}{3} \binom{4}{1}^3 = 1098240 \text{ ways} \Rightarrow p \simeq .423$

Two Pair: $\binom{13}{2} \binom{11}{1} \binom{4}{1}^2 = 123552 \text{ ways} \Rightarrow p \simeq .0475$

Three of a Kind: $\binom{13}{1} \binom{12}{2} \binom{4}{1}^2 = 54912 \text{ ways} \Rightarrow p \simeq .021$

Full House: $\binom{13}{1} \binom{12}{1} \binom{4}{1} \binom{4}{2} = 3744 \text{ ways} \Rightarrow p \simeq .0014$

Four of a Kind: $\binom{13}{1} \binom{12}{1} \binom{4}{4} \binom{4}{1} = 624 \text{ ways} \Rightarrow p \simeq .00024$

For a straight flush, we must first choose the suit (4 ways) then choose the card values (A, 2, 3, ..., 10) that begins the straight. This gives 40 possible hands, so $p \simeq .000015$.

For a regular flush, we choose the suit (4 ways) and then the values ($\binom{13}{5}$ ways), giving 5148, then subtract off the 40 straight flushes, giving 5108 ways, so $p \simeq .0020$.

For a regular straight, we choose the starting value (10 ways), then the suits of each card (45 ways), then again subtract the 40 straight flushes, giving 10200 ways, so $p \simeq .0039$.

2.1.2 Let $G$ be the distribution function of $Y$.

If $a = 0$ then $Y = b$ so $G(y)$ is 0 for $y < b$, 1 for all other values of $y$. 

7
If \( a > 0 \) then:

\[
G(y) = \mathbb{P}(Y \leq y) \\
= \mathbb{P}(aX + b \leq y) \\
= \mathbb{P}(X \leq \frac{y-b}{a}) \\
= F\left(\frac{y-b}{a}\right)
\]

If \( a < 0 \) then:

\[
G(y) = \mathbb{P}(Y \leq y) \\
= \mathbb{P}(aX + b \leq y) \\
= \mathbb{P}(X \geq \frac{y-b}{a}) \\
= 1 - \mathbb{P}\left(X < \frac{y-b}{a}\right) \\
= 1 - \lim_{\epsilon \to 0^-} \mathbb{P}\left(X \leq \frac{y-b}{a} + \epsilon\right) \\
= 1 - \lim_{\epsilon \to 0^-} F\left(\frac{y-b}{a} + \epsilon\right)
\]

2.7.7 The number of passengers not showing up is \( T \quad \text{Bin}(10, \frac{1}{10}) \) for Teeny Weeny and \( B \quad \text{Bin}(20, \frac{1}{10}) \) for Blockbuster. Teeny Weeny is overbooked if \( T = 0 \) and Blockbuster is overbooked if \( B \leq 1 \). \( \mathbb{P}(T = 0) = \left(\frac{9}{10}\right)^{10} \approx .349 \) and \( \mathbb{P}(B \leq 1) = \left(\frac{9}{10}\right)^{20} + 20 \left(\frac{9}{10}\right)^{19} \frac{1}{10} \approx .392 \), so Teeny Weeny is overbooked slightly less often.

2.7.8 The number of heads out of six tosses of a fair coin is \( \text{Bin}(6, \frac{1}{2}) \), so the probability of 5 or more heads is \( \left(\frac{1}{2}\right)^6 + 6 \left(\frac{1}{2}\right)^6 \approx .109 \).

3.1.3 This is identical to the situation where we throw each of \( n \) coins twice and define “success” as coming up heads both times. The probability of success for each coin is \( p^2 \) and there are \( n \) such coins so the sought random variable is distributed as \( \text{Bin}(n, p^2) \) and the mass function is the associated binomial mass function, \( f(k) = \binom{n}{k} p^{2k} (1 - p^2)^{n-k} \).