Independence

Definition
Two events $A$ and $B$ are called independent if their joint probability is equal to the product of their individual probabilities:
$$P(A \cap B) = P(A)P(B).$$
Otherwise, we say that $A$ and $B$ are dependent.

Example
Toss a fair coin 10 times
$E =$ Heads on the first 9 tosses $F =$ Heads on the last toss
Is it true that $P(E \cap F) = P(E)P(F)$?
We get $1/1024 = 2/1024 \times 512/1024$. 
What if you have 3 or more events?

**Definition**
Suppose $A_1, A_2, A_3, \ldots, A_n$ are events, then we say they are independent if

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$$

This implies pairwise independence, but not conversely.
Example

Toss a biased coin repeatedly.

\( p = \) probability of heads \( q = 1-p = \) probability of tails

\( p_n = \) probability of getting an even number of heads after \( n \) tosses

We’ll compute \( p_n \).

\( p_0 = 1 \) is the initial condition.

Let \( A \) be the event ”heads on the \( n \)th toss”.

If the sample space is partitioned into a disjoint union, \( \Omega = B_1 \cup B_2 \cup \ldots \), then we have the following important formula:

\[
P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots
\]

\[
p_n = P(\text{even \# of heads}|A)P(A) + P(\text{even\# of heads}|A^c)P(A^c)
\]

\[
P_n = (1-p_{n-1})p + p_{n-1}q
\]

\[
p_n = p-p(p_{n-1}) + qp_{n-1}.
\]

\[
p_n = p + (q-p)p_{n-1}.
\]
This is a first order difference equation, linear in the \( p_n \)'s. It is not homogenous because of the presence of the constant.

The theory of difference equations is similar to the theories of differential equations. The solution is a sequence of numbers.

A first order difference equation with an initial condition (e.g., \( p_0 = 1 \)) has a unique solution. A second order difference equation with 2 initial conditions has a unique solution.

\[
\begin{align*}
p_n &= p + (q-p)[p + (q-p)p_{n-2}] \\
&= p + (q-p)p + (q-p)^2 p_{n-2} \\
&= p + (q-p)p + (q-p)^2[p + (q-p)p_{n-3}] \\
&= p + (q-p)p + (q-p)^2 p + (q-p)^3 p_{n-3}
\end{align*}
\]

We get

\[
\begin{align*}
p_n &= p + (q-p)p + (q-p)^2 p + \ldots + (q-p)^{n-1} p + (q-p)^n p_0 \\
&= p[1 + (q-p) + (q-p)^2 + \ldots + (q-p)^{n-1}] + (q-p)^n
\end{align*}
\]
We know

\[ 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}, \quad r \neq 1 \]

In our case,

\[
p_n = \frac{p(1-(q-p)^n)}{1-(q-p)} + (q-p)^n
\]

\[
= \frac{p(1-(q-p)^n)}{1-(1-p-p)} + (q-p)^n
\]

\[
= \frac{p(1-(q-p)^n)}{2p} + (q-p)^n
\]

\[
= \frac{1}{2}(1-(q-p)^n) + (q-p)^n
\]

\[
= \frac{1}{2} + \frac{1}{2}(q-p)^n
\]

Note, that for a fair coin, this always equals \( \frac{1}{2} \) because \( p = q \)
Second Order Linear Difference Equations with Constant Coefficients

We have

\[ p_{k+1} + ap_k + bp_{k-1} = c_k \]

where \( a, b \) are constant coefficients and \( c \) can vary with \( k \).
If \( c_k = 0 \), then \( p_{k + 1} + ap_k + bp_{k-1} = 0 \) is a \textit{homogenous} equation.
Suppose initial conditions \( p_0 \) and \( p_1 \) are specified.
A typical solution has the form:
\[ u_k = m^k \text{ for some } m. \]
\[ u_{k+1} = m^{k+1}. \]
\[ m^{k+1} + am^k + bm^{k-1} = 0 \]
\[ m^{k-1}(m^2 + am + b) = 0 \]
Thus, we find \( m \) by finding the roots of quadratic equation \( x^2 + ax + b = 0 \). This equation is also called the characteristic equation.
There are three solution scenarios for a quadratic equation:

1. 2 distinct real roots
2. 1 real root that is repeated
3. 2 complex roots (which will be conjugate)

**Case 1:** Two distinct real roots $m_1$ and $m_2$, then $u_k = m_1^k$ is one solution and $u_k = m_2^k$ is another solution (we are assuming homogenous equations).

If you have two solutions, any linear combination of the two is also a solution. The general solution has the form:

$$p_k = A m_1^k + B m_2^k.$$
Case 2: 1 real repeated root $m$.
$u_k = m^k$ is a solution.
$\nu_k = km^k$ is another solution.
The general solution is $p_k = Am^k + Bkm^k$. 
To review, the steps for solving a second order difference equation are:

1. Write out the homogeneous equation (divide through by the coefficient of $p_{k+1}$)
2. Substitute the $m$'s
3. Write out the characteristic equation
4. Find roots
5. Substitute initial conditions to find $A$ and $B$
Gambler’s Ruin Problem

A player plays a game and on each successive play of the game has the probability of $p$ of winning $1$ and probability of $q$ of losing $1$. If the player’s fortune reaches $0$ or $N$, the game is over. Let $p_k$ be the probability of losing if the player starts with $k$ dollars. 

$$p_k = pp_{k+1} + qp_{k-1}, \ 0 < k < N.$$ 

The initial conditions are $p_0 = 1$ and $p_N = 0$. Note that this is a second order linear equation with constant coefficients:

$$pp_{k+1} - p_k + qp_{k-1} = 0$$
$$p_{k+1} - \frac{p_k}{p} + \frac{q}{p}p_{k-1} = 0$$

$$u_k = m_k$$

$$x^2 - (1/p)x + q/p = 0$$

The roots are $m = 1, q/p$. So $p_k = A + B(q/p)^k$. 