Basic Probability Summer 2020  
NYU Courant Institute  
Practice Problems for the Final Exam

1. Let \( X \) be a continuous random variable with a density function which is symmetric about 0. Show that \( E(X) = 0 \).

3. Let \( X_1, X_2, \ldots, X_{100} \) be independent random variables with the common density \( f(x) = 2 - 2x, \ 0 \leq x \leq 1 \). Let \( S = X_1 + X_2 \ldots X_{100} \). Use the Central Limit Theorem to estimate \( P(S \leq 35) \).

4. Compute the moment generating function of a geometric random variable, and use it to compute the mean and the variance.

5. In a certain community, 4 percent of all adults over the age of 50 have tuberculosis (T.B.). A health service in this community correctly diagnoses 98 percent of all persons with T.B. as having the disease, and incorrectly diagnoses 3 percent of all persons without T.B. as having the disease. Find the probabilities that,

   a) the community health service will diagnose an adult over 50 as having T.B.,

   b) a person over 50 diagnosed by the health service as having T.B. actually has the disease.

6. Suppose the joint probability density of \( X \) and \( Y \) is given by

   \[
   f(x, y) = \begin{cases} 
   24y(1 - x - y) & \text{for } x > 0, \ y > 0, \ x + y < 1, \\
   0 & \text{elsewhere.}
   \end{cases}
   \]

   a) find the marginal density of \( X \).

   b) find the marginal density of \( Y \).

   c) determine if the two variables are independent.
7. Let $X$ have variance $\sigma^2$ and let $m_i = E(X^i)$ denote the $i$th moment. The skewness of the random variable $X$ is defined to be

$$\text{skw}(X) = E((X - m_1)^3)/\sigma^3.$$ 

a) Show that

$$\text{skw}(X) = \frac{m_3 - 3m_1m_2 + 2m_1^3}{\sigma^3},$$

b) Compute the skewness of an exponential variable with parameter $\lambda$ and show that it doesn’t depend on $\lambda$. (The best way to compute the moments is from the moment generating function.)

9. Suppose you play a series of 100 independent games. If you win a game, you win 4 dollars. If you lose a game, you lose 4 dollars. The chances of winning each game is $\frac{1}{2}$. Use the central limit theorem to estimate the chances that you will win more than 50 dollars.

10. Let $X$ and $Y$ be continuous random variables, having joint probability density function

$$f(x, y) = \begin{cases} 24xy & \text{for } 0 < x < 1, 0 < y < 1, 0 < x + y < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the probability density of $Z = X + Y$. 