Solutions - Essentials of Probability Summer 2022
NYU Courant Institute
Quiz

Directions: Closed book, closed notes, show all work to get full credit.

1. A gambler has in her pocket a fair coin and a two-headed coin.
   (a) She selects one of the coins at random, and when she flips it, it shows heads. What is the probability that it is the fair coin?
   **Solution:** Let \( F \) = 'the coin is fair', \( H \) = 'heads on the first toss', \( HH \) = 'heads on first and second toss'.
   Then
   \[
   P(F|H) = \frac{P(F \cap H)}{P(H)} = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|F^c)P(F^c)} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1)(1/2)} = \frac{1}{3}.
   \]
   (b) Suppose that she flips the same coin a second time and again it shows heads? Now what is the probability that it is the fair coin?
   **Solution:**
   Then
   \[
   P(F|HH) = \frac{P(F \cap HH)}{P(HH)} = \frac{P(HH|F)P(F)}{P(HH|F)P(F) + P(HH|F^c)P(F^c)} = \frac{(1/4)(1/2)}{(1/4)(1/2) + (1)(1/2)} = \frac{1}{5}.
   \]
   (c) Suppose that she flips the same coin a third time and it shows tails? Now what is the probability that it is the fair coin?
   **Solution:** The answer is obviously 1, since if the coin shows tails, it can’t be the two headed coin.

2. A mouse has two levers, and is given food if it presses the left hand lever, an electric shock if it presses the right hand lever. The first time there is a 50-50 chance it will press either lever. If it presses the left lever and gets food the first time, then there is a probability of .70 that it will press the left lever the second time. If it presses the right lever and gets a shock the first time, then there is a probability of .80 it will press the left lever the second time.
   (a) What is the probability the mouse will get food the second time?
   **Solution:** Using the obvious notation:
   \[
   P(F_2) = P(F_2|F_1)P(F_1) + P(F_2|S_1)P(S_1) = (.70)(.5) + (.80)(.5) = .75
   \]
   (b) What is the probability that it got food the first time, given that it gets food the second time?
   \[
   P(F_1|F_2) = \frac{P(F_2 \cap F_1)}{P(F_2)} = \frac{P(F_2|F_1)P(F_1)}{P(F_2)} = \frac{.35}{.75} = .4667.
   \]
3. Six different ice cream trucks can park at any of eight different intersections in midtown. What is the probability that at least two trucks park at the same intersection? Assume that the trucks act independently and that all intersections are equally likely for each truck.

**Solution:** The sample space $\Omega$ consists of all the possible ways that the six trucks can park at the eight intersections. There are eight possibilities for each truck so we have $|\Omega| = 8^6$.

In the event $A$ that no two park at the same intersection, there are eight possibilities for the first truck, seven for the second truck, etc. so $|A| = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$. We get $P(A) = \frac{|A|}{|\Omega|} = \frac{P(8,6)}{8^6} = .0769$. The complement of this is the event that at least two do park at the same intersection, so the answer is $1 - .0769 = .9231$.

4. Suppose we toss two fair dice.

(a) Let $E_1$ be the event that the sum is six, and let $F$ be the event that the first die is four. Are $E_1$ and $F$ independent? Justify your answer with a calculation.

**Solution:** We have $P(E_1) = \frac{5}{36}$, $P(F) = \frac{1}{6}$, so $P(E_1)P(F) = \frac{5}{216}$. However $P(E_1 \cap F) = \frac{1}{36}$, which is not the same, so $E_1$ and $F$ are dependent.

(b) Now let $E_2$ be the event that the sum of the dice is seven. Is $E_2$ independent of $F$? Again, show the calculation which justifies your answer.

**Solution:** We have $P(E_2) = \frac{1}{6}$, $P(F) = \frac{1}{6}$, so $P(E_2)P(F) = \frac{1}{36} = P(E_2 \cap F)$ so $E_2$ and $F$ are independent.