RESEARCH STATEMENT

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My main research interests are based in the analysis of partial differential equations (PDEs), particularly those centered about hydrodynamic and geophysical equations, such as the incompressible Navier-Stokes or the quasi-geostrophic equations. I am also interested in chemotaxis equations that, for instance, incorporate the effects from the interaction of an organism with the ambient incompressible fluid, and dispersive equations with weak damping mechanisms. A main source of inspiration in my work derives from the mathematics of turbulence, such as the identification of small length scales, and its applications, for instance, to dissipative dynamical systems or data assimilation; it continues to be a driving force in my research developments. My work employs various tools from harmonic analysis, semigroup theory, approximation theory, control theory, as well as elliptic equations and classical energy methods.

1. Overview of works

My early work involved establishing higher-order regularity for solutions to PDEs via so-called Gevrey-norm techniques. Classically, this technique was used to establish regularity of solutions to PDEs in the analytic class of functions. In the context of turbulence, an immediate consequence of this is that one obtains estimates on the number of degrees of freedom in the flow. It can as well be adapted to establishing *sub-analytic* regularity, which is a natural regularity class for equations that feature a dissipation operator of the form $(-\Delta)^\gamma$, $\gamma < 1/2$. In general, the Gevrey norm approach provides a flexible and efficient way to capture the "optimal" smoothing effect brought upon by such an operator. This approach and these features, in addition to its applications, are explored and developed in several joint works involving A. Biswas, L. Hoang, M.S. Jolly, P. Silva, E.S. Titi, and K. Zhao described below. In particular, in a joint work with A. Biswas and P. Silva, we extend this approach to a general, scaling-critical framework, in which we establish rather powerful harmonic analysis inequalities.

Some of my current work concerns further development of a certain approach to data assimilation and provides the theoretical foundations for its implementation and rigorous support for common practices in data assimilation, e.g., assimilating only surface measurements to synchronize an approximating signal with the true signal in a three-dimensional domain. The approach exploits a feature of certain dissipative systems in which the small scales are asymptotically enslaved to the large scales. My work on this topic involves overcoming analytical difficulties brought on by the presence of fractional diffusion operators, exploring physically relevant modifications of the observation operators, and studying the nature of the synchronization. This approach has also recently found applications to the study of the long-time behavior of such equations. In particular, one can use this idea to reduce the original PDE to an *ordinary differential equation* (ODE) called the "determining form." The works in this vein are joint works involving A. Biswas, M.S. Jolly, E.J. Olson, T. Sadigov, and E.S. Titi.

2. Dissipation length scales & Degrees of freedom

The classical picture for three-dimensional (3D) turbulence posits the existence of an "inertial range" of length scales, where nonlinear effects in the flow are dominant. In this range, the energy decays as the length scale decreases according to a power law (cf. [56]). This inertial range should then extend down to the "dissipation range," where one expects the flow to be governed predominantly by the linear effect of viscous energy consumption. The length scale demarcating the transition from the inertial to the dissipation range is known as the "Kolmogorov dissipation length scale," which shall be denoted as ℓ_{Kol}. One thus imagines that at length scales $\ell \leq \ell_{Kol}$, the viscosity and energy dissipation rate are all that are needed to describe the flow. From this observation, an estimate for the number of degrees of freedom in the system is provided by L^3/ℓ_{Kol}^3, where L is the linear size of the domain (cf. [59]). Under certain universality assumptions, an explicit expression for ℓ_{Kol} can be obtained by heuristic scaling arguments. We seek, therefore, to quantify it through rigorous estimates on the equations of motions themselves

Much literature has been dedicated to this task, namely, of deducing estimates for ℓ_{Kol} directly from the Navier-Stokes equations (NSE), as well as for its analogous two-dimensional (2D) counterpart, ℓ_{Kr}, referred to as the "Kraichnan dissipation length scale" (cf. [2, 17, 24, 34, 37, 38, 39, 42, 58]). The NSE is given by the system

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \nabla\cdot u = 0, \tag{2.1}$$

where u denotes the velocity vector field of the fluid, $\nabla\cdot u = 0$ expresses its incompressibility, p is the scalar pressure field, and f is an external body force. We recall from the Paley-Wiener theorem that a function is real analytic with uniform radius of real analyticity if and only if its Fourier transform decays exponentially. Moreover, the exponential decay rate is proportional to its real analyticity radius. Since ℓ_{Kol} is indicated by an exponential cut-off in the energy spectrum, the radius of real spatial analyticity, ℓ_a, of u satisfies $\ell_a \lesssim \ell_{Kol}$. Therefore, L^3/ℓ_a^3 provides an upper bound estimate for the number of degrees of freedom in the flow.

In my Ph.D. thesis and a joint work with A. Biswas, M.S. Jolly, and E.S. Titi [6], a Gevrey-norm approach is adapted and refined to establish real spatial analyticity and ultimately obtain

$$\ell_a \gtrsim \ell_{Kol}^4 \quad \text{and} \quad \ell_a \gtrsim \ell_{Kr}^2,$$

for turbulent flows that satisfy (2.1) in a periodic domain. The Gevrey norm approach was introduced by Foias and Temam in [38]; it allows one to bypass direct estimation of higher-order derivatives to establish both space and time analyticity. Their method was refined in [24] to obtain a sharper estimate of the analyticity radius in 3D under periodic boundary conditions, while the best-to-date estimates in 2D with these boundary conditions are obtained in [58] by instead resorting to complex-analytic techniques.

Our above estimates improve upon the work of [5], as well as unify the results of [24] in 3D and [58] in 2D *under a single framework*, namely a semigroup framework in analytic Gevrey classes. Moreover, our method exposes *a new path* to lowering the above exponents closer to 1, namely, by improving higher-order estimates of the flow in the long-time average. The mathematics involved in pursuing this avenue appeals to a statistical framework of turbulence and requires further exploration.

3. Gevrey regularity & Asymptotic expansions

3.1. **Supercritical surface quasi-geostrophic (SQG) equation.** For $\gamma \in (0,2)$, the dissipative SQG equation in non-dimensionalized variables is given by

$$\partial_t\theta + \kappa\Lambda^\gamma\theta + u\cdot\nabla\theta = f, \quad u = \mathcal{R}^\perp\theta, \tag{3.1}$$

where θ is the scalar surface temperature of a fluid, which is advected along the velocity field, u, and f is a given external forcing term. The operator $\mathcal{R}^\perp := (R_2, -R_1)$ is determined by the Fourier symbols $i\xi_j/|\xi|$, which ensures that u is divergence-free. The fractional Laplacian operator, Λ^γ, appears with prefactor $\kappa > 0$, and its Fourier symbol is given by $|\xi|^\gamma$.

Equation (3.1) represents the simplest model derived from the 3D NSE of a geophysical fluid in a regime of strong rotation that captures nontrivial dynamics which depart from the regime of geostrophically balanced flows, i.e., where the Earth's rotational effects are balanced by the horizontal pressure gradient (cf. [67]). The regimes $\gamma > 1, \gamma = 1, \gamma < 1$, refer to the subcritical, critical, and supercritical cases, respectively. Since its introduction into the mathematical community by Constantin, Majda, and Tabak [18], the equation has been thoroughly studied, and by now, well-posedness in various function spaces and global regularity has been resolved in all but the supercritical case (cf. [8, 13, 15, 19, 22, 25, 54, 55, 68]). The long-time behavior and existence of a global attractor has also been studied in both the subcritical and critical cases (cf. [9, 12, 16, 21, 19, 51]).

In my Ph.D. thesis and joint work with A. Biswas and P. Silva [7], we establish Gevrey regularity for the supercritical ($\gamma < 1$) SQG equations in critical Besov spaces. These spaces may be viewed as a scale of spaces obtained by interpolating between the classical Sobolev spaces. This work improves upon the results in [3, 11, 27]. More importantly, it *extends the Gevrey norm technique to the more general class of L^p-based Besov spaces* for *subanalytic Gevrey classes*, thereby providing a natural setting in which to study such regularity for solutions to other PDEs with sub-Laplacian dissipative operators.

The Gevrey norm technique was extended to the L^p setting in Lemarié-Rieusset in the case of the analytic Gevrey class by establishing a crucial bilinear estimate in L^p spaces (cf. [60, 61, 62]). However, in our case, the structural changes in the corresponding bilinear operator brought upon by working in a subanalytic Gevrey class necessitates a more delicate treatment of its symbol. By carefully capturing the cancellation in the operator, we are able to prove a powerful commutator estimate in Gevrey classes, which ultimately allows us to establish the requisite a priori bounds. Incidentally, our result would improve upon existing "eventual regularity" results (cf. [23]), provided that the solution becomes sufficiently small in some norm at some future time, thus shedding light on a new avenue to approach this class of problems. Also, an important class of Besov spaces that are not covered by our result are those which have the additional structure of a Banach algebra. Both of these are essentially "endpoint" phenomena and are issues that A. Biswas and I continue to investigate.

3.2. **Keller-Segel-Navier-Stokes model.** In an effort to understand the interplay between certain bacteria swimming in a viscous, incompressible fluid, the oxygen diffusion and consumption, and chemotaxis, a model was proposed in [69] that couples the Navier-Stokes equations with the Keller-Segel model:

$$\begin{aligned} &\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = -n\nabla\phi, \quad \nabla\cdot u = 0,\\ &\partial_t n - D_n\Delta n + (u\cdot\nabla)n + \nabla\cdot(n\chi(c)\nabla c) = 0,\\ &\partial_t c - D_c\Delta c + (u\cdot\nabla)c + nf(c) = 0, \end{aligned} \tag{3.2}$$

where u, p are the fluid velocity and pressure fields, and n, c are the cell density and oxygen concentration, respectively. The constants ν, D_n, C_c are the kinematic viscosity, cell diffusion, and oxygen diffusion coefficients, while χ and f are smooth, nonnegative functions that capture the chemotactic sensitivity and oxygen consumption rate.

From the point of view of hydrodynamic equations, (3.2) is a generalization of the Boussinesq equations. Indeed, if c, χ, ϕ are chosen so that $c, \chi \equiv 0$, $\phi(x_1, x_2) = x_2$, then one recovers precisely the Boussinesq equations. On the other hand, by comparing the nonlinearity and the dissipation, one sees that the problem is also "critical" in the sense that the largest order appearing in the nonlinearity is the same as that of the dissipation, i.e., in the equation for the cell density.

In a joint work with K. Zhao [64], we establish several fundamental results for the system (3.2). In particular, several blow-up criterion are developed in a 3D bounded domain with rather general conditions imposed on f, g and the boundary data, as well as a global regularity result in 2D under certain physical assumptions motivated by [69]. It is also shown that in spite of the high-order derivative appearing in the nonlinearity, solutions to (3.2) instantaneously enter an analytic Gevrey class, in all dimensions $d \geq 2$, in the case of periodic boundary conditions, and remain in that class at least for short-time. However, due to the apparent criticality, this latter result is restricted to a small data regime. Indeed, this criticality can be broken provided that ∇c is replaced by $|\nabla|^\alpha$, where $\alpha < 1$, which in turn yields a short-time, but large-data result; in light of this, it is an issue that deserves further consideration.

3.3. **Asymptotic expansions for the 3D NSE.** In 1984, Foias and Saut [35] proved that in bounded or periodic domains the regular solution of the NSE decays exponentially at an *exact* rate which is an eigenvalue of the Stokes operator (see [35]). Remarkably, they go on to show in [36] that the solution in fact admits an asymptotic expansion which details its long-time behavior with respect to the Sobolev class of functions. This expansion has found applications to statistical solutions of the NSE, decaying turbulence, and the analysis of helicity (cf. [30, 31]).

In a joint work with L. Hoang [46], we consider the 3D NSE in a periodic domain with potential forcing. By exploiting the eventual regularization effect, we are able to strengthen the Foias-Saut expansion to hold in the analytic Gevrey class of functions, and establish explicit decay rates for its remainders with respect to this stronger norm. Our approach via eventual regularization simplifies greatly the original proof of Foias and Saut and ultimately renders it adaptable to other dissipative systems, thus suggesting that the analytic Gevrey class is the natural setting for these expansions. In continuation of this work, L. Hoang, A. Biswas, and I are working to adapt these expansions to the SQG equation and generalize them accommodate the case of a general body force, the latter of which has interesting applications to numerical computation of solutions.

4. Data assimilation & Determining forms

The approach to data assimilation that has been a focus of my recent work is the one proposed by Azouani, Olson, and Titi in [1]. In particular, suppose u is a solution to a physical model over a domain Ω, whose time evolution is governed by the equation

$$\frac{du}{dt} = F(u), \tag{4.1}$$

except that the initial data u_0 has *not* been provided and is thus, unknown. Then consider the following initial value problem:

$$\frac{dw}{dt} = F(w) - \mu I_h(w-u), \quad w(0) = w_0, \tag{4.2}$$

where w_0 is *any* initial condition, $h > 0$, $\mu = \mu(h) > 0$ is the "relaxation parameter", and I_h is a finite-rank linear operator representing the observables. Typically, I_h is a projection onto finitely many nodal or modal values, where h is proportional to the number of spatial nodes or Fourier modes. One would then integrate this modified system (4.2) forward in time to obtain a suitable approximation of the reference solution with which to initialize the original system. Indeed, the term $\mu I_h(u-w)$ serves to relax the large scales of the approximating solution, w, towards the reference solution, u, which in fact suffices to synchronize w with u at the small scales, precisely by exploiting the fact that the many hydrodynamic systems asymptotically enslaves the small scales to the large scales.

4.1. **Data assimilation for 2D SQG equation.** In a joint work with M.S. Jolly and E.S. Titi [49], we establish global well-posedness for the corresponding feedback control system (4.2) of the 2D *subcritical* SQG equation ($\gamma > 1$) with periodic boundary conditions, i.e., $F(w) = -\kappa\Lambda^\gamma w - (\mathcal{R}^\perp w)\cdot\nabla w + f$ and I_h given by projection onto finitely many Fourier modes or local spatial averages. We moreover show that w converges to θ exponentially in time in the topology of square-integrable functions, provided that the relaxation parameter and the number of collected observables are sufficiently large. The key estimates required are fractional Poincaré-type inequalities for the observation operators, I_h. As a consequence, since $\partial_z\psi|_{z=0} = \theta$, where ψ is the streamfunction for the 3D quasi-geostrophic equation, our result immediately implies that in a particular simplified scenario, the approximating streamfunction synchronizes with ψ in the *three-dimensional half-space*. Our result therefore gives rigorous support to the notion that for vertically constrained flows, one need only assimilate data from the two-dimensional boundary. Having isolated the difficulties arising from balancing the nudging term with the fractional dissipation, we will now move on to "physical case," where $\gamma = 1$, i.e., the critical case for the SQG equation.

In a joint work with M.S. Jolly, E.J. Olson, and E.S. Titi [48], we continue to study the 2D subcritical SQG equation, but modify the approach to accommodate the more physical case of *time-averaged modal observables with a delay*, i.e., $\bar{I}_h = \frac{1}{\delta}\int_{t-2\delta}^{t-\delta} I_h$, where I_h is given by projection onto finitely many Fourier modes. Indeed, this is motivated by the fact that the instruments used to collected weather data, e.g. wind velocity, is manifestly averaged in time. We are once again able to ensure the synchronization property, *in spite of the delay* and with the *same* number of modes as the "instantaneous case" above, provided that averaging window, δ, is chosen sufficiently small. We overcome the difficulties introduced by the temporal non-locality, $\bar{I}_h$ by controlling the low modes of the time-derivative, and obtaining a suitable "non-local" Gronwall inequality.

4.2. **Higher-order synchronization for the 2D NSE equation.** In a joint work with A. Biswas [4], we study higher-order synchronization of the approximating solution, w, to the reference solution, u, satisfying (2.1) with periodic boundary conditions. In particular, w satisfies (4.2) supplemented with $\nabla\cdot w = 0$, where $F(w) = \nu\Delta w - w\cdot\nabla w - \nabla q + f$ and I_h is given by projection onto finitely many Fourier modes in (4.2). We are able to show that the synchronization can in fact be upgraded to the topology of uniform convergence *for the same number of modal observables*, up to a prefactor, required in [1], where they were only able to guarantee convergence in the topology of H^1, i.e., L^2 vector fields whose weak derivatives are also L^2. Moreover, by increasing the number of known modal observables, we show that one can in fact ensure synchronization in an *analytic Gevrey norm*. The next step in this direction will be to establish higher-order synchronization when data is presented as local spatial averages or nodal values.

4.3. **Determining form for the subcritical SQG equation.** Motivated by the finite dimensionality of the dynamics of solutions to the 2D NSE, the study of determining forms was initiated in [32, 33] for these equations. A determining form is an ODE in an *infinite-dimensional* Banach space of trajectories, which subsumes the dynamics of the original equation in a certain way. A stronger expression of finite-dimensionality is the existence of an *inertial manifold*, which is a finite-dimensional manifold that contains the global attractor of the original system and, moreover, attracts all solutions at an exponential rate. Restricted to the inertial manifold, the dynamics of the original system reduces to an ODE, known as an *inertial form*, in a *finite-dimensional* phase space. However, the existence of an inertial manifold for the 2D NSE has been an open problem since the 1980s.

In a joint work with M.S. Jolly, T. Sadigov, and E.S. Titi [47], we establish the existence of a determining form for the subcritical SQG equation that is induced by its corresponding feedback control system (4.2). In particular, we show that there exist Banach spaces, X, Y, and a map $W : X \to Y$, which is the solution operator of (4.2) corresponding to a "reference solution," $u \in X$, such that $I_hW : B_X^\rho(0) \to Y$ is *Lipschitz*, for some ball of radius $\rho > 0$, centered at 0 in X, where I_h is given by (smooth) projection onto Fourier modes $|k| \leq 2^{1/h}$. It then follows from [33], for instance, that the equation given by

$$\frac{dv(\cdot)}{d\tau}(\tau) = -\|v(\cdot)(\tau) - I_hW(v(\cdot)(\tau))\|_X^2(v(\cdot)(\tau) - I_h\theta^*), \quad v(0) = v_0 \in \mathcal{B}_X^\rho(0). \tag{4.3}$$

defines an *ODE* in X, where θ^* is a given steady state of (3.1).

Our proof of this result hinges on obtaining uniform estimates in L^∞ for solutions to (4.2) that are *independent* of the number of modes $m \sim 2^{1/h}$. Indeed, the L^p maximum principle that one is able to derive for (4.2), while uniform in t, *depends on the number of modes*, which precludes one from using these bounds to establish the desired Lipschitz property. Nevertheless, we show that by appealing to harmonic analysis tools, one can adapt De Giorgi-type techniques (cf. [8]), to deduce uniform bounds in L^∞ which are indeed independent of m.

5. Unique ergodicity

In the context of wave turbulence, I consider the damped-driven Korteweg-de Vries equation perturbed by a stochastic driving force. A typical assumption made in the study of turbulence is to assume *ergodicity* of the corresponding dynamical system, which is the property that long-time averages can be equated with "ensemble averages," that is, averages over all of the states of the system. This assumption is known as the Ergodic Hypothesis. The goal of "unique ergodicity," then, is to prove that this hypothesis is in fact a valid assumption. This can be done by demonstrating the existence and uniqueness of invariant measures for the system. Then, by virtue of being the unique invariant measure, it is necessarily ergodic. With N. Glatt-Holtz and G. Richards, *we verify the Ergodic Hypothesis in the context of the damped-driven, stochastic KdV equation.* In the course of doing so, we also establish the global approximate controllability of the deterministic damped-driven KdV equation, as well exponential mixing rates of the unique invariant measure in the regime of strong damping.

The model of interest is the stochastically perturbed, damped-driven KdV equation given by

$$du + (\gamma u + u_{xxx} + uu_x)dt = f dt + \sigma \cdot dW, \tag{5.1}$$

equipped with periodic boundary conditions, where u represents an amplitude, $\gamma > 0$ captures damping effects, f is an external, time-independent force, $\sigma \in (C^\infty(\mathbb{T}))^N$, and $W = (W_1, \dots, W_d)$, where $W_j = W_j(t)$ is a one-dimensional Brownian motion for each $j = 1, \dots, N$. Although the deterministic version of (5.1), i.e., when $\sigma \equiv 0$, was originally derived as a model for shallow water waves, it has since been realized as a canonical model equation. Indeed, when $\gamma = 0$ and $\sigma \equiv 0$, (5.1) appears in the study of atmospheric and oceanic internal solitary waves, mid-latitude and equatorial planetary waves, plasma waves, ion-acoustic waves, lattice waves, waves in elastic rods, and many other physical contexts [66].

In an ongoing joint work with N. Glatt-Holtz and G. Richards [40], we establish existence and uniqueness of an invariant, measure for (5.1), in the case where a large, but finite number of modes, $N = N(\gamma) \gg 1$, are forced stochastically, i.e., $\sigma = P_N\sigma$, where P_N is the projection onto Fourier modes $|k| \leq N$. We also establish the global approximate controllability of the system, as well as the regularity of the support of the invariant measure. We note that since ergodic measures are extremal points within the set of invariant measures for (5.1), the unique invariant measure must be ergodic, thereby establishing 'unique ergodicity'. Inspired by recent results [41, 50], we adopt the approach of asymptotic coupling, in which one couples (5.1) with its nudged stochastic counterpart (4.2); the 'coupling' of the statistics of these two equations then suffices to deduce unique ergodicity. Interestingly, the general framework developed by [41] is insufficient to capture the case of (5.1) outside of the regime of large damping due to the lack of stronger estimates required of the framework. In the general case, we therefore make appeal to the variation of the Doob-Khasminskii Theorem established by Hairer and Mattingly, which provide sufficient conditions for deducing uniqueness of invariant measures. Indeed, we are able to verify directly the properties that the Markov transition probabilities $(P_t)_{t\geq 0}$ of (5.1) are both asymptotically strong Feller and satisfy a form of irreducibility.

The case in consideration is known as the 'essentially elliptic' case, where the noise is sufficiently non-degenerate in the sense that all unstable directions, of which there are only finitely many are forced. In the context of turbulence, the number of such modes is finite, albeit large, and represents the number of degrees of freedom. The propagation of the noise through the system in this case is well-understood for the 2D NSE (see for instance [29]) since one can identify the mechanism for the existence of such modes through heat dissipation. The mechanism for the KdV equation is very different since one only has weak dissipation. Our analysis therefore sheds further light on the relation between the contractive property typically expected of this mechanism and proving unique ergodicity.

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