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Drawdowns and Rallies in a Finite Time-horizon

Drawdowns and Rallies

Hongzhong Zhang · Olympia Hadjiliadis

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Abstract In this work we derive the probability that a rally of a units precedes a drawdown of equal units in a random walk model and its continuous equivalent, a Brownian motion model in the presence of a finite time-horizon. A rally is defined as the difference of the present value of the holdings of an investor and its historical minimum, while the drawdown is defined as the difference of the historical maximum and its present value. We discuss applications of these results in finance and in particular risk management.

Keywords Drawdown · Rally · Random walk · Brownian motion

AMS 2000 Subject Classifications Primary 60G40; Secondary 91A60

1 Introduction

In this paper we first derive the probability that a rally of a units $(T_1(a))$ precedes a drawdown $(T_2(a))$ of equal units in a finite time-horizon T. The assumed underlying model considered is a random walk model. For this model we provide a closed-form formula for this probability both in the case of a symmetric random walk and in the

H. Zhang
Department of Mathematics, Graduate Center, C.U.N.Y., 365 Fifth Avenue,
Room 4208, New York, NY 10016-4309, USA
e-mail: hzhang3@gc.cuny.edu

O. Hadjiliadis (✉)
Department of Mathematics, Brooklyn College and the Graduate Center,
C.U.N.Y., New York, NY, USA
e-mail: ohadjiliadis@brooklyn.cuny.edu

case of a non-symmetric random walk. We then derive a closed-form formula for this probability in the case of a Brownian motion model. We use this result to address the problem of computing the probability that a rally of $(100 \times \alpha)\%$ from the running minimum of a stock price occurs before a drawdown of $(100 \times \beta)\%$ from its running minimum, given that the stock price follows a geometric Brownian motion.

Drawdowns provide a dynamic measure of risk in that they measure the drop of a stock price, index or value of a portfolio from its running maximum. They thus provide portfolio managers a tool with which to assess the risk taken by a mutual fund during a given economic cycle, i.e. a peak followed by a trough followed by a peak. The fact that they are reset to 0 every time a cycle of a peak and a trough followed by a peak is completed, renders them unbiased with respect to time, contrary to the maximum drawdown which is a measure that is non-decreasing with respect to time and thus has an increasing bias.

In general, risk management of drawdowns and portfolio optimization with drawdown constraints has become increasingly important among the practitioners. Chekhlov et al. (2005) studied drawdown measures in portfolio optimization. Magdon-Ismail et al. (2004) determined the distribution of the maximum drawdown of Brownian motion, based on which they described another time-adjusted measure of performance based known as the the Calmar ratio (Magdon-Ismail and Atiya 2004). Related works include Vecer (2006, 2007), Pospisil and Vecer (2008) and Pospisil et al. (2009). An overview of the existing techniques for analysis of market crashes as well as a collection of empirical studies of the drawdown and the maximum drawdown please refer to Sornette (2003).

Drawdowns also provide investors a measure of "relative regret" while rallies can be perceived as measures of "relative satisfaction". Thus the first time of a drawdown or a rally of a certain number of units may signal the time in which an investor may choose to change his/her investment position depending on his/her perception of future moves of the market and his/her risk aversion. The probability computed in this paper can be thought of as the probability that an investor who makes decisions based on the relative change in his/her wealth will exit on an upward rally (or a drawdown) of his wealth given a finite time-horizon in which investment can take place. One can view this problem as an extension of the classical gambler's ruin problem (Ross 2008), in which investors with a finite time-horizon make decisions based on the relative wealth process.

This paper extends the results of Hadjiliadis (2005) in discrete time and of Hadjiliadis and Vecer (2006) in the continuous time drifted Brownian motion model. It addresses the same question but in the more realistic setting of a finite time-horizon, which is usually available to investors.

In Section 2 we compute the probability of a rally preceding a drawdown in a finite time-horizon both in the case of a symmetric and in the case of a non-symmetric random walk. In Section 3 we extend these results to the continuous time Brownian motion model. In Section 4 we use the results of Section 3 to address the question of computing the probability that a rally of $(100 \times \alpha)\%$ from the running minimum of a stock price occurs before a drawdown of $(100 \times \beta)\%$ from its running minimum, given that the stock price follows a geometric Brownian motion. We finally conclude with some closing remarks in Section 5.

2 Rallies and Drawdowns in the Random Walk Model

We begin with a mathematical definition of a rally and a drawdown. To this effect consider the following random walk

$$X_n = \sum_{i=1}^{n} Z_i, \ \ X_0 = 0, \tag{1}$$

where

$$Z_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } q. \end{cases}$$

That is, the process $\{X_n\}_{n \geq 1}$ is a simple random walk with parameter p. The upward rally (or rally) and drawdown processes are then defined respectively as

$$UR_n = X_n - \min_{0 \leq k \leq n} X_k, \tag{2}$$

$$DD_n = \max_{0 \leq k \leq n} X_k - X_n. \tag{3}$$

A rally of a units and a drawdown of b units are then defined respectively as

$$T_1(a) = \min\{n \geq 1 | \, UR_n = a\}, \ \ a = 1, 2, \ldots. \tag{4}$$

$$T_2(b) = \min\{n \geq 1 | \, DD_n = b\}, \ \ b = 1, 2, \ldots. \tag{5}$$

In the next theorem we compute the probability that a rally of a units precedes a drawdown of equal units in a pre-specified finite time-horizon T, where $T > a$. It is important to point out that this probability is asymmetric with respect to $T_1(a)$ and $T_2(a)$. This is seen through the fact that it can be expressed as $P(T_1(a) \wedge T < T_2(a) \wedge T)$, or as $P(T_1(a) < T_2(a) \wedge T)$.

Theorem 1 *Let $a, T \in \mathcal{N}^*$, $X_n = \sum_{i=1}^{n} Z_i$ be a simple random walk with parameter p and $T_i(a)$, $i = 1, 2$, be the stopping times of Eqs. 4 and 5 respectively. Define*

$$\wp(T; a, p) = P(T_1(a) \wedge T < T_2(a) \wedge T). \tag{6}$$

The probability that an upward rally of a units proceeds a drawdown of equal units before time $T > a$ is given by

1. *for $a = 1$,*

$$\wp(T; 1, p) = p. \tag{7}$$

2. *for $a = 2$,*

$$\wp(T; 2, p) = p^2 + qp^2 + pqp^2 + \ldots + \underbrace{\ldots qpqp^2}_{(T-1) terms}. \tag{8}$$

3. *for $a \geq 3$,*

$$\wp(T; a, p) = p^a + \sum_{L=a+2}^{T} \sum_{i=1}^{a} \sum_{k=0}^{L-a-1} c_{i,1}^{a,L-a-k-1} \cdot c_{1,a-2}^{a-1,a+k-3} \cdot p^{\frac{L+a-i}{2}} q^{\frac{L-a+i-2}{2}}, \tag{9}$$

where for $m, k, i, j \in \mathcal{N}$,

$$c_{i,j}^{m,k} = \frac{2^{k+1}}{m+1} \sum_{\iota=1}^{m} \left(\cos \frac{\pi \iota}{m+1} \right)^k \sin \frac{i\pi\iota}{m+1} \sin \frac{j\pi\iota}{m+1}. \tag{10}$$

In order to proceed with the proof of this theorem, we will need to make use of two preliminary lemmas. In the first lemma we compute the probability that a random walk, which starts at 0 reaches a specific level $-1 \leq v \leq B$ in N steps, while remaining within a positive strip of a pre-specified height A.

Lemma 1 *For $u, v, A, N \in \mathcal{N}$ and $0 \leq u, v \leq A$, we have*

$$P(X_N = v, 0 \leq X_k \leq A \ \ for \ \forall k \leq N | X_0 = u) = c_{u+1,v+1}^{A+1,N} \cdot p^{\frac{N-u+v}{2}} q^{\frac{N+u-v}{2}}, \tag{11}$$

where $c_{u+1,v+1}^{A+1,N}$ is defined in Eq. 10.

Proof The 1-step transition matrix of a simple random walk on $[0, A]$ is the Toeplitz matrix M_{A+1} generated by column vector c and row vector r, where

$$\mathrm{c} = (\underbrace{0, q, 0, \ldots, 0}_{A+1}) \quad \mathrm{r} = (\underbrace{0, p, 0, \ldots, 0}_{A+1}).$$

The N-step transition matrix is the N-th power of that matrix. The probability in Eq. 11 is the $(u+1, v+1)$-th entry of this N-step transition matrix. Using Theorem 2.3 on page 1064 of Salkuyeh (2006), the result follows. □

In the second lemma we compute the probability that a random walk, which starts at 0 reaches a specific level v in N steps while its minimum reaches the exact level $v - B$ and its maximum never exceeds $v + 1$. We denote this probability by $g(N, v; B)$.

Lemma 2 *For $B, N \in \mathcal{N}$ with $B \leq N$, and $v = -1, 0, \ldots, B$, define*

$$g(N, v; B) = P(X_N = v, \max_{k \leq N} X_k \leq v + 1, \min_{k \leq N} X_k = v - B | X_0 = 0). \tag{12}$$

We have

$$g(N, v; B) = \sum_{k=0}^{N-B} c_{B-v+1,1}^{B+2,N-B-k} \cdot c_{1,B}^{B+1,B+k-1} \cdot p^{\frac{N+v}{2}} q^{\frac{N-v}{2}}, \tag{13}$$

with coefficient $c_{i,j}^{m,k}$ defined in Eq. 10.

Proof With $g(N, v; B)$ as in Eq. 12 we notice that

$$g(N, -1; B) = q \cdot g(N-1, 0; B), \tag{14}$$

$$g(N, B; B) = p \cdot g(N-1, B-1; B) + p \cdot g(N-1, B-1; B-1), \tag{15}$$

and for $-1 < v < B$ that,

$$g(N, v; B) = p \cdot g(N-1, v-1; B) + q \cdot g(N-1, v+1; B). \tag{16}$$

To see Eq. 15, we observe that $g(N, B; B)$ is the probability of an event that only includes paths on which the process remains non-negative. Equation 15 represents the decomposition of these paths into the ones on which the process stays strictly positive after the first upward step, and the ones on which it does not. Equation 16 follows by conditioning on the first step being up or down respectively.

Equations 14, 15, and 16 can be summarized by

$$G_N^{(B)} = M_{B+2} \cdot G_{N-1}^{(B)} + Y_{N-1}^{(B)}, \tag{17}$$

where M_{B+2} is the 1-step transition matrix of a simple random walk on $[-1, B+1]$ which appears in the proof of Lemma 1, $G_N^{(B)}$ and $Y_N^{(B)}$ are the $(B+2) \times 1$ vectors

$$G_N^{(B)} = (g(N, B; B), g(N, B-1; B), \ldots, g(N, -1; B))^{\tau}, \tag{18}$$

and

$$Y_N^{(B)} = (p \cdot g(N, B-1; B-1), 0, \ldots, 0)^{\tau}, \tag{19}$$

respectively, while

$$G_B^{(B)} = Y_{B-1}^{(B)} = (p^B, 0, \ldots, 0)^{\tau}, \tag{20}$$

We can now use Eq. 17 recursively to obtain

$$\begin{aligned} G_N^{(B)} &= [M_{B+2}]^{N-B} \cdot G_B^{(B)} + \sum_{k=0}^{N-B-1} [M_{B+2}]^{N-B-k-1} \cdot Y_{B+k}^{(B)} \\ &= \sum_{k=0}^{N-B} [M_{B+2}]^{N-B-k} \cdot Y_{B+k-1}^{(B)}. \end{aligned} \tag{21}$$

Equation 13 now follows from Eq. 21, Theorem 2.3 on page 1064 of Salkuyeh (2006), and Lemma 1. □

We can now proceed to the proof of Theorem 1.

Proof of Theorem 1 Equations 7 and 8 are easy to see. For $a \geq 3$ it is also easy to see that

$$\wp(a+1; a, p) = p^a. \tag{22}$$

In order to establish Eq. 9, it suffices to determine

$$\begin{aligned} \Delta(T; a, p) &= \wp(T; a, p) - \wp(T-1; a, p) \\ &= P(T_1(a) = T-1, \max_{k \le T-1} DD_k \le a-1), \end{aligned} \tag{23}$$

for any $a, T \in \mathcal{N}^*$ and $T > a + 1 \ge 4$.

We begin by examining the properties of all paths which are included in the event of Eq. 23.

1. For all such paths,

$$X_{T-1} \in \{1, 2, \ldots, a\},$$

for otherwise, a drawdown of a units precedes a rally of equal size, or the range is less than a at time $T-1$.

2. Let us assume $X_{T-1} = x \in \{1, 2, \ldots, a\}$, then

$$\min_{k \le T-1} X_k = x - a.$$

3. Assume $X_{T-1} = x \in \{1, 2, \ldots, a\}$, then

$$X_{T-2} = x-1,\ X_{T-3} = x-2,\ \max_{k \le T-3} X_k \le x-1.$$

This is because the rally (which precedes the drawdown) is achieved by an upward move of the random walk $\{X_n\}_{n \ge 1}$; moreover, the highest position of the random walk before $T-1$ can be at most $x-1$.

These properties give rise to the following representation

$$\Delta(T; a, p) = p^2 \cdot \sum_{v=-1}^{a-2} g(T-3, v; a-2). \tag{24}$$

Using Lemma 2, the result follows. This completes the proof of Theorem 1. □

In the case that an investor is not restricted by a finite time-horizon, the probability that his/her wealth makes a rally of a units before a drawdown of equal units is summarized in the following corollary. This result is easier derived by using martingale arguments (Hadjiliadis 2005) and is displayed for completeness.

Corollary 1 *In the case of an infinite time-horizon we have*

$$P(T_1(a) < T_2(a)) = \frac{\left(\frac{p}{q}\right)^{a+1} - (a+1)\left(\frac{p}{q}\right) + a}{\left[1 - \left(\frac{p}{q}\right)^{a}\right]\left[\left(\frac{q}{p}\right)^{a+1} - 1\right]}, \tag{25}$$

The next corollary draws a connection of our result to the range process which is defined to be the difference of the running maximum and the running minimum.

Corollary 2 *Let $R_n = \max_{k \leq n} X_k - \min_{k \leq n} X_k$ be the range process of a random walk with parameter p. Then for $T > a$, we have*

1. *for $a = 2$,*

$$P(R_{T-1} < 2) = 1 - p^2(1 + q + pq + \ldots + \underbrace{\ldots qpq}_{(T-3)terms}) - q^2(1 + p + qp + \ldots + \underbrace{\ldots pqp}_{(T-3)terms}). \tag{26}$$

2. *for $a \geq 3$,*

$$P(R_{T-1} < a) = 1 - p^a - q^a - \sum_{L=a+2}^{T} \sum_{i=1}^{a} \sum_{k=0}^{L-a-1} \left\{ c_{i,1}^{a,L-a-k-1} \cdot c_{1,a-2}^{a-1,a+k-3} \times (pq)^{\frac{L-2}{2}} \left[p\left(\frac{p}{q}\right)^{\frac{a-i}{2}} + q\left(\frac{q}{p}\right)^{\frac{a-i}{2}} \right] \right\}. \tag{27}$$

Proof We observe that

$$P(R_{T-1} \geq a) = P(T_1(a) \wedge T < T_2(a) \wedge T) + P(T_1(a) \wedge T > T_2(a) \wedge T), \tag{28}$$

where the first term of the right hand side is given in Theorem 1 and the second term of the right hand side is given in Theorem 1 when p is replaced by q. □

The result in Corollary 2 can be compared with Proposition 14 of Vallois (1996).

Remark 1 In the case of a symmetric random walk ($p = q = \frac{1}{2}$) we notice that we can write

$$P(T_1(a) \wedge T < T_2(a) \wedge T) = \frac{1}{2} P(\theta(a) < T), \tag{29}$$

where $\theta(a) = \inf\{n \geq 1 | R_n \geq a\}$. It is now easy to deduce that as $T \to \infty$ Eq. 29 reduces to $\frac{1}{2}$ as expected. Finally, the case of a symmetric random walk ($p = q = \frac{1}{2}$) is summarized in the following corollary for any pre-specified time-horizon T.

Corollary 3 *Let $a, T \in \mathcal{N}^*$. For the symmetric random walk the probability that a rally of a units proceeds a drawdown of equal units before time T is given by*

1. *for $a = 1$,*

$$P(T_1(1) \wedge T < T_2(1) \wedge T) = \frac{1}{2}. \tag{30}$$

Table 1 The probability of Eq. 6 for $T = 30$

$a \downarrow$	$p = 0.3$	$p = 0.5$	$p = 0.7$
5	0.0630	0.4684	0.6382
10	0.0012	0.1040	0.3772
20	1.0945×10^{-8}	1.6319×10^{-4}	0.0272

Table 2 The probability of Eq. 6 for $T = 50$

$a \downarrow$	$p = 0.3$	$p = 0.5$	$p = 0.7$
5	0.0640	0.4981	0.6413
10	0.0023	0.2609	0.4595
20	2.3012×10^{-7}	0.0064	0.2586

2. *for* $a = 2$,

$$P(T_1(2) \wedge T < T_2(2) \wedge T) = \frac{1}{2}\left(1 - \frac{1}{2^{T-1}}\right). \tag{31}$$

3. *for* $a \geq 3$,

$$P(T_1(a) \wedge T < T_2(a) \wedge T) = \frac{1}{2^a} + \frac{1}{2}\sum_{L=a+2}^{T}\sum_{i=1}^{a}\sum_{k=0}^{L-a-1} d_{i,1}^{a,L-a-k-1} \cdot d_{1,a-2}^{a-1,a+k-3} \tag{32}$$

where for $m, k, i, j \in \mathcal{N}$,

$$d_{i,j}^{m,k} = \frac{1}{m+1}\sum_{\iota=1}^{m}\left(\cos\frac{\pi\iota}{m+1}\right)^k \sin\frac{i\pi\iota}{m+1}\sin\frac{j\pi\iota}{m+1}. \tag{33}$$

Proof The proof is seen by substituting $p = q = \frac{1}{2}$. □

In Tables 1 and 2 we calculate the probability of Eq. 6 for specific values of the parameters p, a, and T. We notice that both Tables 1 and 2 increase across rows reflecting the fact that as p increases so does the probability of Eq. 6. On the other hand, as the threshold a increases, the probability of Eq. 6 typically decreases. However, in the case that $p = 0.7 (> 0.5)$ the probability of Eq. 6 experiences a slight increase from $a = 1$ to $a = 2$ followed by a dramatic decrease. This is seen in Fig. 1. Finally, as the time-horizon T increases the probability of Eq. 6 increases as well. However, for small values of a, the increase is not as dramatic as for larger values of a.

We now proceed to the continuous time case.

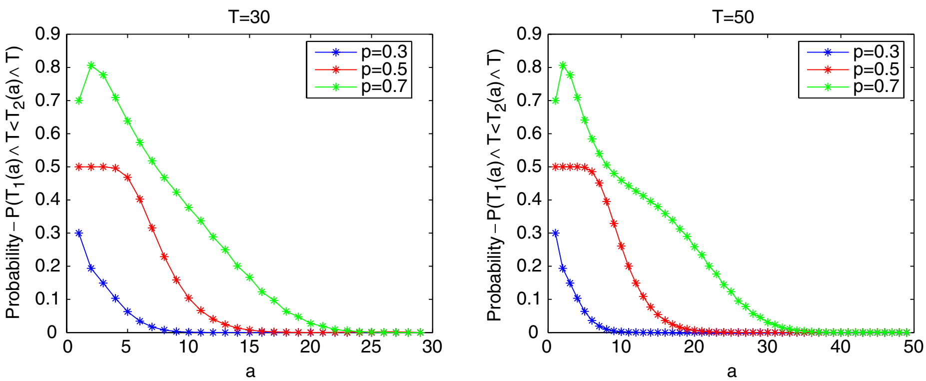

Fig. 1 A graph of the probability of Eq. 6 for *Left:* $T = 30$ and *Right:* $T = 50$

3 Rallies and Drawdowns in a Brownian Motion Model

In this section we consider the case of a continuous time Brownian motion with drift parameter ν and diffusion parameter σ. In particular, let

$$dX_t = \nu dt + \sigma dW_t, \;\; X_0 = 0, \tag{34}$$

where $\nu \in \mathcal{R}$ is the drift coefficient and $\sigma > 0$ is the diffusion coefficient.

Similarly to the discrete-time random walk model, we define an upward rally (or rally) process as

$$X_t - \inf_{s \le t} X_s, \tag{35}$$

and a drawdown process as

$$\sup_{s \le t} X_s - X_t. \tag{36}$$

A rally of a units and a drawdown of b units are then defined respectively as

$$T_1(a) = \inf\{t \ge 0 |\, X_t - \inf_{s \le t} X_s = a\}, \;\; a \in \mathcal{R}_+ \tag{37}$$

$$T_2(b) = \inf\{t \ge 0 |\, \sup_{s \le t} X_s - X_t = b\}, \;\; b \in \mathcal{R}_+. \tag{38}$$

In the theorem that follows we compute the probability that a rally of a units precedes a drawdown of equal units in a pre-specified finite time-horizon T. This probability is asymmetric with respect to $T_1(a)$ and $T_2(a)$ since it can be expressed as $P(T_1(a) \wedge T < T_2(a) \wedge T)$, or as $P(T_1(a) < T_2(a) \wedge T)$.

Theorem 2 *Let $dX_t = \nu dt + \sigma dW_t$ be the Brownian motion with drift coefficient ν and diffusion coefficient σ, and let $T_i(a)$, $i = 1, 2$, be stopping times of Eqs. 37 and 38 respectively. Define*

$$\wp(T; a, \nu, \sigma) = P(T_1(a) \wedge T < T_2(a) \wedge T). \tag{39}$$

Then,

$$\begin{aligned}\wp(T; a, \nu, \sigma)& \\ = \sum_{n=1}^{\infty} \frac{2n^2\pi^2}{C_n^2} &\left\{ (1 - (-1)^n e^{\frac{\nu a}{\sigma^2}}) \left(1 - \frac{4\nu^2 a^2}{\sigma^4 C_n}\right) - (-1)^n \frac{\nu a}{\sigma^2} e^{\frac{\nu a}{\sigma^2}} - \exp\left(-\frac{\sigma^2 C_n}{2a^2} T\right) \right. \\ &\left. \times \left[(1 - (-1)^n e^{\frac{\nu a}{\sigma^2}}) \left(1 + \frac{n^2\pi^2\sigma^2 T}{a^2} - \frac{4\nu^2 a^2}{\sigma^4 C_n}\right) - (-1)^n \frac{\nu a}{\sigma^2} e^{\frac{\nu a}{\sigma^2}} \right] \right\},\end{aligned} \tag{40}$$

where $C_n = n^2\pi^2 + \nu^2 a^2/\sigma^4$, $n \in \mathcal{N}$.

The proof of the above theorem makes use of the following proposition:

Proposition 1 *For $t > 0$ and $0 < x \leq a$, we have*

$$P(T_1(a) \in dt, T_2(a) > t, X_t \in dx) = g(t, x; a, \nu, \sigma)dtdx, \tag{41}$$

where

$$\begin{aligned} g(t, x; a, \nu, \sigma) = \frac{\sigma^2}{a^5} \sum_{n=1}^{\infty} n\pi \exp\left(-\frac{\sigma^2 C_n}{2a^2} t + \frac{\nu}{\sigma^2} x\right) \\ \times \left\{(n^2\pi^2\sigma^2 t - 2a^2) \sin\left(\frac{n\pi}{a} x\right) - n\pi a x \cos\left(\frac{n\pi x}{a}\right)\right\}, \end{aligned} \tag{42}$$

with C_n , $n \in \mathcal{N}$ defined as above.

In order to prove Proposition 1 and Theorem 2, we will need the following lemma.

Lemma 3 *For $0 < x \leq a$, define*

$$\tau_x = \inf\{t \geq 0 | X_t = x\}. \tag{43}$$

We have

$$P(\tau_x \in dt, \inf_{s \leq t} X_s \geq x - a) = h(t, x; a, \nu, \sigma)dt, \tag{44}$$

where

$$\begin{aligned} h(t, x; a, \nu, \sigma) &= \frac{1}{\sigma t^{\frac{3}{2}}} \exp\left(\frac{\nu}{\sigma^2} x - \frac{\nu^2}{2\sigma^2} t\right) \sum_{k=-\infty}^{\infty} (x + 2ka)\phi\left(\frac{x + 2ka}{\sigma\sqrt{t}}\right) \\ &= \frac{\sigma^2}{a^2} \exp\left(\frac{\nu}{\sigma^2} x - \frac{\nu^2}{2\sigma^2} t\right) \sum_{n=1}^{\infty} (n\pi) \exp\left(-\frac{n^2\pi^2\sigma^2}{2a^2} t\right) \sin\left(\frac{n\pi x}{a}\right). \end{aligned} \tag{45}$$

Proof The proof follows by recognizing that, $h(t, x; a, \nu, \sigma)$ appears in Anderson (1960), Theorem 5.1. In particular, $h(t, x; a, \nu, \sigma)$ is $dP_1(t)/dt$ of Eq. 5.3 with parameters $\gamma_1 = x/\sigma, \gamma_2 = (x - a)/\sigma$ and $\delta_1 = \delta_2 = -\nu/\sigma$. More specifically, after substitution and some algebra, we obtain

$$\begin{aligned} &\frac{1}{t^{\frac{3}{2}}} \phi\left(\frac{\delta_1 t + \gamma_1}{\sqrt{t}}\right) \sum_{k=0}^{\infty} e^{-(2k/t)[(k+1)\gamma_1 - k\gamma_2][\delta_1 t + \gamma_1 - (\delta_2 t + \gamma_2)]} [(2k + 1)\gamma_1 - 2k\gamma_2] \\ &= \frac{1}{\sigma t^{\frac{3}{2}}} \exp\left(\frac{\nu}{\sigma^2} x - \frac{\nu^2}{2\sigma^2} t\right) \sum_{k=0}^{\infty} (x + 2ka)\phi\left(\frac{x + 2ka}{\sigma\sqrt{t}}\right), \end{aligned}$$

while

$$\frac{1}{t^{\frac{3}{2}}}\phi\left(\frac{\delta_1 t+\gamma_1}{\sqrt{t}}\right)\sum_{k=0}^{\infty}e^{-[2(k+1)/t][k\gamma_1-(k+1)\gamma_2][\delta_1 t+\gamma_1-(\delta_2 t+\gamma_2)]}[(2k+1)\gamma_1-2(k+1)\gamma_2]$$

$$=\frac{1}{\sigma t^{\frac{3}{2}}}\exp\left(\frac{\nu}{\sigma^2}x-\frac{\nu^2}{2\sigma^2}t\right)\sum_{k=0}^{\infty}[-x+2(k+1)a]\phi\left(\frac{x-2(k+1)a}{\sigma\sqrt{t}}\right).$$

By combining the above two identities we obtain the upper expression in Eq. 45. The last expression in Eq. 45 is obtained by a Fourier transform. □

We now proceed to the proof of Proposition 1.

Proof of Proposition 1 Observe that

$$\{T_1(a)\in dt,\, T_2(a)>t,\, X_t\in dx\}=\{\tau_x\in dt,\, \inf_{s\le t}X_s\in x-da\}, \tag{46}$$

from which we obtain

$$g(x,t;a,\nu,\sigma)=\frac{\partial}{\partial a}h(t,x;a,\nu,\sigma). \tag{47}$$

This completes the proof of Proposition 1. □

We can now proceed to the proof of Theorem 2.

Proof of Theorem 2 We use Proposition 1 to obtain

$$P(T_1(a)\wedge T<T_2(a)\wedge T)=\int_0^T\int_0^a P(T_1(a)\in dt,\, T_2(a)>t,\, X_t\in dx), \tag{48}$$

which completes the proof of Theorem 2. □

In the case that an investor is not restricted by a finite time-horizon, the probability that his/her wealth makes a rally of a units before a drawdown of equal units in the model of Eq. 34 is summarized in the following corollary. This result is easier derived by using martingale arguments (Hadjiliadis 2005; Hadjiliadis and Vecer 2006) and is displayed here for completeness.

Corollary 4 *In the case of an infinite time-horizon we have*

$$P(T_1(a)<T_2(a))=\frac{e^{\frac{2\nu}{\sigma^2}a}-\frac{2\nu}{\sigma^2}a-1}{e^{\frac{2\nu}{\sigma^2}a}+e^{-\frac{2\nu}{\sigma^2}a}-2}.$$

The next corollary draws a connection of our result to the range process of a Brownian motion.

Corollary 5 *Let $R_t = \sup_{s\le t} X_s - \inf_{s\le t} X_s$ be the range process of Eq. 34. Then*

$$P(R(T) \le a) = \sum_{n=1}^{\infty} \frac{4n^2\pi^2}{C_n^2} \exp\left(-\frac{\sigma^2 C_n}{2a^2} T\right) \Big\{ \big(1-(-1)^n \cosh(\nu a/\sigma^2)\big) \times \left(1+\frac{n^2\pi^2\sigma^2}{a^2}T - \frac{4\nu^2 a^2}{\sigma^4 C_n}\right) - (-1)^n \frac{\nu a}{\sigma^2}\sinh(\nu a/\sigma^2)\Big\}. \quad (49)$$

Proof Define the first passage time of range process R_t by

$$\theta(a) = \inf\{t \ge 0 | R_t = a\}, \quad (50)$$

then

$$\theta(a) = T_1(a) \wedge T_2(a). \quad (51)$$

Therefore, we have

$$\begin{aligned} P(R(T) \le a) &= P(\theta(a) \ge T) = 1 - P(\theta(a) < T) \\ &= 1 - P(T_1(a) \wedge T < T_2(a) \wedge T) - P(T_1(a) \wedge T > T_2(a) \wedge T) \\ &= 1 - \wp(T; a, \nu, \sigma) - \wp(T; a, -\nu, \sigma). \end{aligned} \quad (52)$$

The result follows from Theorem 2. □

The result in Corollary 5 is also seen in Tanré and Vallois (2006). The density of the range in the special case of a Brownian motion without a drift can be found in Eq. 3.6 of Feller (1951).

The case of a Brownian motion without a drift is summarized in the following corollary:

Corollary 6

$$P(T_1(a) \wedge T < T_2(a) \wedge T) = \frac{1}{2} - \sum_{n\ge 1, odd} \frac{4}{n^2\pi^2} \exp\left(-\frac{n^2\pi^2\sigma^2}{2a^2}T\right) \cdot \left(1 + \frac{n^2\pi^2\sigma^2}{a^2}T\right). \quad (53)$$

We notice that Eq. 53 of Corollary 6 reduces to $\frac{1}{2}$ as $T \to \infty$ as expected.

We now proceed to apply these results in the case of a geometric Brownian motion model.

4 Applications

Consider the case of a stock with geometric Brownian motion dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \; S_0 = 1. \quad (54)$$

Table 3 The probability of Eq. 60 for $\sigma = 15\%$ and $T = 1$

$100 \times \alpha \downarrow$	$\nu = 10\%$	$\nu = 12\%$	$\nu = 15\%$
20%	0.6173	0.6610	0.7228
30%	0.3449	0.3906	0.4623
50%	0.0575	0.0736	0.1040

Table 4 The probability of Eq. 60 for $\sigma = 20\%$ and $T = 1$

$100 \times \alpha \downarrow$	$\nu = 12\%$	$\nu = 15\%$	$\nu = 17\%$
20%	0.6603	0.7002	0.7254
30%	0.5508	0.6028	0.6367
50%	0.2053	0.2468	0.2770

Using Theorem 2, we are in a position to address the following question:

What is the probability that this stock would rise by $(100 \times \alpha)$**% before it incurs a drop of** $(100 \times \beta)$**% in a pre-specified time-horizon** T**?**

First observe that

$$d \log S_t = \nu dt + \sigma dW_t, \ \ \log S_0 = 0, \tag{55}$$

where $\nu = \mu - \frac{1}{2}\sigma^2$ represents the logarithm of the return of the stock (Luenberger 1998).

Now define the running maximum and the running minimum of the stock process $\{S_t\}$

$$M_t = \sup_{s \leq t} S_s, \tag{56}$$

$$N_t = \inf_{s \leq t} S_s. \tag{57}$$

We also let $U_1(\alpha)$ be the first time the stock rises by $(100 \times \alpha)\%$ from its historical low and $U_2(\beta)$ the first time that the stock drops by an amount equal to $(100 \times \beta)\%$ from its historical high, where $(1 + \alpha)(1 - \beta) = 1$. That is,

$$U_1(\alpha) = \inf\{t \geq 0 | \ S_t = (1 + \alpha)N_t\}, \tag{58}$$

$$U_2(\beta) = \inf\{t \geq 0 | \ S_t = (1 - \beta)M_t\}. \tag{59}$$

Thus, it is possible to calculate the exact expression for the probability that a percentage relative rise of $(100 \times \alpha)\%$ precedes a relative drop of $(100 \times \beta)\%$ $(\beta = \frac{\alpha}{1+\alpha})$ by noticing that

$$P(U_1(\alpha) \wedge T < U_2(\beta) \wedge T) = P(T_1(\log(1 + \alpha)) \wedge T < T_2(-\log(1 - \beta)) \wedge T), \tag{60}$$

the latter of which is given in Theorem 2.

In Tables 3 and 4 we calculate the probability of Eq. 60 for specific values of the yearly logarithmic return ν of a stock and its volatility σ. We fix the time-horizon to $T = 1$ year and give $(100 \times \alpha)\%$ the values 20%, 30%, and 50% respectively. We notice that both Tables 3 and 4 increase across rows reflecting the fact that as the return of the stock increases so does the probability of Eq. 60. Moreover, we notice

Table 5 The probability of Eq. 60 for $\sigma = 15\%$ and $T = 1$

$100 \times \alpha \downarrow$	$\nu = -10\%$	$\nu = -12\%$	$\nu = -15\%$
20%	0.1990	0.1695	0.1312
30%	0.0561	0.0441	0.0302
50%	0.0023	0.0016	0.0008

Table 6 The probability of Eq. 60 for $\sigma = 20\%$ and $T = 1$

$100 \times \alpha \downarrow$	$\nu = -12\%$	$\nu = -15\%$	$\nu = -17\%$
20%	0.3163	0.2785	0.2547
30%	0.1800	0.1486	0.1300
50%	0.0280	0.0204	0.0164

that observing the process of Eq. 55 from one unit of time (say [0, 1]) is equivalent to observing the process $\{Y_t\}$ on the interval $[0, \sigma^2]$, where

$$dY_t = \frac{\nu}{\sigma^2}dt + d\tilde{W}_t, \ \ Y_0 = 0, \tag{61}$$

and $\{\tilde{W}_t\}$ is a re-scaled Brownian motion given by $\tilde{W}_{\sigma^2 t} = \sigma W_t$, because they have the same law. Thus, in the case in which $\nu > 0$ the effect of increasing the volatility decreases the drift of the process $\{Y_t\}$ while increasing the interval of observation. A smaller drift delays an upward rally, while a longer period of observation increases the probability of observing the rally in a finite time-horizon. The numerical results of Tables 3 and 4 indicate that the latter effect is typically stronger (especially for bigger α). In the case in which $\nu < 0$, the effect of increasing the volatility increases the drift and the interval of observation of $\{Y_t\}$. Thus, in this case, increasing the volatility results in higher values of the probability in Eq. 60. This fact is clearly seen by comparing the entries of Tables 5 and 6. The effect of increasing the threshold α on the probability of Eq. 60 is seen in Fig. 2 for the case in which the volatility σ is set to 15% (Left) and $\sigma = 20\%$ (Right). In both figures it is seen that the probability of Eq. 60 decreases roughly exponentially as α increases. However, an interesting feature which is seen in Fig. 2 (Right) is that for the smaller values of α in the neighborhood of 0.20 to 0.22, the probability of Eq. 60 initially experiences a small increase before exponentially decreasing to 0 as α increases. This fact reflects that if the volatility of a stock is high, it may be beneficial to an investor who makes decisions based on a the relative change of his/her wealth to favor rallies of moderate levels before withdrawing his/her investment as this is more likely to precede a

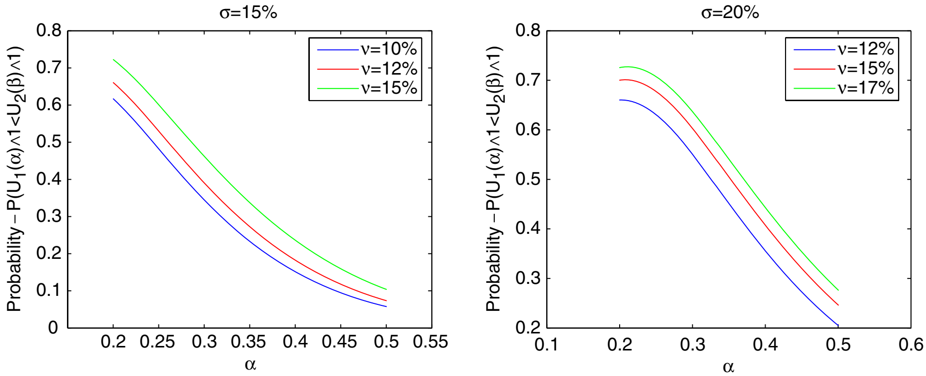

Fig. 2 A graph of the probability of Eq. 60 for $T = 1$ year, *Left:* $\sigma = 15\%$ and $\nu = 10\%$ (*blue line*), $\nu = 12\%$ (*red line*), $\nu = 15\%$ (*green line*); and *Right:* $\sigma = 20\%$, and $\nu = 12\%$ (*blue line*), $\nu = 15\%$ (*red line*), $\nu = 17\%$ (*green line*) [colors can be seen in .pdf version]

drawdown of a given level $\beta = \frac{\alpha}{1+\alpha}$ and that this rally will be realized before a pre-specified time-horizon T.

5 Concluding Remarks

In this paper we derive a closed-form expression for the probability that a rally of a units precedes a drawdown of equal units in a pre-specified finite time-horizon for a non-symmetric random walk model. We then generalized this result to the drifted Brownian motion model, a model for which we are able to also derive a closed-form expression for the probability of the relevant event. We then apply this result to address the question of what is the probability that a rally of $(100 \times \alpha)$ % precedes a drawdown of $(100 \times \beta)$ % in a pre-specified time-horizon under geometric Brownian motion dynamics. An investor usually has a finite time-horizon in which to make decisions regarding withdrawing or investing more of his/her wealth in a fund. In this paper we derive closed-form expressions for the probability that his/her wealth may incur a relative rise of a certain percentage before a relative drop of a certain percentage in a finite time-horizon. Although the geometric Brownian motion model is very restrictive and in many cases fails to provide an objective model of stock returns, it has historically been used as a prototypical model. Moreover, our derivation may be used as a benchmark in more realistic general dependence models for which the derivation of a closed form expression for the probability in Eq. 60 would be impossible.

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