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Drawdowns and the Speed of Market Crash

Hongzhong Zhang · Olympia Hadjiliadis

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Abstract In this paper we examine the probabilistic behavior of two quantities closely related to market crashes. The first is the drawdown of an asset and the second is the duration of time between the last reset of the maximum before the drawdown and the time of the drawdown. The former is the first time the current drop of an investor's wealth from its historical maximum reaches a pre-specified level and has been used extensively as a path-dependent measure of a market crash in the financial risk management literature. The latter is the speed at which the drawdown occurs and thus provides a measure of how fast a market crash takes place. We call this the speed of market crash. In this work we derive the joint Laplace transform of the last visit time of the maximum of a process preceding the drawdown, the speed of market crash, and the maximum of the process under general diffusion dynamics. We discuss applications of these results in the pricing of insurance claims related to the drawdown and its speed. Our applications are developed under the drifted Brownian motion model and the constant elasticity of variance (CEV) model.

Keywords Drawdown · Speed of market crash · Diffusions · Drawdown insurance

AMS 2000 Subject Classifications Primary 60G40; Secondary 91A60

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H. Zhang
Department of Statistics, Columbia University, 1255 Amsterdam ave,
Room SSW 1010, New York, NY 10027, USA
e-mail: hz2244@columbia.edu

O. Hadjiliadis (✉)
Department of Mathematics, Brooklyn College
and the Graduate Center C.U.N.Y., Brooklyn, NY, USA
e-mail: ohadjiliadis@brooklyn.cuny.edu

1 Introduction

In this paper we derive the joint Laplace transform of the last visit time of the maximum of a process preceding the drawdown, the speed of market crash, and the maximum of the process under general diffusion dynamics. The drawdown of an asset is the first time that its drawdown process reaches a pre-specified level K and is denoted by τ_K^D. The drawdown process of the process $\{X_t\}_{t\geq 0}$ is defined as its current drop from the running maximum $\overline{X}_t$. We define the speed of a market crash as the difference between the drawdown and the last reset of the maximum preceding it. The last reset of the maximum preceding the drawdown is denoted by ρ and is the last time preceding the drawdown at which the maximum was reset. A key result in our derivations is the fact that conditional on the level of the process attained at ρ, X_ρ, the process can be decomposed into two independent pieces the path before and after ρ. We then separately study the probabilistic behavior of the process before and after ρ. Using this path decomposition we are also able to derive analytical formulas for the Laplace transforms of the random time ρ and of the speed of market crash conditional on X_ρ , namely the difference $\mathcal{S} = \tau_K^D - \rho$ under general diffusion dynamics. We then combine these results to derive our joint Laplace transform. We finally discuss applications of our results in the pricing of insurance claims based on the drawdown and its speed under a drifted Brownian motion model and a constant elasticity of variance (CEV) model (see Jeanblanc et al. 2009).

Our results extend the work of Taylor (1975), Lehoczky (1977) and Nikeghbali (2006) from which it is possible to extract the Laplace transform of the random variable X_ρ under drifted Brownian motion dynamics and general diffusion dynamics. In our derivation of the Laplace transform of ρ we use the method of progressive enlargement of filtration developed in Jeulin and Yor (1978, 1985) and Jeulin (1980). Related work also includes Kardaras (2010) who is concerned with the projection of the random times on the natural filtration. However, none of these works are concerned with random times related to the drawdown or the drawdown itself. In Tanré and Vallois (2006) the method of path decomposition is used to derive the last visit time of the extreme values of a drifted Brownian motion before the first range time. The drawdown and its probabilistic properties have been extensively studied in the standard Brownian motion model by Douady et al. (2000) and in the drifted Brownian motion by Graversen and Shiryaev (2000) as well as Magdon-Ismail et al. (2004). Quantities of interest related to the joint distribution of the drawdown and the drawup have been extensively studied and derived in Hadjiliadis and Vecer (2006), where a closed-form formula is derived for the probability that the drawdown precedes the drawup in a drifted Brownian motion model. This result was later extended to diffusion processes in Pospisil et al. (2009). In Zhang and Hadjiliadis (2010), the authors obtain the probability that the drawup of a units precedes the drawdown of equal units in a drifted Brownian motion model in a finite time-horizon. This result was later extended in Zhang and Hadjiliadis (2011) to the analytical derivation of the joint distribution of the drawdown when it precedes the drawup in general diffusions. Another work related to the joint distribution of the maximum drawdown and the maximum drawup in a drifted Brownian motion model is Salminen and Vallois (2007). Other important properties of drawdowns that renders them extremely important in the financial risk management literature have also been studied for instance by Meilijson (2003) who proved that the drawdown can be viewed as the optimal exercise time of a certain type of look-back American put option.

Yet, the most important reason for the popularity of the drawdowns in financial risk management is due to the fact that they provide a dynamic measure of risk. Related works discussing precisely this aspect include Magdon-Ismail and Atiya (2004) who described a closely related time-adjusted measure of risk based on the drawdown known as the the Calmar ratio. Other relevant literature includes Vecer (2006, 2007). Drawdowns measure the drop of a stock price, index or value of a portfolio from its running maximum and thus provide portfolio managers a tool with which to assess the risk taken by a mutual fund during a given economic cycle, i.e. a peak followed by a trough followed by a peak. It is for this reason that drawdown processes have also been used as constraints in portfolio optimization (see, for example, Grossman and Zhou 1993, Cvitanic and Karatzas 1995, Chekhlov et al. 2005). Due to their dynamic and path-dependent nature, they can also be used as a way to describe market crashes. An overview of the existing techniques for analysis of market crashes related to the drawdown and the maximum drawdown can be found in Sornette (2003). It is also conceivable that a portfolio or hedge-fund manager may want to insure against such market crashes as measured by large realizations of the drawdown. The pricing and dynamic hedging of insurance claims based are developed in Pospisil and Vecer (2010). Recently, static replication strategies for hedging digital options based on large realizations of the drawdown are developed in Carr et al. (2010). Yet, the issue of how fast a market crash occurred is of vital importance to investors and portfolio or hedge-fund managers. This is because a slow transition from the maximum-to-date to a drop of a pre-specified level (i.e. a drawdown) is far easier to absorb or react to than a dramatic one. Therefore, the speed at which the drawdown is realized is a very relevant quantity in the description of a market crash. This is precisely the motivation of our paper which for the first time studies quantities related to the joint distribution of the drawdown and the speed at which it is realized. Our work thus provides the analytical basis for the pricing of insurance claims based on the drawdown and its speed and which can be used to hedge against dramatic market crashes.

In Section 2 we begin by the mathematical set-up of our problem and provide the main result, which is an analytical formula of the joint Laplace tranform of X_ρ, ρ and $\mathcal{S} = \tau_K^D - \rho$. In Section 3 we provide the proof of the main results using the tools of progressive enlargement of filtrations and path decomposition. We also derive analytical formulas for the conditional Laplace transforms of ρ and of the speed of market crash $\mathcal{S}$ given X_ρ. As applications, in Section 4 we propose and price an innovative drawdown insurance using Carr's randomization (Carr and Madan 1999). We provide analytical formula for the randomized claim price in the special cases of a drifted Brownian motion and a constant elasticity of variance model. We finally conclude with some closing remarks in Section 5.

2 Mathematical Formulation and Main Results

We begin with a filtered probability space $(\Omega, \mathbb{F}, P)$ with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. Let $X_\cdot = \{X_t\}_{t\geq 0}$ be an one dimensional linear diffusion on interval $I = (a, b) \subset \mathbb{R}$ on this probability space:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \;\; X_0 = x. \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion with respect to $\mathbb{F}$, $\mu(\cdot)$ and $\sigma(\cdot)$ are real-valued continuous functions and $\sigma(u) > 0$ for all $u \in I$. We assume that the boundaries a and b are either natural or entrance (see, for example, Mckean 1956). We denote the running maximum of $X_\cdot$ by

$$\overline{X}_t := \sup_{s\in[0,t]} X_s. \tag{2}$$

The drawdown process of $X_\cdot$, which is denoted by $D_\cdot$, is defined as the drop of X_t from its running maximum $\overline{X}_t$. That is,

$$D_t := \overline{X}_t - X_t, \;\; t \geq 0. \tag{3}$$

In this work, we will denote by τ_L^Y the first hitting time[1] to a level L of any continuous process $Y_\cdot$:

$$\tau_L^Y := \inf\{t \geq 0 | Y_t = L\}. \tag{4}$$

For any $K > 0$ such that $x - K \in I$, the drawdown of level K is defined as the first time at which the drawdown process $D_\cdot$ reaches level K:

$$\tau_K^D = \inf\{t \geq 0 | D_t = K\}. \tag{5}$$

The last visit time of the maximum before the stopping time τ_K^D is denoted by ρ. That is,

$$\rho := \sup\left\{t \in \left[0, \tau_K^D\right] | \overline{X}_t = X_t\right\}. \tag{6}$$

The time elapsed between ρ and τ_K^D is called the speed of market crash, which we denote by $\mathcal{S}$:

$$\mathcal{S} := \tau_K^D - \rho. \tag{7}$$

The random variable X_ρ is well studied (see Lehoczky 1977, Nikeghbali 2006). However, the fact that the random time ρ is not a stopping time makes it difficult to analyze. The main contribution of this work is the derivation of an analytical formula for the joint Laplace transform of random variables ρ, $\mathcal{S}$ and X_ρ:

$$E_x \exp(-\alpha\rho - \beta \cdot \mathcal{S} - \gamma X_\rho).$$

[1] As usual, we set $\inf \emptyset = \infty$.

Here $P_x(\cdot) = P(\cdot|X_0 = x)$ defines the distribution on a suitable path space of the diffusion process started at x. In particular, we have:

Theorem 1 *Let $\alpha, \beta, \gamma > 0$ be positive numbers, then for the diffusion in Eq. 1, we have*

$$E_x \exp(-\alpha\rho - \beta \cdot \mathcal{S} - \gamma X_\rho) = \int_x^b \frac{e^{-\gamma m} s'(m)}{g_\beta(m-K)h_\beta(m) - g_\beta(m)h_\beta(m-K)}$$
$$\cdot \exp\left(- \int_x^m \frac{g_\alpha(u-K)h'_\alpha(u) - g'_\alpha(u)h_\alpha(u-K)}{g_\alpha(u-K)h_\alpha(u) - g_\alpha(u)h_\alpha(u-K)} du \right) dm, \ \forall x \in (a,b), \tag{8}$$

where $s(\cdot)$ is a scale function of the diffusion in Eq. 1, $g_\lambda(\cdot)$ and $h_\lambda(\cdot)$ $(\lambda = \alpha, \beta)$ are any two independent solutions of $\frac{1}{2}\sigma^2(u) f'' + \mu(u) f' = \lambda f$, such that their Wronskian satisfies

$$g_\lambda(u)h'_\lambda(u) - g'_\lambda(u)h_\lambda(u) = s'(u), \ \forall u \in (a,b). \tag{9}$$

It is worth pointing out that, by letting $\alpha = \beta = \lambda$ in Eq. 8, we obtain Lehoczky's joint Laplace transform of τ_K^D and X_ρ (1977):

Corollary 1 *let $\lambda, \gamma > 0$ be positive numbers, $s(\cdot)$, $g_\lambda(\cdot)$ and $h_\lambda(\cdot)$ be the same as in Theorem 1, then we have*

$$E_x \exp(-\lambda\tau_K^D - \gamma X_\rho) = \int_x^b \frac{e^{-\gamma m} s'(m)}{g_\lambda(m-K)h_\lambda(m) - g_\lambda(m)h_\lambda(m-K)} \cdot$$
$$\cdot \exp\left(- \int_x^m \frac{g_\lambda(u-K)h'_\lambda(u) - g'_\lambda(u)h_\lambda(u-K)}{g_\lambda(u-K)h_\lambda(u) - g_\lambda(u)h_\lambda(u-K)} du \right) dm, \ \forall x \in (a,b). \tag{10}$$

In the next section, we will use the method of progressive enlargement of filtration developed in Jeulin and Yor (1978, 1985), Jeulin (1980), to study ρ, $\mathcal{S}$, as well as the path decomposition of $X_\cdot$ before and after the random time ρ. To this end, we prove our main result in Eq. 8.

3 Proof of the Main Results

In this section we prove the main result through optional projection and path decomposition. More specifically, we consider the optional projection of the random process $\mathbb{I}_{\{\rho > t\}}$ on the natural filtration $\mathbb{F}$,

$$Y_t^\rho := P_x(\rho > t|\mathcal{F}_t). \tag{11}$$

The fact that $\mathbb{I}_{\{\rho > t\}}$ is non-increasing implies that the process $Y_\cdot^\rho = \{Y_t^\rho\}_{t \geq 0}$ is a supermartingale, so a decomposition of the Doob–Meyer type exists. That is,

$$Y_t^\rho = M_t^\rho + A_t^\rho, \tag{12}$$

where $\{M_t^\rho\}_{t\geq 0}$ is a local martingale and $\{A_t^\rho\}_{t\geq 0}$ is a predictable non-increasing process. Using the scale function for linear diffusions (see for example, Borodin and Salminen 2002), it is convenient to derive analytical formulas for Y^ρ, M^ρ and A^ρ. In particular, we have

Proposition 1 *Let s be a scale function of the process in Eq. 1, then*

$$Y_t^\rho = \frac{s(X_t) - s(\overline{X}_t - K)}{s(\overline{X}_t) - s(\overline{X}_t - K)} \mathbb{I}_{\{t<\tau_K^D\}}, \tag{13}$$

$$M_t^\rho = 1 + \int_0^{t\wedge\tau_K^D} \frac{s'(X_u)\sigma(X_u)dW_u}{s(\overline{X}_u) - s(\overline{X}_u - K)}, \tag{14}$$

$$A_t^\rho = \int_0^{t\wedge\tau_K^D} \frac{-s'(\overline{X}_u)d\overline{X}_u}{s(\overline{X}_u) - s(\overline{X}_u - K)}. \tag{15}$$

Proof We notice that, $\{\rho > t\}$ means that, $\{t < \tau_K^D\}$ and the path of $X_\cdot$ will revisit $\overline{X}_t$ before it reaches $\overline{X}_t - K$. Let s be a scale function of $X_\cdot$, then $\{s(X_u)\}_{u\geq t}$ is a local martingale. So we have:

$$Y_t^\rho = P_x(\rho > t|\mathcal{F}_t) = \frac{s(X_t) - s(\overline{X}_t - K)}{s(\overline{X}_t) - s(\overline{X}_t - K)} \mathbb{I}_{\{t<\tau_K^D\}}. \tag{16}$$

This proves Eq. 13. We then apply Itô's lemma to process $Y_\cdot^\rho$ to obtain that, for any $t < \tau_K^D$

$$dY_t^\rho = \frac{d[s(X_t) - s(\overline{X}_t - K)]}{s(\overline{X}_t) - s(\overline{X}_t - K)} - \frac{s(X_t) - s(\overline{X}_t - K)}{[s(\overline{X}_t) - s(\overline{X}_t - K)]^2} d[s(\overline{X}_t) - s(\overline{X}_t - K)].$$

It is easily seen that $ds(X_t) = s'(X_t)\sigma(X_t)dW_t$, and

$$d[s(\overline{X}_t) - s(\overline{X}_t - K)] = [s'(\overline{X}_t) - s'(\overline{X}_t - K)]d\overline{X}_t.$$

Since the measure $d\overline{X}_t$ is supported on $\{t|X_t = \overline{X}_t\}$, we further have that

$$\begin{aligned} dY_t^\rho &= \frac{s'(X_t)\sigma(X_t)dW_t}{s(\overline{X}_t) - s(\overline{X}_t - K)} - \frac{s'(\overline{X}_t - K)d\overline{X}_t}{s(\overline{X}_t) - s(\overline{X}_t - K)} - \frac{s'(\overline{X}_t) - s'(\overline{X}_t - K)}{s(\overline{X}_t) - s(\overline{X}_t - K)} d\overline{X}_t \\ &= \frac{s'(X_t)\sigma(X_t)dW_t}{s(\overline{X}_t) - s(\overline{X}_t - K)} - \frac{s'(\overline{X}_t)d\overline{X}_t}{s(\overline{X}_t) - s(\overline{X}_t - K)}. \end{aligned} \tag{17}$$

Equations 14 and 15 then follow from the fact that $\lim_{t\to\tau_D^K} Y_t^\rho = 0$. □

The random time ρ is an honest time. This is because, on the event $\{\rho \leq t\}$, one has $\rho = \sup\{s \leq t | X_{s\wedge\tau_K^D} = \overline{X}_{t\wedge\tau_K^D}\}$, which is $\mathcal{F}_t$-measurable.[2] To enlarge the filtration in order to make ρ a stopping time, we define

$$\mathcal{F}_t^\rho := \mathcal{F}_t \vee \sigma\{\rho \wedge t\},\ t \geq 0. \tag{18}$$

Under the enlarged filtration $\mathbb{F}^\rho := \{\mathcal{F}_t^\rho\}$, a square integrable $\mathbb{F}$-martingale is a semimartingale since ρ is an honest time. In particular, we have:

[2] See page 373 of Protter (2005).

Lemma 1 *Let $\{N_t\}_{t\geq 0}$ be a square integrable $\mathbb{F}$-martingale, then it is a $\mathbb{F}^\rho$-semimartingale. Moreover, N_t has a Doob–Meyer decomposition*

$$N_t = \left(N_t - \int_0^{t\wedge\rho} \frac{1}{Y_s^\rho} d\langle N, M^\rho\rangle_s + \mathbb{1}_{\{t\geq\rho\}} \int_\rho^t \frac{1}{1-Y_s^\rho} d\langle N, M^\rho\rangle_s\right)$$
$$+\left(\int_0^{t\wedge\rho} \frac{1}{Y_s^\rho} d\langle N, M^\rho\rangle_s - \mathbb{1}_{\{t\geq\rho\}} \int_\rho^t \frac{1}{1-Y_s^\rho} d\langle N, M^\rho\rangle_s\right). \tag{19}$$

Here the first line of the right hand size is a $\mathbb{F}^\rho$-martingale, and the second line of the right hand size is a process with finite variation.

Proof See Theorem 18 on page 375 of Protter (2005). □

As a result of Lemma 1, the driving Brownian motion of a diffusion process is now a semimartingale. Using Lévy's characterization of Brownian motion (see Revuz and Yor 1999), we can see that the martingale part of this semimartingale is in fact a standard $\mathbb{F}^\rho$-Brownian motion. This will enable us to separately study the law of the diffusion path in Eq. 1 during the period $[0, \rho]$ and the period $[\rho, \tau_K^D]$, conditional on the event $\{X_\rho = M\}$.

In particular, we can prove the following result:

Proposition 2 *Conditionally on $X_\rho = M$, $\{X_t\}_{t\in[0,\rho]}$ is a process with the same law as the unique weak solution of the following stochastic differential equation, stopped at the first hitting time of a level M, τ_M^Z.*

$$dZ_t = \left(\mu(Z_t) + \frac{s'(Z_t)\sigma^2(Z_t)}{s(Z_t) - s(\overline{Z}_t - K)}\right)dt + \sigma(Z_t)dB_t,\ Z_0 = x, \tag{20}$$

where $\{B_t\}_{t\geq 0}$ is a standard $\mathbb{F}^\rho$-Brownian motion.

Proof Using Eq. 19 we know that for any $t \in [0, \rho]$, the process defined as

$$B_t = W_t - \int_0^t \frac{s'(X_u)\sigma(X_u)du}{s(X_u) - s(\overline{X}_u - K)}, \tag{21}$$

is a standard $\mathbb{F}^\rho$-Brownian motion. Therefore, for any $t \in [0, \rho]$, we have

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + \frac{s'(X_t)\sigma^2(X_t)dt}{s(X_t) - s(\overline{X}_t - K)}$$
$$= \left(\mu(X_t) + \frac{s'(X_t)\sigma^2(X_t)}{s(X_t) - s(\overline{X}_t - K)}\right)dt + \sigma(X_t)dB_t. \tag{22}$$

To finish the proof, we need to show that the stochastic differential equation in Eq. 20 admits a unique weak solution. To this effect, consider a stopping time for any given $\epsilon > 0$:

$$\tau_\epsilon = \inf\{t \geq 0 | \overline{Z}_t - Z_t \geq K - \epsilon\}. \tag{23}$$

Then the Theorem 5.5.15 of Karatzas and Shreve (1991) implies that there is a unique weak solution $\{Z_t\}_{t\geq 0}$ satisfying Eq. 20 and initial condition $Z_0 = x$, until time τ_ϵ. We let $\epsilon \to 0^+$ and the assertion is proved. This concludes the proof of Proposition 2. □

A consequence of Proposition 2 is that, the Laplace transform of the random time ρ conditional on $\{X_\rho = M\}$ is the same as the Laplace transform of the stopping time τ_M^Z. More specifically, we have:

Proposition 3 *Conditionally on $X_\rho = M$, the Laplace transform of the random time ρ is given by*

$$\begin{aligned} &E_x\{e^{-\lambda\rho}|X_\rho = M\} \\ &\quad= \exp\left(\int_M^x \left(\frac{-s'(u)}{s(u)-s(u-K)} + \frac{g(u-K)h'(u)-g'(u)h(u-K)}{g(u-K)h(u)-g(u)h(u-K)}\right)du\right), \end{aligned} \tag{24}$$

where $g(\cdot)$ and $h(\cdot)$ are any two independent solutions of $\frac{1}{2}\sigma^2(u)f'' + \mu(u)f' = \lambda f$.

Proof We apply the Feynman–Kac theorem (see Karatzas and Shreve 1991) to the "inhomogeneous" diffusion in Eq. 20. To this effect, we search for a locally bounded function on $\{z, m \in (a,b)|m \geq z, m-z < K\}$ satisfying the following partial differential equation for any given $\lambda > 0$:

$$\frac{1}{2}\sigma^2(z)\frac{\partial^2 f_\lambda}{\partial z^2} + \left(\mu(z) + \frac{s'(z)\sigma^2(z)}{s(z)-s(m-K)}\right)\frac{\partial f_\lambda}{\partial z} = \lambda f_\lambda(z,m), \tag{25}$$

$$\left.\frac{\partial f_\lambda}{\partial m}\right|_{m=z} = 0. \tag{26}$$

Let us consider a function

$$\begin{aligned} f_\lambda(z,m) = \exp&\left(\int^m \left(\frac{-s'(u)}{s(u)-s(u-K)} + \frac{g(u-K)h'(u)-g'(u)h(u-K)}{g(u-K)h(u)-g(u)h(u-K)}\right)du\right) \\ &\cdot\frac{g(m-K)h(z)-g(z)h(m-K)}{g(m-K)h(m)-g(m)h(m-K)} \cdot \frac{s(m)-s(m-K)}{s(z)-s(m-K)}, \end{aligned} \tag{27}$$

Then it is straightforward to check that $f_\lambda(\cdot)$ is a locally bounded solution of the partial differential equation in Eq. 25. Moreover, it can be shown that f_λ in Eq. 27 satisfies boundary condition in Eq. 26. In terms of this solution, we can now express the conditional Laplace transforms of ρ as:

$$E_x\{e^{-\lambda\rho}|X_\rho = M\} = E_x\{e^{-\lambda\tau_M^Z}\} = \frac{f_\lambda(x,x)}{f_\lambda(M,M)}, \tag{28}$$

which gives Eq. 24. □

On the other hand, the Brownian path during the drawdown, $[\rho, \tau_K^D]$, can be similarly described by conditioning on $\{X_\rho = M\}$. In particular, we give the law of $\{X_{t+\rho} - M\}_{t\in[0,\mathcal{S}]}$.

Proposition 4 *Conditionally on $X_\rho = M$, the law of $\{M - X_{t+\rho}\}_{t\in[0,\mathcal{S}]}$ is the same as the unique positive weak solution of the following stochastic differential equation, stopped at the first hitting time to level K, τ_K^J.*

$$dJ_t = \left(-\mu(M-J_t) + \frac{s'(M-J_t)\sigma^2(M-J_t)}{s(M)-s(M-J_t)}\right)dt - \sigma(M-J_t)dB_t,\ J_0 = 0 \tag{29}$$

where $\{B_t\}_{t\geq 0}$ *is a standard* $\{\mathcal{F}^{\rho}_{\rho+t}\}$*-Brownian motion. Moreover, the processes* $\{X_t\}_{t\in[0,\rho]}$ *and* $\{X_t\}_{t\in[\rho,\tau^D_K]}$ *are independent.*

Proof Using Eq. 19 we know that for any $t \in [0, \mathcal{S}]$, the process defined as

$$B_{t+\rho} = W_{t+\rho} - W_\rho + \int_\rho^{t+\rho} \frac{s^{'}(X_u)\sigma(X_u)du}{s(M) - s(X_u)}, \tag{30}$$

is a $\{\mathcal{F}_{t+\rho}\}$-standard Brownian motion. Therefore, for any $t \in [\rho, \tau^D_K]$, we have

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t - \frac{s^{'}(X_t)\sigma^2(X_t)}{s(M) - s(X_t)}dt \\ &= \left(\mu(X_t) - \frac{s^{'}(X_t)\sigma^2(X_t)}{s(M) - s(X_t)}\right)dt + \sigma(X_t)dB_t. \end{aligned} \tag{31}$$

Since $\{B_{t+\rho}\}_{t\geq 0}$ is independent of $\mathcal{F}^\rho_\rho$, $\{B_{t+\rho}\}_{t\geq 0}$ is independent of X_ρ and $\{X_t\}_{t\in[0,\rho]}$. To finish the proof, we need to show that the stochastic differential equation in Eq. 29 has a unique positive weak solution. This is seen using Theorem 5.5.15 of Karatzas and Shreve (1991). □

As a consequence of Proposition 4, the Laplace transform of the random variable $\mathcal{S}$ can be derived by solving for the Laplace transform of the stopping time τ^J_K. Therefore we have:

Proposition 5 *Conditionally on* $X_\rho = M$*, the Laplace transform of the random variable* $\mathcal{S}$ *is given by*

$$E_x\{e^{-\lambda\mathcal{S}}|X_\rho = M\} = \frac{s(M) - s(M-K)}{g(M-K)h(M) - g(M)h(M-K)}, \tag{32}$$

where $g(\cdot)$ *and* $h(\cdot)$ *are any two independent solutions of* $\frac{1}{2}\sigma^2(u)f^{''} + \mu(u)f^{'} = \lambda f$*, such that their Wronskian satisfies*

$$g(u)h^{'}(u) - g^{'}(u)h(u) = s^{'}(u),\ \forall u \in (a, b). \tag{33}$$

Proof We apply the Feynman–Kac theorem (see Karatzas and Shreve 1991) to the process in Eq. 29 and search for a locally bounded function on $(0, M-a)$ which satisfies the following partial differential equation for any given $\lambda > 0$:

$$\frac{1}{2}\sigma^2(M-p)\frac{\partial^2 f_\lambda}{\partial p^2} + \left(-\mu(M-p) + \frac{s^{'}(M-p)\sigma^2(M-p)}{s(M) - s(M-p)}\right)\frac{\partial f_\lambda}{\partial p} = \lambda f_\lambda(p). \tag{34}$$

Let us consider a function defined as

$$f_\lambda(p) = \frac{g(M-p)h(M) - g(M)h(M-p)}{s(M) - s(M-p)}. \tag{35}$$

Then it is straightforward to check that $f_\lambda(\cdot)$ is a locally bounded solution of the partial differential equation in Eq. 34. In terms of this solution, we can express the conditional Laplace transform of $\mathcal{S}$ as:

$$E_x\{e^{-\lambda \mathcal{S}} | X_\rho = M\} = E_x\{e^{-\lambda \tau_K^J}\} = \frac{f_\lambda(0)}{f_\lambda(K)}, \tag{36}$$

which gives Eq. 32. □

Finally, using the independence of path before and after the random time ρ, as well as individual Laplace transforms of ρ and $\mathcal{S}$ given X_ρ, we are able to compute their joint Laplace transform in Eq. 8 from Eqs. 28 and 36. In particular, notice that

Lemma 2 *For the linear diffusion in Eq. 1, we have that*

$$P_x(X_\rho \geq M) = \exp\left(- \int_x^M \frac{s'(u)}{s(u) - s(u-K)} du \right), \ \forall M > x. \tag{37}$$

Proof See Eq. 3 on page 602 of Lehoczky (1977), or Eq. 4.4 on page 930 of Nikeghbali (2006). □

By applying the results in Propositions 2, 4, and Lemma 2, we obtain our main result in Eq. 8.

4 Applications

In this section we propose and price an innovative claim which can be used as a means of insurance against market crashes. More specifically, let $X_\cdot$ denote the underlying price process. We consider a perpetual barrier claim with a pre-specified knockout barrier $H > X_0 = x$ and a pre-specified strike $T > 0$. This claim will pay one dollar at τ_K^D, if and only if, both the underlying has not been knocked out by time τ_K^D, and the speed of market crash of the underlying is smaller than the strike T. In the case $\tau_K^D = \infty$, the claim is regarded as delivering zero payoff.

In the presence of no arbitrage, no transaction cost, and a constant risk-free rate $r \geq 0$, the time-0 price of this claim can be evaluated using a risk-neutral measure P. In particular, let us denote by $V_H(T)$ the time-0 price of the above claim, then

$$V_H(T) = E_x\{e^{-r\tau_K^D} \cdot \mathbb{1}_{\{\mathcal{S} < T, X_\rho < H\}}\}, \tag{38}$$

where x is the time-0 price of the underlying.

Using Carr's randomization (Carr and Madan 1999) in T, i.e., letting $T = \zeta$ be an exponentially distributed random variable with parameter $c > 0$, which is independent of the underlying price process $X_\cdot$, we obtain that

$$\begin{aligned}
E_x\{V_H(\zeta)\} &= E_x\{e^{-r\tau_K^D} \cdot \mathbb{1}_{\{\mathcal{S}<\zeta, X_\rho<H\}}\} \\
&= E_x\{e^{-r\tau_K^D} \cdot \mathbb{1}_{\{X_\rho<H\}} \cdot P_x(\mathcal{S} < \zeta | \mathcal{S})\} \\
&= E_x\{e^{-r\tau_K^D - c\mathcal{S}} \cdot \mathbb{1}_{\{X_\rho<H\}}\} \\
&= \int_x^H E_x\{e^{-r\rho}|X_\rho = m\} \cdot E_x\{e^{-(r+c)\mathcal{S}}|X_\rho = m\} \cdot P_x(X_\rho \in dm) \qquad (39)
\end{aligned}$$

The price $V_H(T)$ for a pre-specified $T > 0$, is then obtained through Laplace inversion of $E_x\{V_H(\zeta)\}$:

$$V_H(T) = \mathcal{L}_c^{-1}\left(\frac{1}{c}E_x\{V_H(\zeta)\}\right), \qquad (40)$$

where $\mathcal{L}_c^{-1}$ is the inversion Laplace operator.

In what follows we consider two examples of dynamics for the underlying. in the first example, we consider a drifted Brownian motion as the logarithm of the underlying price process and we are able to provide a closed-form formula for the randomized price in Eq. 39. We also consider a constant elasticity of variance (CEV) model in the second example, a prototypical model for a strict local martingale.

4.1 Brownian Motion with Drift

We consider a Brownian motion with drift $\mu(\cdot) \equiv \mu \neq 0$ and diffusion coefficient $\sigma(\cdot) \equiv \sigma > 0$:

$$X_t = x + \mu t + \sigma W_t, \; x \in (-\infty, \infty). \qquad (41)$$

It is known from Borodin and Salminen (2002) that, $\forall u \in (-\infty, \infty)$,

$$s(u) = e^{-2\delta u}, \qquad (42)$$

$$g_\lambda(u) = e^{(-\delta + S^\lambda_{\delta,\sigma})u}, \qquad (43)$$

$$h_\lambda(u) = \frac{\delta}{S^\lambda_{\delta,\sigma}} e^{(-\delta - S^\lambda_{\delta,\sigma})u}, \qquad (44)$$

where

$$\delta := \frac{\mu}{\sigma^2}, \quad S^\lambda_{\delta,\sigma} := \sqrt{\delta^2 + \frac{2\lambda}{\sigma^2}}. \qquad (45)$$

Applying Propositions 3, 5, and Lemma 2 we obtain that, $\forall M > x$,

$$E_x\{e^{-\lambda\rho}|X_\rho = M\} = e^{-[S^\lambda_{\delta,\sigma} \cdot \coth(S^\lambda_{\delta,\sigma} \cdot K) - \delta \cdot \coth(\delta \cdot K)](M-x)}, \qquad (46)$$

$$E_x\{e^{-\lambda\mathcal{S}}|X_\rho = M\} = \frac{S^\lambda_{\delta,\sigma}}{\delta} \cdot \frac{\sinh(\delta \cdot K)}{\sinh(S^\lambda_{\delta,\sigma} \cdot K)}, \qquad (47)$$

$$P_x(X_\rho \geq M) = \exp\left(-\frac{2\delta(M-x)}{e^{2\delta K} - 1}\right). \qquad (48)$$

Using Eq. 39 we have:

$$E_x\{V_H(\zeta)\} = \frac{S_{\delta,\sigma}^{r+c}}{\sinh(S_{\delta,\sigma}^{r+c}\cdot K)}\cdot e^{-\delta K}\cdot\frac{1-e^{-(S_{\delta,\sigma}^{r}\coth(S_{\delta,\sigma}^{r}\cdot K)-\delta)(H-x)}}{S_{\delta,\sigma}^{r}\coth(S_{\delta,\sigma}^{r}\cdot K)-\delta}. \tag{49}$$

Notice that we only need to invert the first factor in Eq. 49 to get the price $V_H(T)$. And the result of this inversion is available in analytical form (see Borodin and Salminen 2002).

4.2 CEV Model

In this section we consider a zero drift constant elasticity variance (CEV) model with parameter 2. More specifically, we consider a nonnegative diffusion with drift coefficient $\mu(\cdot)\equiv 0$ and diffusion coefficient $\sigma(u)=\sigma u^2$:

$$dX_t = \sigma X_t^2 dW_t,\ \ X_0 = x > K > 0. \tag{50}$$

where $\sigma > 0$ is a constant. It is worth pointing out that, the above CEV model can be expressed as the strict local martingale $\{(R_{\sigma^2 t})^{-1}\}$, where $R_{\cdot} = \{R_t\}_{t\ge 0}$ is a three dimensional Bessel process starting at $\frac{1}{x} > 0$.

It is known from Jeanblanc et al. (2009), and Lebedev (1965) that, $\forall u\in(0,\infty)$,

$$s(u) = -u, \tag{51}$$

$$g_\lambda(u) = \sqrt{u}K_{\frac{1}{2}}\left(\frac{\sqrt{2\lambda}}{\sigma u}\right) = \sqrt{\frac{\pi}{2}\cdot\frac{1}{S_{0,\sigma}^{\lambda}}}\cdot u\cdot\exp\left(-\frac{S_{0,\sigma}^{\lambda}}{u}\right), \tag{52}$$

$$h_\lambda(u) = \sqrt{u}I_{\frac{1}{2}}\left(\frac{\sqrt{2\lambda}}{\sigma u}\right) = \sqrt{\frac{2}{\pi}\cdot\frac{1}{S_{0,\sigma}^{\lambda}}}\cdot u\cdot\sinh\left(\frac{S_{0,\sigma}^{\lambda}}{u}\right), \tag{53}$$

where $S_{0,\sigma}^{\lambda}$ is given in Eq. 45, $I_{\frac{1}{2}}(\cdot)$ and $K_{\frac{1}{2}}(\cdot)$ are respectively the modified Bessel functions of the first and second kind of order $\frac{1}{2}$.

Applying Propositions 3, 5, and Lemma 2 we obtain that, $\forall M > x$,

$$E_x\{e^{-\lambda\rho}|X_\rho = M\} = \frac{x}{M}\exp\left(\frac{M-x}{K} - \int_x^M \frac{S_{0,\sigma}^{\lambda}}{u^2}\coth\left(\frac{S_{0,\sigma}^{\lambda}\cdot K}{u(u-K)}\right)du\right) \tag{54}$$

$$E_x\{e^{-\lambda\mathcal{S}}|X_\rho = M\} = \frac{S_{0,\sigma}^{\lambda}\cdot K}{M(M-K)}\cdot\frac{1}{\sinh\left(\frac{S_{0,\sigma}^{\lambda}\cdot K}{M(M-K)}\right)}, \tag{55}$$

$$P_x(X_\rho \ge M) = \exp\left(-\frac{M-x}{K}\right). \tag{56}$$

Using Eq. 39 we have:

$$E_x\{V_H(\zeta)\} = \int_x^H \frac{S_{0,\sigma}^{r+c}}{\sinh\left(\frac{S_{0,\sigma}^{r+c}\cdot K}{m(m-K)}\right)}\cdot\frac{x\exp\left(-\int_x^m \frac{S_{0,\sigma}^{r}}{u^2}\coth\left(\frac{S_{0,\sigma}^{r}\cdot K}{u(u-K)}\right)du\right)}{m^2(m-K)}dm. \tag{57}$$

5 Concluding Remarks

In this paper, we derive an analytical formula for the joint Laplace transform of the last visit time of the maximum preceding the drawdown, the speed of market crash, and the maximum at the drawdown for a general diffusion process. Using Carr's randomization, we apply this result to price an innovative perpetual claim that can be used as a means of insurance against market crashes. We present formulas of the randomized claim price in both the drifted Brownian motion model and a CEV model. A possible extension is to consider finite maturity counterpart of the above drawdown insurance. This would require a combination of our results and double Carr's randomization in both the strike and the maturity. The computational cost of the involved Laplace inversion will however be slightly expensive.

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