Insurance against Market Crashes

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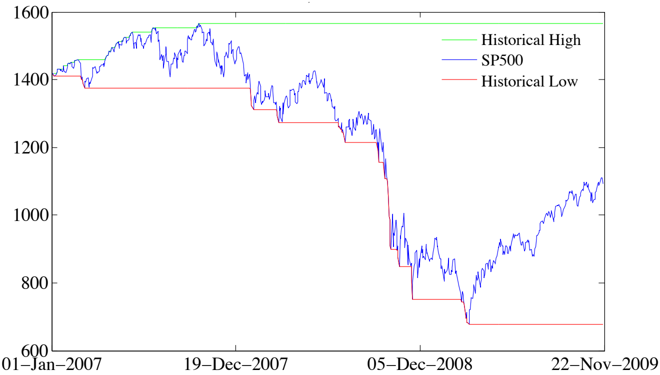
Market Turbulence



- How to insure?
- How much is the insurance?

Setup

- Filtered probability space $(\Omega, \mathbb{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$
- Price/value process: $\{S_t\}_{t\geq 0}$: $\frac{dS_t}{S_t} = rdt + \sigma dW_t$
- Log price/value process: $\{X_t\}_{t\geq 0}$ where $X_t = \log S_t$ and $x = X_0$
- Drawdown process: $D_t = \overline{X}_t - X_t$, where $\overline{X}_t = \overline{x} \vee \left(\sup_{s\in[0,t]} X_s\right)$
- Drawup process: $U_t = X_t - \underline{X}_t$, where $\underline{X}_t = \underline{x} \wedge \left(\inf_{s\in[0,t]} X_s\right)$
- Reference period's low & high: $\underline{x} \leq \overline{x} < \underline{x} + k$
- First hitting times of the drawdown/drawup process:

$$\tau_D(k) = \inf\{t \geq 0 | D_t \geq k\}$$

$$\tau_U(k) = \inf\{t \geq 0 | U_t \geq k\}$$

- A market crash is modeled as $\tau_D(k)$!

A snapshot of log S&P index

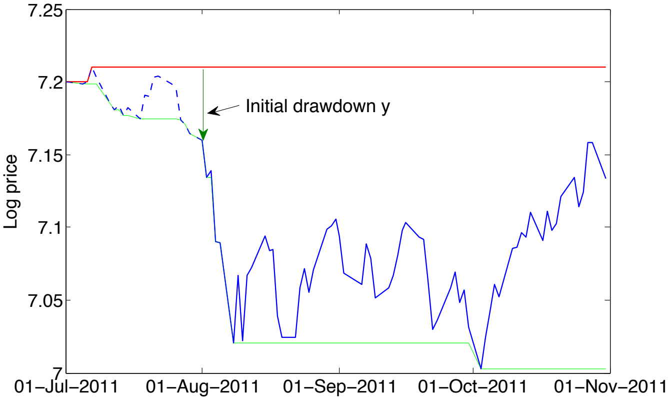


Figure: July of 2011 is the reference period. $\overline{x} = 7.21$ and $x = 7.16$. Initial drawdown $y = D_0 = 0.05$. The large drawdown in August is due to the downgrade of US debt by S&P.

Insurance claims against a market crash

Let $r \geq 0$ be the risk-free interest rate, $\mathbb{Q}$ the risk-neutral measure

- Drawdown insurance: time-0 value is (seen from the protection buyer)

$$V_0(p) = E^{\mathbb{Q}}\left\{ - \int_0^{\tau_D(k)\wedge T} pe^{-rt}dt + \alpha e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)\leq T\}} \right\}$$

- Ways to terminate a drawdown insurance when necessary
  - Callable drawdown insurance: the time-0 value is (seen from the protection buyer, τ is the cancelation time)

$$V_0^c(p) = \sup_{0\leq\tau<T} E^{\mathbb{Q}}\left\{ - \int_0^{\tau_D(k)\wedge\tau} pe^{-rt}dt + \alpha e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)\leq\tau\}} - ce^{-r\tau}\mathbb{I}_{\{\tau<\tau_D(k)\}} \right\}$$

  - Drawdown insurance contingent on drawups: time-0 value is (seen from the protection buyer)

$$V_0^U(p) = E^{\mathbb{Q}}\left\{ - \int_0^{\tau_D(k)\wedge\tau_U(k)\wedge T} pe^{-rt}dt + \alpha e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)\leq\tau_U(k)\wedge T\}} \right\}$$

Fair evaluation

- The premium is the rate $P^\star$ such that the time-0 value of a insurance is zero:

$$v_0(P^\star) = 0$$

- Value calculation:

$$\begin{aligned} v_0(p) =& E^{\mathbb{Q}}\left\{ -\int_0^{\tau_D(k)\wedge T} pe^{-rt}dt + \alpha e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)\leq T\}} \right\} \\ =& E^{\mathbb{Q}}\left\{ \left(\frac{p}{r} + \alpha\mathbb{I}_{\{\tau_D(k)\leq T\}}\right) e^{-r(\tau_D(k)\wedge T)} - \frac{p}{r} \right\} \end{aligned}$$

If perpetual $T = \infty$, then

$$v_0(p) = \frac{p}{r} - \left(\alpha + \frac{p}{r}\right)\xi(D_0) := -f(D_0, p)$$

where $\xi(y) = E^{\mathbb{Q}}\{e^{-r\tau_D(k)} | D_0 = y\}$

The conditional Laplace transform $\xi(y)$

For $\mu = r - \frac{1}{2}\sigma^2$, $0 \le y_1, y_2 < k$, $\Xi^r_{\mu,\sigma} = \sqrt{\frac{2r}{\sigma^2} + \frac{\mu^2}{\sigma^4}}$

- The quantity $\xi(y) = E^{\mathbb{Q}}\{e^{-r(\tau_D(k))} | D_0 = y\}$ satisfies functional equation:

$$\xi(y_2) = e^{\frac{\mu}{\sigma^2}(y_2-k)} \frac{\sinh(\Xi^r_{\mu,\sigma}(y_2 - y_1))}{\sinh(\Xi^r_{\mu,\sigma}(k - y_1))} + e^{\frac{\mu}{\sigma^2}(y_2-y_1)} \frac{\sinh(\Xi^r_{\mu,\sigma}(k - y_2))}{\sinh(\Xi^r_{\mu,\sigma}(k - y_1))} \xi(y_1)$$

  Equivalently,

$$\Lambda(y_2) - \lambda(y_1) = \frac{e^{-\frac{\mu k}{\sigma^2}} \sinh(\Xi^r_{\mu,\sigma}(y_2 - y_1))}{\sinh(\Xi^r_{\mu,\sigma}(k - y_1)) \sinh(\Xi^r_{\mu,\sigma}(k - y_2))}$$

- $\xi(0)$ is calculated by H. Taylor 1975.

More properties of $\xi(y)$

- $\xi(y)$ is increasing over $[0, k]$: continuity of path and Markov property
- Neumann condition at 0: $\xi^{'}(0) = 0$
- ODE: Feymann-Kac

$$\frac{1}{2}\sigma^2\xi^{''}(y) - \mu\xi^{'} = r\xi(y)$$

- $\xi(y)$ is strictly convex, i.e., $\xi^{''}(y) > 0$ for all $y \in (0, k)$

Pricing a cancellable drawdown insurance

- Callable drawdown insurance: recall that the time-0 value is (seen from the protection buyer, τ is the cancellation time, c is the cancellation fee)

$$V_0^c(p) =$$

$$\sup_{0\le\tau<T} E^{\mathbb{Q}}\left\{ -\int_0^{\tau_D(k)\wedge\tau} pe^{-rt}dt + \alpha e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)\le\tau\}} - ce^{-r\tau}\mathbb{I}_{\{\tau<\tau_D(k)\}} \right\}$$

- To find the fair premium $p^\star$, we need to first solve the above optimal stopping problem to find the value function $V_0^c(p)$, and then solve for $P^\star$ in

$$V_0^c(P^\star) = 0$$

Premium of cancellation

- To avoid unnecessary complications, we consider perpetual insurances, i.e., $T = \infty$
- Notice that, for any cancellation time $\tau < \tau_D(k)$,

$$
\begin{aligned}
& -\int_0^{\tau_D(k)\wedge\tau} pe^{-rt}dt + \alpha e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)\le\tau\}} - ce^{-rt}\mathbb{I}_{\{\tau<\tau_D(k)\}} \\
= \; & -\int_0^{\tau_D(k)} pe^{-rt}dt + \alpha e^{-r\tau_D(k)} \\
& + \underbrace{\int_{\tau_D(k)\wedge\tau}^{\tau_D(k)} pe^{-rt}dt - ce^{-r\tau}\mathbb{I}_{\{\tau<\tau_D(k)\}} - \alpha e^{-r\tau_D(k)}\mathbb{I}_{\{\tau<\tau_D(k)\}}}_{\text{Extra premium from cancellation}}
\end{aligned}
$$

- Let $V_0(p)$ be the time-0 value of a perpetual drawdown insurance, then necessarily,

$$V_0(p) \le V_0^c(p).$$

The cancellable drawdown insurance as an American call type contract

- Recall that:

$$V_0(p) = -f(D_0, p), \text{ where } f(y) := \frac{p}{r} - \left(\alpha + \frac{p}{r}\right)\xi(y)$$

- The value function of the cancellable drawdown insurance can also be computed:

$$V_0^c(p) = V_0(p) + \sup_{\tau \in \mathcal{S}} E^{\mathbb{Q}}\{e^{-r\tau}(f(D_\tau) - c)\}, \ \mathcal{S} = \{\tau | 0 \le \tau < \tau_D(k)\}$$

- Since $\xi(\cdot)$ is increasing, $f(\cdot)$ is decreasing. To avoid trivial optional cancellation strategy ($\tau^\star \equiv \infty$), it is necessary to have $f(0) > 0$. In other words,

$$\textbf{Cond} : p > \frac{r(c + \alpha\xi(0))}{1 - \xi(0)} \ge 0$$

- Under condition **Cond**, we seek the optimal exercise time: $\sup_{\tau \in \mathcal{S}} E^{\mathbb{Q}}\{e^{-r\tau}\tilde{f}(D_\tau)\mathbb{I}_{\{\tau < \tau_D(k)\}}\}$, with $\tilde{f} = f - c$

Method of solution

- Conjecture a stopping time of the form $\tau^{\theta} := \tau_D^-(\theta) \wedge \tau_D(k) \in \mathcal{S}$, where

$$\tau_D^-(\theta) = \inf\{t \geq 0 | D_t \leq \theta\}, \; 0 < \theta < k$$

- We seek a θ^* through smooth pasting

$$\frac{\partial}{\partial y}\bigg|_{y=\theta} E^{\mathbb{Q}}\{e^{-r\tau^{\theta}} \tilde{f}(D_{\tau^{\theta}}) \mathbb{I}_{\{\tau^{\theta} < \tau_D(k)\}} | D_0 = y\} = \tilde{f}'(\theta)$$

- Let $V(\theta^*, y) = E^{\mathbb{Q}}\{e^{-r\tau^{\theta^*}} \tilde{f}(D_{\tau^{\theta^*}}) \mathbb{I}_{\{\tau^{\theta^*} < \tau_D(k)\}} | D_0 = y\}$, show that $\{e^{-r(t \wedge \tau_D(k))} V(\theta^*, D_{t \wedge \tau_D(k)})\}_{t \geq 0}$ is the smallest supermartingale dominating $\{\tilde{f}(D_{t \wedge \tau_D(k)})\}_{t \geq 0}$
- Verify the cancellation strategy based on θ^* is indeed optimal

Smooth pasting

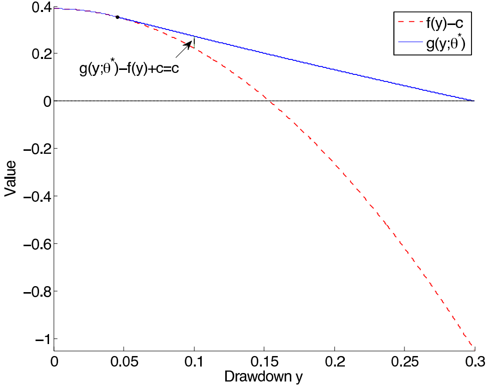


Figure: Model parameters:
$r = 2\%, p = P^{\star} = 1.5245, \sigma = 30\%, k = 30\%, \alpha = 1$, $c = 0.05$ and $D_0 = 10\%$.
The "intrinsic function" $\tilde{f}(\cdot)$ is shown in red dash line, the optimal extra premium from cancellation is shown in blue solid line. The only point determined by smooth pasting is $\theta^{*} \approx 5\%$

Theorem

Under the proposed model, there exists a unique solution $\theta^* \in (0, \theta_0)$ to equation

$$\frac{\partial}{\partial y}\bigg|_{y=\theta} E^{\mathbb{Q}}\{e^{-r\tau^{\theta}}\tilde{f}(D_{\tau^{\theta}})\mathbb{I}_{\{\tau^{\theta}<\tau_D(k)\}}|D_0 = y\} = \tilde{f}'(\theta).$$

Moreover, for any $\theta \in (\theta^*, k)$,

$$E^{\mathbb{Q}}\{e^{-r\tau^{\theta^*}}\tilde{f}(D_{\tau^{\theta^*}})\mathbb{I}_{\{\tau^{\theta^*}<\tau_D(k)\}}|D_0 = \theta\} > \tilde{f}(\theta)$$

Here $\theta_0 \in (0, k)$ is the unique root to equation $f(\theta) = 0$.

- Mean value theorem implies existence
- Uniqueness: We use properties of $\Lambda(\cdot)$ and representation $\tilde{f}(\theta) = (\alpha + \frac{p}{r})(\xi(\theta_0) - \xi(\theta))$ to prove it.
- The last result in the theorem asserts that $\{e^{-r(t\wedge\tau_D(k))}V(\theta^*, D_{t\wedge\tau_D(k)})\}_{t\geq 0}$ is the smallest supermartingale dominating $\{f(D_{t\wedge\tau_D(k)})\}_{t\geq 0}$

Determine the fair premium implicitly

- If $\tilde{f}(0) \leq 0$, the fair premium is obtained from

$$V_0(P^\star) = 0$$

- If $\tilde{f}(0) > 0$, the fair premium is obtained from

$$V_0(P^\star) + E^{\mathbb{Q}}\{e^{-r\tau^{\theta^*}} f(D_{\tau^{\theta*}})\mathbb{I}_{\{\tau^{\theta*} < \tau_D(k)\}}\} = 0$$

  Notice that in this case, $\theta^* = \theta^*(P^\star)$ depends on $P^\star$.
- To see the dependence of $P^\star$ on the size of drawdown k, we plot the value function V_0^c on a grid of (k, p), and then find the zero contour.

The value function and the fair premium p^* vs. k (B-S model)

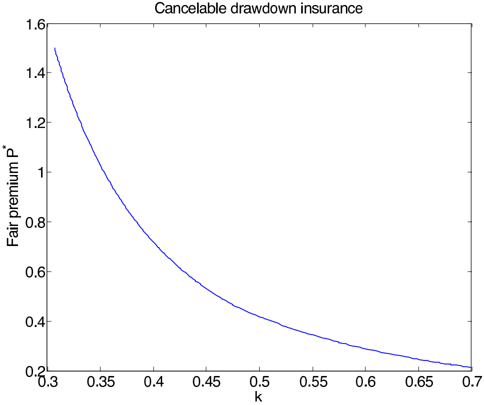


Figure: The fair premium of the cancelable drawdown insurance decreases with respect to the drawdown strike level k. Model parameters: $r = 2\%, \sigma = 30\%, \alpha = 1$, $c = 0.05$ and $D_0 = 10\%$.

Cancellable vs. non-cancellable

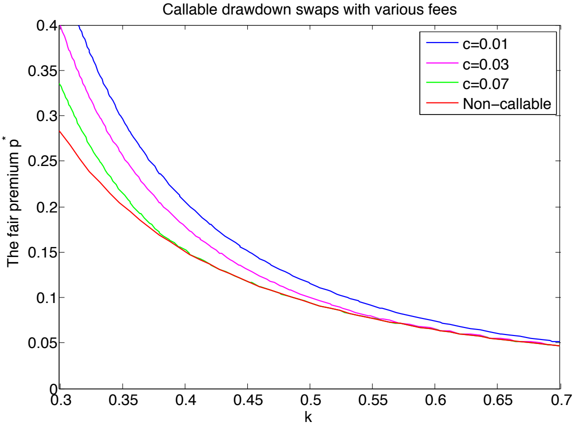


Figure: Model parameters: $r = 1\%, \sigma = 15\%, \alpha = 1$, and $D_0 = 10\%$. The fair premium $p^*(c = \infty)$ for the non-callable drawdown insurance is shown in red. It is seen that the fair premium $P^\star(c)$ is decreasing in the cancellation fee c.

The fair premium of drawdown insurances with contingency

- Recall that

$$V_0^U(p) = \left(\alpha \mathbb{I}_{\{\tau_D(k) \le \tau_U(k) \wedge T\}} + \frac{p}{r}\right) E^{\mathbb{Q}}\{e^{-r(\tau_D(k) \wedge \tau_U(k) \wedge T)}\} - \frac{p}{r}$$

- For drawdown insurance contingent on drawups:

$$P^{\star} = \frac{r\alpha E^{\mathbb{Q}}\{e^{-r\tau_D(k)} \mathbb{I}_{\{\tau_D(k) \le \tau_U(k) \wedge T\}}\}}{1 - E^{\mathbb{Q}}\{e^{-r(\tau_D(k) \wedge \tau_U(k) \wedge T)}\}}$$

- Examine the dependence of $P^{\star}$ on interest rate, volatility, maturity and other model parameters.

Finite time-horizon

- Using Zhang&Hadjiliadis '10, the following probability can be obtained for drifted Brownian motion X

$$\mathbb{Q}\{\tau_D(k) \leq \tau_U(k) \wedge T\}, \; \mathbb{Q}\{\tau_D(k) \wedge \tau_U(k) \leq T\}$$

- Explicit computation of the fair premium

$$P^\star = \frac{r\alpha \int_0^T e^{-rt} \left(\frac{\partial}{\partial t} \mathbb{Q}\{\tau_D(k) \leq \tau_U(k) \wedge t\} \right) dt}{1 - \int_0^T e^{-rt} \left(\frac{\partial}{\partial t} \mathbb{Q}\{\tau_D(k) \wedge \tau_U(k) \leq t\} \right) dt}$$

Large-time and infinite time-horizons

- For a large time-horizon T, it is known that

$$P^\star(T) \to P^\star(\infty), \text{ as } T \to \infty$$

  where $P^\star(\infty)$ is the fair premium for perpetual insurance
- Using Zhang&Hadjiliadis '09, the following Laplace transform can be obtained for a general regular linear diffusion X

$$E^{\mathbb{Q}}\{e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)<\tau_U(k)\}}\}, \; E^{\mathbb{Q}}\{e^{-r(\tau_D(k)\wedge\tau_U(k))}\}$$

- Explicit computation of the fair premium for perpetual drawdown insurance

$$P^\star = P^\star(\infty) = \frac{r\alpha E^{\mathbb{Q}}\{e^{-r\tau_D(k)}\mathbb{I}_{\{\tau_D(k)<\tau_U(k)\}}\}}{1 - E^{\mathbb{Q}}\{e^{-r(\tau_D(k)\wedge\tau_U(k))}\}}$$

The fair premium $P^\star$ vs. the size of drawdown k

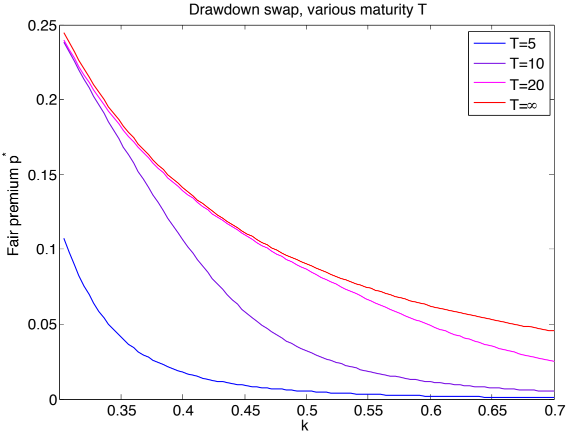


Figure: Model parameters: $r = 1\%, \sigma = 15\%, \alpha = 1$. The fair premium $P^\star(\infty)$ is shown in red. It is seen that $P^\star(k, T)$ is increasing in T and decreasing in k.

The fair premium p^* vs. interest rate r

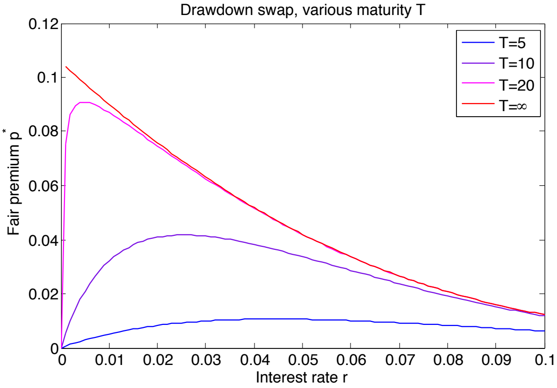


Figure: Model parameters: $\sigma = 15\%, k = 50\%, \alpha = 1$. The fair premium $P^{\star}(\infty)$ is shown in red. It is seen that, the fair premium $P^{\star}(r)$ is eventually decreasing in r.

The fair premium $P^\star$ vs. volatility σ

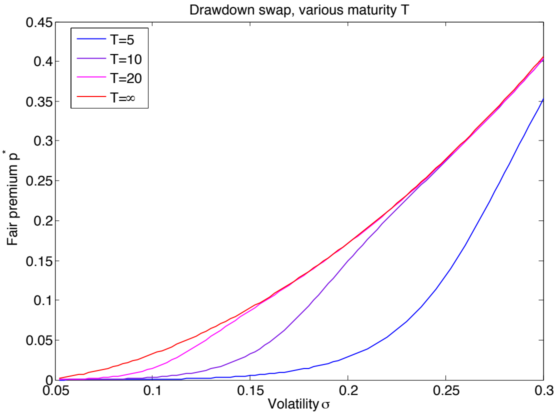


Figure: Model parameters: $r = 1\%, k = 50\%, \alpha = 1$. The fair premium $P^\star(\infty)$ is shown in red. Like most derivatives, the fair premium $P^\star(\sigma, T)$ is increasing both in σ and in T.

Thank You!

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