Multivariable Calculus/Vector Analysis More Review Problems Summer 2009
Courant Institute

1. Using the Lagrange multiplier method, find the dimensions of a cylindrical can, with lid, which is to contain 1 liter of water, and uses the minimum amount of metal. (i.e. minimize the surface area subject to the constraint that the volume is equal to 1).

2. Use Stokes Theorem to compute the surface integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + yz\mathbf{k}$ and S is the portion of the paraboloid $z = x^2 + y^2$ satisfying $0 \leq z \leq 1$.

3. Use Stokes Theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F} = 2z\mathbf{i} + (8x - 3y)\mathbf{j} + (3x + y)\mathbf{k}$ and C is the boundary of the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$ (traversed in that order).

4. Let W be the solid cylinder bounded by $x^2 + y^2 = 4$, $z = 0$, and $z = 3$. If $\mathbf{F} = (x^3 + \tan yz)\mathbf{i} + (y^3 - e^{xz})\mathbf{j} + (3z + x^3)\mathbf{k}$, find the flux of $\mathbf{F}$ across $S = \partial W$. (Hint: Evaluating the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly would be hard. Use the Divergence Theorem instead. Also, you'll want to use cylindrical coordinates.)

5. Use the Divergence Theorem to compute $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = 3x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$ and W is the ball $x^2 + y^2 + z^2 \leq 9$.

6. Use Green's Theorem to find the area of the region enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

7. Let

$$\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j} = P\mathbf{i} + Q\mathbf{j}.$$

Show that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$. Let C be the unit circle and show that $\int_C P\,dx + Q\,dy \neq 0$. Why doesn't this contradict Green's Theorem?

8. Find the volume of the solid that is bounded above by the sphere $\rho = a$, and below by the cone $\phi = \alpha$, where a and α are constants, and ϕ is the

angle made with the positive z-axis. (Hint: this is just a "rectangular box" in spherical coordinates, i.e. use change of variables to switch to spherical coordinates and then the triple integral will have constant limits.)

9. Evaluate $\iint_R y\, dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$. Use the change of variables $x = r\cos\theta$, $y = r\sin\theta$.

10. Evaluate $\iint_R \sin(y^3)\, dA$ where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$. (Hint: remember, you can change the order of integration.)

11. Suppose the temperature T on a circular plate $\{(x, y) : x^2 + y^2 \leq 1\}$ is given by $T(x, y) = 2x^2 + y^2 - y$. Find the hottest and coldest spots.