

Multivariable Calculus/Vector Analysis More Review
Problems – Solutions Summer 2009
Courant Institute

1. Using the Lagrange multiplier method, find the dimensions of a cylindrical can, with lid, which is to contain 1 liter of water, and uses the minimum amount of metal. (i.e. minimize the surface area subject to the constraint that the volume is equal to 1).

2. Use Stokes Theorem to compute the surface integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + yz\mathbf{k}$ and S is the portion of the paraboloid $z = x^2 + y^2$ satisfying $0 \leq z \leq 1$.

Solution: By Stokes Theorem, $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{S} = \int_C y dx - x dy + yz dz$ where C is the boundary of S , which we can parameterize as follows: $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 1$. The line integral becomes $\int_0^{2\pi} ((\sin t)(-\sin t) - (\cos t)(\cos t) + 0) dt = -2\pi$.

3. Use Stokes Theorem to evaluate the line integral $\int_C \mathbf{F} \cdot ds$ where $\mathbf{F} = 2z\mathbf{i} + (8x - 3y)\mathbf{j} + (3x + y)\mathbf{k}$ and C is the boundary of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$ (traversed in that order).

Solution: By Stokes Theorem, $\int_C \mathbf{F} \cdot ds = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where S is the surface which is the triangle with the given vertices, which happens to be the graph of $z = 2 - 2x - 2y$ over the triangle D in the xy -plane with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. We can compute the surface integral as follows. First, $\text{curl } \mathbf{F} = \mathbf{i} - \mathbf{j} + 8\mathbf{k}$, and we get $\iint_D (1, -1, 8) \cdot (2, 2, 1) dA = 4$.

4. Let W be the solid cylinder bounded by $x^2 + y^2 = 4$, $z = 0$, and $z = 3$. If $\mathbf{F} = (x^3 + \tan yz)\mathbf{i} + (y^3 - e^{xz})\mathbf{j} + (3z + x^3)\mathbf{k}$, find the flux of \mathbf{F} across $S = \partial W$. (Hint: Evaluating the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly would be hard. Use the Divergence Theorem instead. Also, you'll want to use cylindrical coordinates.)

Solution: By the Divergence Theorem (Gauss' Thm.) $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div } \mathbf{F} dV$, where S is the boundary of the solid cylinder W . We have $\text{div } \mathbf{F} = 3x^2 + 3y^2 + 3$ so we get $\int_0^2 \int_0^{2\pi} \int_0^3 (3x^2 + 3y^2 + 3)r dz d\theta dr = 108\pi$.

5. Use the Divergence Theorem to compute $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = 3x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$ and W is the ball $x^2 + y^2 + z^2 \leq 9$.

Solution: $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div } \mathbf{F} dV$. Now, $\text{div } \mathbf{F} = 3 - 2 + 4 = 5$ so we get $5 \iiint_W dV = 180\pi$.

6. Use Green's Theorem to find the area of the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

7. Let

$$\mathbf{F} = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j} = P\mathbf{i} + Q\mathbf{j}.$$

Show that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$. Let C be the unit circle and show that $\int_C P dx + Q dy \neq 0$. Why doesn't this contradict Green's Theorem?

Solution: A straightforward calculation shows that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$. The line integral is computed by

$$\int_C \frac{y}{x^2 + y^2} dx - \frac{x}{x^2 + y^2} dy = \int_0^{2\pi} [(\sin t)(-\sin t) - (\cos t)(\cos t)] dt = -2\pi \neq 0.$$

The reason this doesn't contradict Green's Theorem, is that there is no region D , on which C is the boundary curve, and on which \mathbf{F} is defined. The curve C is the boundary of the unit disk, but the unit disk contains the origin, and \mathbf{F} isn't defined at the origin.

8. Find the volume of the solid that is bounded above by the sphere $\rho = a$, and below by the cone $\phi = \alpha$, where a and α are constants, and ϕ is the angle made with the positive z -axis. (Hint: this is just a "rectangular box" in spherical coordinates, i.e. use change of variables to switch to spherical coordinates and then the triple integral will have constant limits.)

Solution:

$$\int_0^\alpha \int_0^{2\pi} \int_0^a \rho^2 \sin \phi d\rho d\theta d\phi = \frac{2}{3}\pi a^3(1 - \cos \alpha).$$

9. Evaluate $\iint_R y dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos \theta)$. Use the change of variables $x = r \cos \theta$, $y = r \sin \theta$.

10. Evaluate $\iint_R \sin(y^3) dA$ where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$. (Hint: remember, you can change the order of integration.)

Solution: The point of this exercise is that the integral is hard if you do it one way, and easy if you do it the other way. The easy way is

$$\int_0^2 \int_0^{y^2} \sin(y^3) dx dy = \int_0^2 \sin(y^3) y^2 dy.$$

(Now just use u -substitution.)

11. Suppose the temperature T on a circular plate $\{(x, y) : x^2 + y^2 \leq 1\}$ is given by $T(x, y) = 2x^2 + y^2 - y$. Find the hottest and coldest spots.