Multivariable Calculus/Vector Analysis **Midterm Review**
Summer 2009 **Courant Institute**

1. Determine if the following limits exist and if so, evaluate the limit.

a) $$\lim_{(x,y)\to(-1,2)} \frac{xy - y^3}{(x+y+1)^2}$$

b) $$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}$$

c) $$\lim_{(x,y)\to(0,0)} \frac{xy + y^3}{x^2+y^2}$$

2. The surface $9x^2 + 4y^2 - z = 0$ is coated by a thin material, whose density at each point is given by $f(x, y, z) = 3x + 2y + z + 5$. Find the spot(s) where the density is a minimum. How do you know your answer is a minimum, and not a maximum?

3. Compute the volume of the solid region in the first octant bounded by the surface $z = e^{x-y}$, the plane $x + y = 1$, and the coordinate planes.

4. Consider the planes

$$x + 2y - 2z = 3 \qquad \text{and} \qquad 2x + 4y - 4z = 7$$

If these planes are parallel, find the distance between them. If they are not parallel, find the equation of their line of intersection.

5. A function $u = f(x, y)$ with continuous second partial derivatives satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called *harmonic*. Show that $u = x^3 - 3xy^2$ is harmonic.

6. Find the equation of the tangent plane to the surface $xz^2 + yz = 6$ at the point $(2, -1, 2)$.

7. a) Compute the directional derivative of $f(x, y, z) = x^2y + ye^{xz}$ at the point $(1, 2, 0)$ in the direction of $\vec{v} = (12, 4, -3)$.

b) In which direction is f increasing the fastest at the point $(1, 2, 0)$ and what is the directional derivative in this direction?

8. Set up and evaluate the triple integral which gives the volume of the solid bounded by $x = 0$, $y = 0$, $z = 0$, and $z = -2x - 2y + 4$.

9. A rectangular box without a top is to have a fixed volume of 4000 cm^3. What should its dimensions be to minimize its total surface area?

10. a) Find all the critical points of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$.

b) For each critical point, determine if it is a local maximum, local minimum, or saddle point. Are any of the local extrema also global extrema?

11. The centers of two three-dimensional balls both of radius a are $2b$ units apart with $b \leq a$. Set up but *do not evaluate* the triple integral giving the volume of their intersection.

12. Find the distance from the point $(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

13. Consider the function $f : \mathbf{R^3} \to \mathbf{R}$ given by $f(x, y, z) = x^2 + y^2 - z^2$. Describe and sketch the level surfaces of this funtion corresponding to the values $c = -1, 0, 1$.

14. What is the volume of the parallelepiped spanned by the vectors $\mathbf{i} - \mathbf{j} + \mathbf{2k}$, $\mathbf{j} + \mathbf{4k}$, and $-\mathbf{i} + \mathbf{3j} + \mathbf{k}$?

15. Let $f(x, y, z) = x^2e^{-yz}$.

a) Compute ∇f.

b) Compute the directional derivative in the direction of $\vec{u} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

c) In which direction is f increasing the fastest and what is the directional derivative in this direction?

16. Compute the first and second order Taylor approximations for the function $f(x, y) = \frac{1}{x^2 + y^2 + 1}$, where $x_0 = 0$, $y_0 = 0$.

17. Let $f(x, y, z) = xyz$ and $(x_0, y_0, z_0) = (1, 1, 1)$.

a) Calculate ∇f at (x_0, y_0, z_0).

b) Find the equation of the tangent plane to the level surface of f at (x_0, y_0, z_0).

18. Let $f(x, y) = x^2 + 2y^2 - 2x + 3$ on the disk $D = \{(x, y)|x^2 + y^2 \leq 10\}$. Find the maximum and minimum values that f attains on D.

19. For the function $f(x, y) = 3x^2 - 6xy + 5y^2 + y^3$, determine all the critical points, and for each one determine if it is a local maximum, local minimum, or saddle point.

20. Compute $\iiint_W \sqrt{x^2 + y^2 + z^2}\, e^{-(x^2+y^2+z^2)}\, dV$, where W is the solid bounded by the two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $0 < b < a$.

21. The temperature T on the spherical surface $x^2 + y^2 + z^2 = 1$ is given by $T(x, y, z) = xz + yz$. Find all the hot spots (i.e. maximize T subject to the constraint $x^2 + y^2 + z^2 = 1$).

22. Show that $\mathbf{F} = y(\cos x)\mathbf{i} + x(\sin y)\mathbf{j}$ is *not* a gradient vector field (HINT: compute the curl).

23. Set up *but do not evaluate* the triple integral which gives the total mass of the solid bounded by $x = 0$, $y = 0$, $z = 0$, and $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ where the density at each point is given by $\rho(x, y, z) = x + yz$.

24. Compute the volume under the surface $z = e^{x^2+y^2}$ and lying above the region in the first quadrant bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 9$, $y = 0$, and $x = y$. Use a double integral with polar coordinates. (Remember, $dA = r dr d\theta$.)

25. Compute the volume of the solid region bounded by the surfaces $x^2 + 2y^2 = 2$, $z = 0$, and $x + y + 2z = 2$.