Multivariable Calculus/Vector Analysis G63.1002/V63.0224 Midterm
SOLUTIONS Summer 2009
Courant Institute

Closed book, closed notes, show all work to get full credit.

1. Let $f(x, y, z) = x^2 + y^2 + 2z^2$ and $(x_0, y_0, z_0) = (1, 2, 3)$.

a) Calculate ∇f at (x_0, y_0, z_0).

Solution: $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k}$ which at the point $(1, 2, 3)$ is $\nabla f = 2\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$

b) Find the equation of the tangent plane to the level surface of f at (x_0, y_0, z_0).

$2(x - 1) + 4(y - 2) + 12(z - 3) = 0$.

2. For the function $f(x, y) = 3x^3 + y^2 - 9x + 4y$, determine all the critical points, and for each one determine if it is a local maximum, local minimum, or saddle point. Are any of the local extrema also global extrema?

Solution: Compute the first and second order partials: $\dfrac{\partial f}{\partial x} = 9x^2 - 9$; $\dfrac{\partial f}{\partial y} = 2y + 4$; $\dfrac{\partial^2 f}{\partial x^2} = 18x$; $\dfrac{\partial^2 f}{\partial x \partial y} = 0$; $\dfrac{\partial^2 f}{\partial y^2} = 2$.

Now set the two first order partials equal to 0 and solve to find that there are two critical points at $(1, -2)$ and $(-1, -2)$. The discriminant at $(1, -2)$ is positive, and $\dfrac{\partial^2 f}{\partial x^2} > 0$ so by the second derivative test, $(1, -2)$ is a local minimum. The discriminant at $(-1, -2)$ is negative, this is a saddle point.

3. Compute the first and second order Taylor approximations for the function $f(x, y) = e^{(x-1)^2} \cos y$, where $x_0 = 1$, $y_0 = 0$.

Solution: $\dfrac{\partial f}{\partial x} = 2(x-1)e^{(x-1)^2} \cos y, \dfrac{\partial f}{\partial y} = e^{(x-1)^2}(-\sin y)$, $\dfrac{\partial^2 f}{\partial x^2} = (\cos y)[4(x-1)^2 e^{(x-1)^2} + 2e^{(x-1)^2}]$, $\dfrac{\partial^2 f}{\partial x \partial y} = 2(x - 1)^2 e^{(x-1)^2}(-\sin y)$, and $\dfrac{\partial^2 f}{\partial y^2} = (-\cos y)e^{(x-1)^2}$.

Evaluated at the point $(1, 0)$ these give $\dfrac{\partial f}{\partial x}(1, 0) = 0$, $\dfrac{\partial f}{\partial y}(1, 0) = 0$, $\dfrac{\partial^2 f}{\partial x^2}(1, 0) = 2$, $\dfrac{\partial^2 f}{\partial x \partial y}(1, 0) = 0$, and $\dfrac{\partial^2 f}{\partial y^2}(1, 0) = -1$. Also we have $f(1, 0) = 1$. This gives a first order approximation of $f(1 + h_1, h_2) = 1 + R_1$ and a second order aproximation of $f(1 + h_1, h_2) = 1 + h_1^2 - (1/2)h_2^2 + R_2$. (We did *not* cover any explicit formulas for the remainders R_1 and R_2.)

4. Let $f(x, y) = x^2 + y^2$ on the domain D consisting of all points in $\mathbf{R^2}$ satisfying $x^2 + 2y^2 \leq 1$. Find the maximum and minimum values that f attains on D. (HINT: on the boundary of

D, which is an ellipse, you can parameterize it by $\vec{c}(t) = (\cos t, (1/\sqrt{2})\sin t)$, or you can use Lagrange multipliers.)

Solution: Setting $\nabla f = 0$ we get the only critical point is $(0,0)$. On the boundary we have $g(t) = f(\mathbf{c}(t)) = \cos^2 t + \frac{1}{2}\sin^2 t$. Then, $g'(t) = 2\cos t(-\sin t) + \sin t \cos t = 0$ which gives $(0, \frac{1}{\sqrt{2}})$, $(0, -\frac{1}{\sqrt{2}})$, $(1,0)$, $(-1,0)$. Checking the value of the function at these points and the critical point, we get that the maxima occur at $(1,0)$ and $(-1,0)$ and the minimum occurs at $(0,0)$.

5. Let $\mathbf{F}(x,y,z)$ be the vector field defined by $\mathbf{F}(x,y,z) = x^2y\mathbf{i} + z\mathbf{j} + xyz\mathbf{k}$.

a) Compute the divergence of $\mathbf{F}$.

Solution: $\operatorname{div}\mathbf{F} = 2xy + 0 + xy = 3xy$.

b) Compute the curl of $\mathbf{F}$.

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \overline{\partial x} & \overline{\partial y} & \overline{\partial z} \\ x^2y & z & xyz \end{vmatrix} = (xz-1)\mathbf{i} - (yz-0)\mathbf{j} + (0-x^2)\mathbf{k} = (xz-1)\mathbf{i} - yz\mathbf{j} - x^2\mathbf{k}.$$

6. Use a double integral to calculate the volume within the cyclinder $x^2+y^2=4$ between the planes $y+z=9$ and $z=0$. (Hint: you may use the formula $\int_{-b}^{b}\sqrt{b^2-x^2}\,dx = (1/2)\pi b^2$.)

Solution:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (9-y)\,dy\,dx$$

$$= \int_{-2}^{2}\left[9y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy$$

$$= 18\int_{-2}^{2}\sqrt{4-x^2}\,dy = 36\pi.$$

(You can also integrate in the other order).

7. Evaluate

$$\int\int\int_W \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

where W is the solid bounded by the two spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, where $0<b<a$ (Hint: use change of variables to convert to a triple integral in spherical coordinates).

Solution:

$$\begin{aligned}\iiint_W \frac{dV}{(x^2+y^2+z^2)^{3/2}} &= \int_b^a \int_0^{2\pi} \int_0^{\pi} \frac{\rho^2 \sin\phi}{(\rho^2)^{3/2}} d\phi\, d\theta\, d\rho \\ &= (2\pi) \int_b^a \frac{d\rho}{\rho} \int_0^{\pi} \sin\phi\, d\phi \\ &= (2\pi)(\ln a - \ln b)(2) = 4\pi \ln(a/b).\end{aligned}$$