Multivariable Calculus/Vector Analysis Review Problems
Solutions Summer 2009
Courant Institute

1. Determine if the following limits exist and if so, evaluate the limit.

a) $$\lim_{(x,y)\to(-1,2)} \frac{xy - y^3}{(x+y+1)^2}$$

Solution: $\dfrac{-5}{2}$.

b) $$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}$$

Solution: 1.

c) $$\lim_{(x,y)\to(0,0)} \frac{xy + y^3}{x^2+y^2}$$

Solution: Does not exist.

2. The surface $9x^2 + 4y^2 - z = 0$ is coated by a thin material, whose density at each point is given by $f(x, y, z) = 3x + 2y + z + 5$. Find the spot(s) where the density is a minimum. How do you know your answer is a minimum, and not a maximum?

Solution: $\nabla f = \lambda \nabla g$. We get

$$3 = \lambda(18)x$$
$$2 = \lambda(8)y$$
$$1 = \lambda(-1)$$

Solve to get $(x, y, z) = (-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2})$. Clearly, this is not a maximum as $f \to \infty$ as x and y go to infinity.

3. Compute the volume of the solid region in the first octant bounded by the surface $z = e^{x-y}$, the plane $x + y = 1$, and the coordinate planes.

Solution:

$$\int_0^1 \int_0^{1-y} e^{x-y}\, dx\, dy = \int_0^1 \left[e^{x-y}\right]_0^{1-y} dy = \int_0^1 (e^{1-2y} - e^{-y})\, dy = \frac{e^{-1}}{2} + \frac{e}{2} - 1.$$

4. Consider the planes

$$x + 2y - 2z = 3 \qquad \text{and} \qquad 2x + 4y - 4z = 7$$

If these planes are parallel, find the distance between them. If they are not parallel, find the equation of their line of intersection.

Solution: Parallel, the distance is $\frac{1}{6}$.

5. A function $u = f(x, y)$ with continuous second partial derivatives satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called *harmonic*. Show that $u = x^3 - 3xy^2$ is harmonic.

6. Find the equation of the tangent plane to the surface $xz^2 + yz = 6$ at the point $(2, -1, 2)$.

Solution: Let $g(x, y, z) = xz^2 + yz = 6$. Then $\nabla g(2, -1, 2) = (4, 2, 7)$ and the tangent plane is $4(x - 2) + 2(y + 1)7(z - 2) = 0$.

7. a) Compute the directional derivative of $f(x, y, z) = x^2y + ye^{xz}$ at the point $(1, 2, 0)$ in the direction of $\vec{v} = (12, 4, -3)$.

b) In which direction is f increasing the fastest at the point $(1, 2, 0)$ and what is the directional derivative in this direction?

8. Set up and evaluate the triple integral which gives the volume of the solid bounded by $x = 0$, $y = 0$, $z = 0$, and $z = -2x - 2y + 4$.

Solution:

$$\begin{aligned} V &= \int_0^2 \int_0^{2-y} \int_0^{-2x-2y+4} dz\,dx\,dy \\ &= \int_0^2 \int_0^{2-y} (-2x-2y+4)\,dx\,dy \\ &= \int_0^2 \left[-x^2-2xy+4x\right]_0^{2-y} dy \\ &= \int_0^2 -(2-y)^2-2(2-y)y+4(2-y)\,dy \\ &= \frac{8}{3}. \end{aligned}$$

9. A rectangular box without a top is to have a fixed volume of 4000 cm^3. What should its dimensions be to minimize its total surface area?

Solution: Let $V = wlh = 4000$ and $A(w,l,h) = wl + 2wh + 2lh$. Then $\nabla A = \lambda \nabla V$ or

$$\begin{aligned} l + 2h &= \lambda lh \\ w + 2h &= \lambda wh \\ 2w + 2l &= \lambda wl \end{aligned}$$

Solving, we get $l = w$ and $h = w/2$. Using $4000 = wlh$ we get $l = w = 20$ and $h = 10$.

10. a) Find all the critical points of $f(x,y) = 3x^2 - 2xy + y^2 - 8y$.

b) For each critical point, determine if it is a local maximum, local minimum, or saddle point. Are any of the local extrema also global extrema?

11. The centers of two three-dimensional balls both of radius a are $2b$ units apart with $b \leq a$. Set up but *do not evaluate* the triple integral giving the volume of their intersection.

Solution:

$$2\int_b^a \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \int_{-\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}} dx\,dy\,dz.$$

12. Find the distance from the point $(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution: $\frac{17}{7}$.

13. Consider the function $f : \mathbf{R^3} \to \mathbf{R}$ given by $f(x, y, z) = x^2 + y^2 - z^2$. Describe and sketch the level surfaces of this function corresponding to the values $c = -1, 0, 1$.

Solution: For $c = -1$ it is a hyperboloid of two sheets, for $c = 0$ it is a cone, for $c = 1$ it is a hyperboloid of one sheet.

14. What is the volume of the parallelepiped spanned by the vectors $\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $\mathbf{j} + 4\mathbf{k}$, and $-\mathbf{i} + 3\mathbf{j} + \mathbf{k}$?

Solution:

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \\ -1 & 3 & 1 \end{vmatrix} = 5.$$

15. Let $f(x, y, z) = x^2 e^{-yz}$.

a) Compute ∇f at the point $(x_0, y_0, z_0) = (1, 1, 1)$.

Solution: $\nabla f = 2xe^{-yz}\mathbf{i} + x^2e^{-yz}(-z)\mathbf{j} + x^2e^{-yz}(-y)\mathbf{k}$. Evaluated at $(1, 1, 1)$ gives $\nabla f = 2e^{-1}\mathbf{i} - e^{-1}\mathbf{j} - e^{-1}\mathbf{k}$.

b) Compute the directional derivative in the direction of $\vec{u} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Solution: 0.

c) In which direction is f increasing the fastest and what is the directional derivative in this direction?

Solution:

In the direction of $2e^{-1}\mathbf{i} - e^{-1}\mathbf{j} - e^{-1}\mathbf{k}$, at rate of $||2e^{-1}\mathbf{i} - e^{-1}\mathbf{j} - e^{-1}\mathbf{k}|| = \sqrt{6e^{-2}}$.

16. Compute the first and second order Taylor approximations for the function $f(x, y) = \frac{1}{x^2 + y^2 + 1}$, where $x_0 = 0$, $y_0 = 0$.

17. Let $f(x, y, z) = xyz$ and $(x_0, y_0, z_0) = (1, 1, 1)$.

a) Calculate ∇f at (x_0, y_0, z_0).

Solution: $\nabla f(1, 1, 1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

b) Find the equation of the tangent plane to the level surface of f at (x_0, y_0, z_0).

Solution: $(x-1)+(y-1)+(z-1)=0$.

18. Let $f(x,y)=x^2+2y^2-2x+3$ on the disk $D=\{(x,y)|x^2+y^2\leq 10\}$. Find the maximum and minimum values that f attains on D.

19. For the function $f(x,y)=3x^2-6xy+5y^2+y^3$, determine all the critical points, and for each one determine if it is a local maximum, local minimum, or saddle point.

Solution: The critical points are $(0,0)$ and $(-4/3,-4/3)$. Using the second derivative test we get that the first is a local minimum, and the second is a saddle point.

20. Compute $\iiint_W \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dV$, where W is the solid bounded by the two spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $0<b<a$.

Solution: Changing to spherical coordinates we get

$$\begin{aligned}\iiint_W \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dV &= \int_0^{\pi}\int_0^{2\pi}\int_b^a \rho e^{-\rho^2}\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\ &= \Big(\int_0^{2\pi} d\theta\Big)\Big(\int_0^{\pi}\sin\phi\, d\phi\Big)\Big(\int_b^a \rho e^{-\rho^2}\rho^2\, d\rho\Big) \\ &= 2\pi[e^{-a^2}(-a^2-1)-e^{-b^2}(-b^2-1)].\end{aligned}$$

21. The temperature T on the spherical surface $x^2+y^2+z^2=1$ is given by $T(x,y,z)=xz+yz$. Find all the hot spots (i.e. maximize T subject to the constraint $x^2+y^2+z^2=1$).

Solution: $\nabla T = z\mathbf{i}+z\mathbf{j}+(x+y)\mathbf{k}$ and $\nabla g = x^2+y^2+z^2-1$. The equation $\nabla g=\lambda\nabla f$ gives the equations

$$\begin{aligned}2x &= \lambda z\\ 2y &= \lambda z\\ 2z &= \lambda(x+y)\\ x^2+y^2+z^2 &= 1\end{aligned}$$

Solving, we get the points $(1/2,1/2,1/\sqrt{2})$, $(1/2,1/2,-1/\sqrt{2})$, $(-1/2,-1/2,1/\sqrt{2})$, $(-1/2,-1/2,-1/\sqrt{2})$. The maximam value of T occurs at the first and fourth of these, and is $T=\dfrac{1}{\sqrt{2}}$.

22. Show that $\mathbf{F} = y(\cos x)\mathbf{i} + x(\sin y)\mathbf{j}$ is *not* a gradient vector field (HINT: compute the curl).

23. Set up *but do not evaluate* the triple integral which gives the total mass of the solid bounded by $x = 0$, $y = 0$, $z = 0$, and $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ where the density at each point is given by $\rho(x, y, z) = x + yz$.

Solution:

$$\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} (x + yz)\, dz\, dy\, dx$$

24. Compute the volume under the surface $z = e^{x^2+y^2}$ and lying above the region in the first quadrant bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 9$, $y = 0$, and $x = y$. Use a double integral with polar coordinates. (Remember, $dA = r dr d\theta$.)

Solution:

$$\int_1^3 \int_0^{\frac{\pi}{4}} e^{r^2} r\, d\theta\, dr = \frac{\pi}{8}(e^9 - e^1).$$

25. Compute the volume of the solid region bounded by the surfaces $x^2 + 2y^2 = 2$, $z = 0$, and $x + y + 2z = 2$.

Solution:

$$\int_{-1}^1 \int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} \left[1 - x/2 - y/2\right] dx\, dy = \int_{-1}^1 \left[x - \frac{x^2}{4} - \frac{xy}{2}\right]_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} dy = 2\sqrt{2}\pi.$$