Basic Probability: Problem Set 1

1.3.1 We have $A \cap B \subset B \Rightarrow \mathbb{P}(A \cap B) \leq \mathbb{P}(B)$. This upper bound on $\mathbb{P}(A \cap B)$ is attained if $B \subset A \Rightarrow A \cap B = B \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(B)$. The similar fact for A is true as for B.

From the definition of a probability measure we have $\mathbb{P}(A \cap B) \geq 0$. We also have from the inclusion-exclusion principle that

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \\ &\geq \mathbb{P}(A) + \mathbb{P}(B) - 1\end{aligned}$$

So we have

$$\max(\mathbb{P}(A) + \mathbb{P}(B) - 1, 0) \leq \mathbb{P}(A \cap B) \leq \min(\mathbb{P}(A), \mathbb{P}(B))$$

or, in this case,

$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$$

For examples of attaining each bound, let $\Omega = \{1, 2, \ldots, 12\}$ let $\mathcal{F} = P(\Omega)$, and let $\mathbb{P}(\omega) = \frac{1}{12}$ for each $\omega \in \Omega$. Further, let $A = \{\omega | \omega \leq 9\}$.

When $B = \{\omega | \omega \geq 9\}$, $\mathbb{P}(A \cap B) = \frac{1}{12}$ and for $B = \{\omega | \omega \leq 4\}$, $\mathbb{P}(A \cap B) = \frac{1}{3}$. In each case, $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$.

For comparable bounds on $\mathbb{P}(A \cup B)$, note that from the above we have

$$\max(\mathbb{P}\left(A^C\right) + \mathbb{P}\left(B^C\right) - 1, 0) \leq \mathbb{P}\left(A^C \cap B^C\right) \leq \min(\mathbb{P}\left(A^C\right), \mathbb{P}\left(B^C\right))$$

But this implies that

$$\max(1 - \mathbb{P}(A) + 1 - \mathbb{P}(B) - 1, 0) \leq 1 - \mathbb{P}(A \cup B) \leq \min(1 - \mathbb{P}(A), 1 - \mathbb{P}(B))$$

or

$$1 - \min(\mathbb{P}(A) + \mathbb{P}(B), 1) \leq 1 - \mathbb{P}(A \cup B) \leq 1 - \max(\mathbb{P}(A), \mathbb{P}(B))$$

or

$$\min(\mathbb{P}(A) + \mathbb{P}(B), 1) \geq \mathbb{P}(A \cup B) \geq \max(\mathbb{P}(A), \mathbb{P}(B))$$

1.4.3 Let 0, 1, and 2 be the events that the coin has 0, 1, and 2 heads, respectively. Let H_L (resp. H_U) be the events that the lower (upper) side is heads. Note that $H_U \cap H_L = 2$ (the event 2) and $\mathbb{P}(H_L) = \mathbb{P}(H_U)$

$$\begin{aligned}\mathbb{P}(H_L) &= \mathbb{P}(H_L|0)\,\mathbb{P}(0) + \mathbb{P}(H_L|1)\,\mathbb{P}(1) + \mathbb{P}(H_L|2)\,\mathbb{P}(2) \\ &= 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5} + 1 \cdot \frac{2}{5} \\ &= \frac{3}{5}\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}(H_L|H_U) &= \frac{\mathbb{P}(H_L \cap H_U)}{\mathbb{P}(H_U)} \\
&= \frac{\mathbb{P}(2)}{\mathbb{P}(H_L)} \\
&= \frac{2/5}{3/5} \\
&= \frac{2}{3}
\end{aligned}
$$

Let $H_U H_L$ be the event that the first toss is heads up and the second is heads down.

$$
\begin{aligned}
\mathbb{P}(H_U H_L|H_U) &= \frac{\mathbb{P}(H_U H_L \cap H_U)}{\mathbb{P}(H_U)} \\
&= \frac{\mathbb{P}(H_U H_L)}{3/5} \\
&= \frac{5}{3}\left(\mathbb{P}(H_U H_L|0)\,\mathbb{P}(0) + \mathbb{P}(H_U H_L|1)\,\mathbb{P}(1) + \mathbb{P}(H_U H_L|2)\,\mathbb{P}(2)\right) \\
&= \frac{5}{3}\left(0 \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{2}{5} + 1 \cdot \frac{2}{5}\right) \\
&= \frac{5}{6}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(H_U H_L|H_U H_U) &= \frac{\mathbb{P}(H_U H_L \cap H_U H_U)}{\mathbb{P}(H_U H_U)} \\
&= \frac{\mathbb{P}(2)}{\mathbb{P}(H_U H_U)} \\
&= \frac{\mathbb{P}(2)}{\mathbb{P}(H_U H_L)} \\
&= \frac{2/5}{1/2} \\
&= \frac{4}{5}
\end{aligned}
$$

We do the final question in two parts. First, we condition on what the first coin was. First, suppose the first coin was 1 (it cannot have been 0).

$$
\begin{aligned}
\mathbb{P}(H_U H_U H_U|1 \cap H_U H_U) &= \mathbb{P}(H_U H_U H_U|10 \cap H_U H_U)\,\mathbb{P}(10 \cap H_U H_U|1 \cap H_U H_U) \\
&\quad + \mathbb{P}(H_U H_U H_U|11 \cap H_U H_U)\,\mathbb{P}(11 \cap H_U H_U|1 \cap H_U H_U) \\
&\quad + \mathbb{P}(H_U H_U H_U|12 \cap H_U H_U)\,\mathbb{P}(12 \cap H_U H_U|1 \cap H_U H_U) \\
&= 0 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} \\
&= \frac{5}{8}
\end{aligned}
$$

Next, suppose it was 2.

$$\begin{aligned}
\mathbb{P}(H_U H_U H_U | 2 \cap H_U H_U) &= \mathbb{P}(H_U H_U H_U | 20 \cap H_U H_U)\,\mathbb{P}(20 \cap H_U H_U | 2 \cap H_U H_U) \\
&\quad + \mathbb{P}(H_U H_U H_U | 21 \cap H_U H_U)\,\mathbb{P}(21 \cap H_U H_U | 2 \cap H_U H_U) \\
&\quad + \mathbb{P}(H_U H_U H_U | 22 \cap H_U H_U)\,\mathbb{P}(22 \cap H_U H_U | 2 \cap H_U H_U) \\
&= 0 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

We have $\mathbb{P}(2|H_U H_U) = \mathbb{P}(H_U H_L | H_U H_U) = \frac{4}{5}$, and since $\mathbb{P}(0|H_U H_U) = 0$, we must have $\mathbb{P}(1|H_U H_U) = \frac{1}{5}$

Finally, combining the results:

$$\begin{aligned}
\mathbb{P}(H_U H_U H_U | H_U H_U) &= \mathbb{P}(H_U H_U H_U | 1 \cap H_U H_U) \cdot \mathbb{P}(1|H_U H_U) \\
&\quad + \mathbb{P}(H_U H_U H_U | 2 \cap H_U H_U) \cdot \mathbb{P}(2|H_U H_U) \\
&= \frac{5}{8} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{4}{5} \\
&= \frac{21}{40}
\end{aligned}$$

1.5.1

$$\begin{aligned}
\mathbb{P}\left(A^C\right)\mathbb{P}(B) &= (1 - \mathbb{P}(A))\mathbb{P}(B) \\
&= \mathbb{P}(B) - \mathbb{P}(A)\,\mathbb{P}(B) \\
&= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
&= \mathbb{P}\left(A^C \cap B\right)
\end{aligned}$$

By the exact same logic, replacing B with A^C and A with B in the equations above, A^C and B^C are independent.

1.5.2 For pairwise independence, consider A_{ij} and $A_{k\ell}$, reordering subscripts so that possibly $i = k$ but definitely $j \neq \ell$. If the i^{th} roll comes up x and the k^{th} roll is y (possibly with $x = y$), then A_{ij} is the event that the j^{th} roll is x and $A_{k\ell}$ is the event that the ℓ^{th} roll is y. Since the rolls are independent, each of these events has conditional probability $\frac{1}{6}$ and the conditional probability of both is $\frac{1}{36}$, *no matter what x and y are.* Thus, the probability of each event is $\frac{1}{6}$ and the probability of both is $\frac{1}{36}$.

For independence of all the events, note that having all the pairs be equal is equivalent to having all the dice be equal, and that there are six ways to get this outcome, each having probability $\frac{1}{6^n}$, so we have $\mathbb{P}\left(\bigcap_{i<j} A_{ij}\right) = \frac{1}{6^{n-1}}$. On the other hand, there are $\frac{n(n-1)}{2}$ pairs $i < j$ so $\prod_{i<j} \mathbb{P}(A_{ij}) = \frac{1}{6^{n(n-1)/2}}$. For $n > 2$, these are not equal to each other.

1.5.9 Following the proof given in class:

There are 36 equally likely outcomes of the roll, and six of them ($\{(n, 7-n) | n = 1, 2, \ldots, 6\}$) result in a sum of 7. Therefore, the probability that the sum of the dice is 7 is $\frac{1}{6}$.

For each $1 \le n \le 6$, there are 6 ways for the first roll to be n (therefore probability $\frac{1}{6}$), exactly one of which (namely, $(n, 7-n)$) intersects the first event (therefore probability of both is $\frac{1}{36}$).

Since $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$, the two events are independent.

1.8.1 (a) Define the events: $A_1 = 6$ turns up on first die and $A_2 = 6$ turns up on second die, $A = 6$ turns up exactly once.

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}((A_1 \cup A_2) - (A_1 \cap A_2)) \\
&= \mathbb{P}(A_1 \cup A_2) - \mathbb{P}(A_1 \cap A_2) \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) - 2\mathbb{P}(A_1 \cap A_2) \\
&= \frac{1}{6} + \frac{1}{6} - 2 \cdot \frac{1}{36} \\
&= \frac{5}{18}
\end{aligned}$$

(b) $A_1 =$ first number is odd, $A_2 =$ second number is odd. The rolls are independent, so A_1 and A_2 are.

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \\
&= \frac{1}{2} \cdot \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

(c) There are three ways for this to happen: $A = \{(1, 3), (2, 2), (3, 1)\}$. Each way is equally as likely as every other of the 36 joint possibilities, so $\mathbb{P}(A) = \frac{1}{12}$.

(d) Of the numbers 1 through 6, two each are congruent to 0, 1, and 2 modulo 3. If the first roll is x modulo 3, then *regardless of* x there is a one-third chance that the second is congruent to $-x$ modulo 3 (i.e. that their sum is 0 modulo 3). Thus the probability that the sum is divisible by 3 is $\frac{1}{3}$

1.8.4 Let $P(S)$ denote the power set of a set S. When $\mathcal{F} = P(\Omega)$ it is sufficient to define $\mathbb{P}(\omega)$ for each $\omega \in \Omega$.

Let p be the probability of heads on a given flip. Let $h(\omega)$ for $\omega \in \Omega$ be the number of total heads (e.g. $h(HHT) = 2$).

$$\begin{aligned}
\Omega &= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \\
\mathcal{F} &= P(\Omega) \\
\mathbb{P}(\omega) &= p^{h(\omega)}(1-p)^{3-h(\omega)}
\end{aligned}$$

(b) Let $d(\omega)$ be the indicator that the first and second balls have different colors (1 if different, 0 o.w.). The color of the first ball is one half probability either way, and for the second ball to have the same color as the first has probability $\frac{1}{3}$.

$$\begin{aligned}\Omega &= \{UU, UV, VU, VV\}\\ \mathcal{F} &= P(\Omega)\\ \mathbb{P}(\omega) &= \frac{1+d(\omega)}{6}\end{aligned}$$

(c) Define $\omega = n$ as the event that heads comes up first on toss n, and $\omega = \infty$ if it never comes up.
The probability of heads coming up first on toss n ($\omega = n$) is the probability of $n-1$ tails followed by a heads, in other words $(1-p)^{n-1}p$. The probability of heads never coming up ($\omega = \infty$, countable intersection of the nested events "tails comes up on each of the first n tosses") is the limit of the probabilities of those events, $\lim_{n\to\infty}(1-p)^n$, which is 0 if $p > 0$ and 1 otherwise.

$$\begin{aligned}\Omega &= \{1,2,3,\ldots\} \cup \{\infty\}\\ \mathcal{F} &= P(\Omega)\end{aligned}$$

and $\mathbb{P}$ as defined above.

1.8.15 Let D_i be the value of the i^{th} die.

(a) If $S = 4$ then $N \leq 4$ so our sum in the following stops at $N = 4$.

$$\begin{aligned}\mathbb{P}(N=2|S=4) &= \frac{\mathbb{P}(S=4|N=2)\,\mathbb{P}(N=2)}{\sum_{i=1}^{4}\mathbb{P}(S=4|N=i)\,\mathbb{P}(N=i)}\\ &= \frac{\frac{1}{12}\cdot\frac{1}{4}}{\frac{1}{6}\cdot\frac{1}{2}+\frac{1}{12}\cdot\frac{1}{4}+\frac{3}{216}\cdot\frac{1}{8}+\frac{1}{6^4}\cdot\frac{1}{16}}\\ &\simeq 0.197\end{aligned}$$

(b) Again, we must have $N \leq 4$. Let E be the event that N is even.

$$\begin{aligned}\mathbb{P}(S=4|E) &= \frac{\mathbb{P}(S=4\cap E)}{\mathbb{P}(E)}\\ &= \frac{\mathbb{P}(S=4|N=2)\,\mathbb{P}(N=2)+\mathbb{P}(S=4|N=4)\,\mathbb{P}(N=4)}{\sum_{i=1}^{\infty}\mathbb{P}(N=2i)}\\ &= \frac{\frac{1}{12}\cdot\frac{1}{4}+\frac{1}{6^4}\cdot\frac{1}{16}}{\sum_{i=1}^{\infty}(1/4)^i}\\ &= 3\left(\frac{1}{12}\cdot\frac{1}{4}+\frac{1}{6^4}\cdot\frac{1}{16}\right)\\ &\simeq 0.063\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{P}(N=2|S=4\cap D_1=1) &= \frac{\mathbb{P}(S=4\cap D_1=1|N=2)\,\mathbb{P}(N=2)}{\sum_{i=1}^4 \mathbb{P}(S=4\cap D_1=1|N=i)\,\mathbb{P}(N=i)} \\
&= \frac{\frac{1}{36}\cdot\frac{1}{4}}{0\cdot\frac{1}{2}+\frac{1}{36}\cdot\frac{1}{4}+\frac{2}{216}\cdot\frac{1}{8}+\frac{1}{6^4}\cdot\frac{1}{16}} \\
&\simeq 0.852
\end{aligned}$$

(d) For $0 \le r \le 6$, define:

$$\begin{aligned}
f(r) &= \mathbb{P}\left(\bigcap_{i=1}^N D_i \le r\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{r}{6}\right)^n 2^{-n} \\
&= \sum_{n=1}^{\infty}\left(\frac{r}{12}\right)^n \\
&= \frac{r/12}{1-r/12} \\
&= \frac{r}{12-r}
\end{aligned}$$

Note $f(6)=1$. Then the probability that the highest roll for $1 \le r \le 6$ is exactly r for $1 \le r \le 6$ is $f(r)-f(r-1)$, or $\frac{12}{(12-r)(13-r)}$

1.8.22 Model this problem as randomly ordering the cherries from 1 to 20, letting the first 15 be those that the pig eats, and the sixteenth be the one we pick up of the remaining 5.

(a) This is the probability that cherry 16 has a stone, the same as the probability any other of them does, $\frac{1}{4}$.

(b) By the above, there are $20!/4$ total orderings of the cherries having a stone in place 16. There are $15!\cdot 5!$ total orderings having all stones in places 16 through 20. Thus the probability the pig ate no stones given cherry 16 is a stone is $\frac{4}{\binom{20}{5}}$, or .000258. The probability he *did* eat a stone given the cherry we pick is a stone, then is $1-.000258=.999742$.

1.8.30 Note 1991 is not a leap year so it has 365 days.

Inductive proof that the probability no students have the same birthday is $q(m)=\frac{365!}{(365-m)!365^m}$. $q(1)=1$, so the base case works. For the inductive step, assume the formula works for $m-1$. Then there are 365 equally likely choices for the m^{th} student who enters the room, of which $365-(m-1)$ do not intersect with the set of already represented birthdays. Thus the new probability of no common birthdays is

$$\begin{aligned}
q(m-1)\cdot\frac{365-(m-1)}{365} &= \frac{365!}{((365-(m-1))!/(365-(m-1))365^{m-1}365} \\
&= \frac{365!}{(365-m)!365^m} \\
&= q(m)
\end{aligned}$$

Thus the formula works for all m, and it follows that the probability of the complement of this event, that at least 2 students have the same birthday, is $1-q(m)$. In particular, for $m=23$, $q(m)=.493$ and $1-q(m)=.507>\frac{1}{2}$.

1.8.33 There are $\binom{52}{5}=2,598,960$ total hands of poker, so to compute the probability of a given hand type, we simply divide the number of ways to get that hand type by 2,598,960.

First consider the hands other than flushes and straights. We first choose the duplicated values (i.e. 2,3,...,A), then the unduplicated values, then the suits of each type.

Pair: $\binom{13}{1}\binom{12}{3}\binom{4}{2}\binom{4}{1}^3=1098240$ ways $\Rightarrow p\simeq .423$

Two Pair: $\binom{13}{2}\binom{11}{1}\binom{4}{2}^2\binom{4}{1}=123552$ ways $\Rightarrow p\simeq .0475$

Three of a Kind: $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}^2=54912$ ways $\Rightarrow p\simeq .021$

Full House: $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}=3744$ ways $\Rightarrow p\simeq .0014$

Four of a Kind: $\binom{13}{1}\binom{12}{1}\binom{4}{4}\binom{4}{1}=624$ ways $\Rightarrow p\simeq .00024$

For a straight flush, we must first choose the suit (4 ways) then choose the card values (A, 2, 3, ..., 10) that begins the straight. This gives 40 possible hands, so $p\simeq .000015$.

For a regular flush, we choose the suit (4 ways) and then the values ($\binom{13}{5}$ ways), giving 5148, then subtract off the 40 straight flushes, giving 5108 ways, so $p\simeq 0.0020$.

For a regular straight, we choose the starting value (10 ways), then the suits of each card (4^5 ways), then again subtract the 40 straight flushes, giving 10200 ways, so $p\simeq 0.0039$.

1.8.35 Let T be the event that the passerby is a tourist, B that he is Bandrikan, C_E that the correct answer is East, C_W that West is correct, and, for example, EEE that the first three answers given are all East.

For any sequence of answers a, define

$$\begin{aligned}
e_T(a) &= \mathbb{P}(a|T\cap C_E)\\
w_T(a) &= \mathbb{P}(a|T\cap C_W)\\
e_B(a) &= \mathbb{P}(a|B\cap C_E)\\
w_B(a) &= \mathbb{P}(a|B\cap C_W)\\
e(a) &= \mathbb{P}(a|C_E)=\frac{2e_T(a)+e_B(a)}{3}\\
w(a) &= \mathbb{P}(a|C_W)=\frac{2w_T(a)+w_B(a)}{3}
\end{aligned}$$

Assume that absent any information it is equally likely that the exit is East or West.

(a) We have

$$
\begin{aligned}
e_T(E) &= 3/4 \\
w_T(E) &= 1/4 \\
e_B(E) &= 0 \\
w_B(E) &= 1 \\
e(E) &= 1/2 \\
w(E) &= 1/2
\end{aligned}
$$

So $\mathbb{P}(C_E|E) = \frac{e(E)/2}{w(E)/2+e(E)/2} = 1/2$.

(b) We have:

$$
\begin{aligned}
e_T(EE) &= 9/16 \\
w_T(EE) &= 1/16 \\
e_B(EE) &= 0 \\
w_B(EE) &= 1 \\
e(EE) &= 3/8 \\
w(EE) &= 3/8
\end{aligned}
$$

So $\mathbb{P}(C_E|EE) = \frac{e(EE)/2}{w(EE)/2+e(EE)/2} = 1/2$.

(c) We have:

$$
\begin{aligned}
e_T(EEE) &= 27/64 \\
w_T(EEE) &= 1/64 \\
e_B(EEE) &= 0 \\
w_B(EEE) &= 1 \\
e(EEE) &= 9/32 \\
w(EEE) &= 11/32
\end{aligned}
$$

So $\mathbb{P}(C_E|EEE) = \frac{e(EEE)/2}{w(EEE)/2+e(EEE)/2} = 9/20$.

(d) We have:

$$
\begin{aligned}
e_T(EEEE) &= 81/256 \\
w_T(EEEE) &= 1/256 \\
e_B(EEEE) &= 0 \\
w_B(EEEE) &= 1 \\
e(EEEE) &= 27/128 \\
w(EEEE) &= 43/128
\end{aligned}
$$

So $\mathbb{P}(C_E|EEEE) = \frac{e(EEEE)/2}{w(EEEE)/2+e(EEEE)/2} = 27/70$.

(e) We have:

$$\begin{aligned}
e_T(EEEW) &= 27/256 \\
w_T(EEEW) &= 3/256 \\
e_B(EEEW) &= 0 \\
w_B(EEEW) &= 0 \\
e(EEEW) &= 9/128 \\
w(EEEW) &= 1/128
\end{aligned}$$

So $\mathbb{P}(C_E|EEEW) = \frac{e(EEEW)/2}{w(EEEW)/2+e(EEEW)/2} = 9/10$.

2.1.2 Let G be the distribution function of Y.

If $a = 0$ then $Y = b$ so $G(y)$ is 0 for $y < b$, 1 for all other values of y.

If $a > 0$ then:

$$\begin{aligned}
G(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(aX + b \leq y) \\
&= \mathbb{P}\left(X \leq \frac{y-b}{a}\right) \\
&= F\left(\frac{y-b}{a}\right)
\end{aligned}$$

If $a < 0$ then:

$$\begin{aligned}
G(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(aX + b \leq y) \\
&= \mathbb{P}\left(X \geq \frac{y-b}{a}\right) \\
&= 1 - \mathbb{P}\left(X < \frac{y-b}{a}\right) \\
&= 1 - \lim_{\epsilon \to 0^-} \mathbb{P}\left(X \leq \frac{y-b}{a} + \epsilon\right) \\
&= 1 - \lim_{\epsilon \to 0^-} F\left(\frac{y-b}{a} + \epsilon\right)
\end{aligned}$$

2.7.7 The number of passengers not showing up is $T \sim \text{Bin}(10, \frac{1}{10})$ for Teeny Weeny and $B \sim \text{Bin}(20, \frac{1}{10})$ for Blockbuster. Teeny Weeny is overbooked if $T = 0$ and Blockbuster is overbooked if $B \leq 1$. $\mathbb{P}(T = 0) = \left(\frac{9}{10}\right)^{10} \simeq .349$ and $\mathbb{P}(B \leq 1) = \left(\frac{9}{10}\right)^{20} + 20\left(\frac{9}{10}\right)^{19}\frac{1}{10} \simeq .392$, so Teeny Weeny is overbooked slightly less often.

2.7.8 The number of heads out of six tosses of a fair coin is $\text{Bin}(6, \frac{1}{2})$, so the probability of 5 or more heads is $\left(\frac{1}{2}\right)^6 + 6\left(\frac{1}{2}\right)^6 \simeq .109$.

3.1.3 This is identical to the situation where we throw each of n coins twice and define "success" as coming up heads both times. The probability of success for each coin is p^2 and there are n such coins so the sought random variable is distributed as $\mathrm{Bin}(n, p^2)$ and the mass function is the associated binomial mass function, $f(k) = \binom{n}{k} p^{2k}(1-p^2)^{n-k}$.