Basic Probability Summer 2009
NYU Courant Institute
Midterm Exam with Solutions

1. Suppose that an airplane engine will fail, when in flight, with probability $1-p$ independently from engine to engine; suppose that the airplane will make a successful flight if at least 50% of its engines remain operational. If $p = 3/4$, which is preferable, a four-engine plane or a two-engine plane? What about if $p = 1/2$?

Let X be the binomial variable which is the number of engines that don't fail. For the first part we have $p = 3/4$ and $n = 4$ and we want

$$\begin{aligned} P(X \geq 2) &= P(X=2) + P(X=3) + P(X=4) \\ &= \binom{4}{2}(3/4)^2(1/4)^2 + \binom{4}{3}(3/4)^3(1/4)^1 + \binom{4}{4}(3/4)^4 = .9492. \end{aligned}$$

For the two engine plane we have $n = 2$ and we want

$$P(X \geq 1) = P(X=1) + P(X=2) = \binom{2}{1}(3/4)^1(1/4)^1 + \binom{2}{2}(3/4)^2 = .9375$$

So the four-engine plane is more likely to have a successful flight.

Now, what if $p = 1/2$?

For $n = 4$ we get

$$\begin{aligned} P(X \geq 2) &= P(X=2) + P(X=3) + P(X=4) \\ &= \binom{4}{2}(1/2)^2(1/2)^2 + \binom{4}{3}(1/2)^3(1/2)^1 + \binom{4}{4}(1/2)^4 = .6875 \end{aligned}$$

and for $n = 2$ we get

$$P(X \geq 1) = P(X=1) + P(X=2) = \binom{2}{1}(1/2)^1(1/2)^1 + \binom{2}{2}(1/2)^2 = .75$$

so the two-engine plane is safer. The conclusion is that, depending on p, it may *not* be the case that the more engines the better!

2. Each customer that enters 'Reasonably Honest Dave's Appliance Store' will purchase a TV with probability p. If the number of customers entering the store is Poisson distributed with mean λ, what is the probability that Dave will sell k TVs? If X is the number of TVs sold, what kind of distribution does X have, and what is the mean of X? (Hint: compute $P(X = k)$ by conditioning on N, the number of customers entering the store.)

The number of TVs sold is

$$\begin{aligned}
P(X=k) = \sum_{n=k}^{\infty} P(X=k|N=n)P(N=n) &= \sum_{n=k}^{\infty} \binom{n}{k} p^k q^{n-k} \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k q^{n-k} \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \frac{e^{-\lambda}p^k}{k!} \sum_{n=k}^{\infty} \frac{q^{n-k}\lambda^n}{(n-k)!} \\
&= \frac{e^{-\lambda}p^k\lambda^k}{k!} \sum_{n=k}^{\infty} \frac{q^{n-k}\lambda^{n-k}}{(n-k)!} \\
&= \frac{e^{-\lambda}(p\lambda)^k}{k!} \sum_{i=0}^{\infty} \frac{(q\lambda)^i}{i!} \\
&= \frac{e^{-\lambda}(p\lambda)^k}{k!} e^{q\lambda} \\
&= \frac{e^{-p\lambda}(p\lambda)^k}{k!}
\end{aligned}$$

so X is Poisson with parameter $p\lambda$, and the mean is $p\lambda$.

3. A TV purchased from 'Reasonably Honest Dave's Appliance Store' will require repair on the average once every two years. Assuming that the times between repairs are exponentially distributed, what is the probability that the TV will work at least 3 years without requiring repairs?

The cumulative distribution function of an exponential variable T is

$$P(T \le t) = F(t) = 1 - e^{-\lambda t}$$

where $\frac{1}{\lambda}$ is the mean. Thus $\frac{1}{\lambda} = 2$ or $\lambda = \frac{1}{2}$ and we get

$$P(T \ge 3) = 1 - F(3) = e^{-3/2} = .2231.$$

4. Let X and Y be independent Bernoulli variables with parameter $p = 1/2$. Show that $X+Y$ and $|X-Y|$ are uncorrelated but not independent.

The following table lists all the possibilities (there are only four), with their probabities:

Outcome	P	$X+Y$	$\lvert X-Y\rvert$	$(X+Y)\lvert X-Y\rvert$
$X=0, Y=0$	1/4	0	0	0
$X=1, Y=0$	1/4	1	1	1
$X=0, Y=1$	1/4	1	1	1
$X=1, Y=1$	1/4	2	0	0

From this we immediately see

that $E(X+Y)=1$, $E(|X-Y|)=1/2$, and $E((X+Y)|X-Y|)=1/2$. Thus, $E((X+Y)|X-Y|)=E(X+Y)E(|X-Y|)$ so they are uncorrelated. To see that the are not independent, notice that the conditional probability is $P(|X-Y|=0|X+Y=0)=1$, whereas the marginal probability is $P(|X-Y|=0)=1/2$.

5. Compute the mean and variance of a random variable with a gamma distribution with parameters α and λ.

Let X be gamma with parameters α and λ.

$$E(X)=\int_{-\infty}^{\infty} xf(x)\,dx=\int_0^{\infty}\frac{xx^{\alpha-1}\lambda^{\alpha}e^{-\lambda x}}{\Gamma(\alpha)}\,dx$$

$$=\int_0^{\infty}\frac{x^{\alpha}\lambda^{\alpha}e^{-\lambda x}}{\Gamma(\alpha)}\,dx$$

$$=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda}\int_0^{\infty}\frac{x^{\alpha}\lambda^{\alpha+1}e^{-\lambda x}}{\Gamma(\alpha+1)}\,dx$$

$$=\frac{\alpha}{\lambda}$$

The variance is similar and we get:

$$Var(X)=\frac{\alpha}{\lambda^2}.$$

6. Let U be a uniform random variable on $[0,1]$, and let $V=\dfrac{1}{U}$.

a) Find the density function of V. (HINT: first compute the cdf of V, then differentiate.)

First we find the cdf:

$$F_V(v)=P(V\le v)=P(1/U\le v)=P(U\ge 1/v)=\int_{1/v}^{1}\,dv=1-1/v.$$

Thus,

$$f_V(v)=F_V'(v)=1/v^2.$$

b) What is the mean of V?

$$E(V)=\int_{-\infty}^{\infty} vf_V(v)\,dv=\int_1^{\infty} v/v^2\,dv=\ln v|_1^{\infty}=\infty.$$