

Basic Probability Summer 2009
NYU Courant Institute
Practice Problems - Solutions

1. Which is more likely: 9 heads in 10 tosses of a fair coin or 18 heads in 20 tosses?

Solution: Use the binomial distribution. The first is

$$\binom{10}{9} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} = .00976.$$

The second is

$$\binom{20}{18} \left(\frac{1}{2}\right)^{18} \left(\frac{1}{2}\right)^{20-18} = .00018,$$

so the first is larger.

2. Let X be a binomially distributed random variable with n trials and probability p of success on each trial. For which value of k is $P(X = k)$ maximized? (Hint: Consider $\frac{P(X=k)}{P(X=k-1)}$, the ratio of successive terms. If this ratio is bigger than one, then the probability is still going up with k , if it is smaller than one, then it is going down.)

Solution: Consider the ratio

$$\frac{P(X = k)}{P(X = k - 1)} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-(k-1)}} = \frac{n!(k-1)!(n-(k-1))! p^k q^{n-k}}{k!(n-k)! n! p^{k-1} q^{n-(k-1)}} = \frac{n-k+1}{k} \frac{p}{q}.$$

The question is: when is this positive? We get

$$(n - k + 1)p > k(1 - p)$$

or

$$(n + 1)p > k.$$

So, if $(n + 1)p$ is an integer, then $P(X = k)$ increases monotonically, reaching its largest value at $(n + 1)p - 1$ and $(n + 1)p$, then decreasing monotonically. If $(n + 1)p$ is not an integer, then $P(X = k)$ increases monotonically until it reaches its largest value at the integer k such that $(n + 1)p - 1 < k < (n + 1)p$, then decreases monotonically.

In either case, we can say that $P(X = k)$ reaches its largest value at roughly $(n + 1)p$.

3. Three identical fair coins are thrown simultaneously until all three show the same face. What is the probability that they are thrown more than three times?

Solution: The probability of all three showing the same face on one toss is $1/4$. Letting X equal the number of tosses required for this to occur, we have a geometric variable X with parameter $p = 1/4$ and the problem is to compute

$$P(X > 3) = 1 - [P(X = 1) + P(X = 2) + P(X = 3)] = 1 - \left[\frac{1}{4}\left(\frac{3}{4}\right)^0 + \frac{1}{4}\left(\frac{3}{4}\right)^1 + \frac{1}{4}\left(\frac{3}{4}\right)^2\right] = \frac{27}{64}.$$

4. Let X and Y be jointly continuous random variables with joint density

$$f(x, y) = \begin{cases} \frac{6}{7}(x + y)^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find (i) $P(X > Y)$, (ii) $P(X + Y \leq 1)$.

Solution: $P(X > Y)$ is the double integral of the density function over the region $\{x > y\}$.

$$\text{So } P(X > Y) = \int_0^1 \int_0^x \frac{6}{7}(x+y)^2 dy dx = \frac{6}{7} \int_0^1 \frac{(x+y)^3}{3} \Big|_0^x dx = \frac{2}{7} \int_0^1 [(2x)^3 - x^3] dx = 2 \int_0^1 x^3 dx = \frac{1}{2}.$$

$$\text{Also, } P(X + Y \leq 1) = \int_0^1 \int_0^{1-x} \frac{6}{7}(x+y)^2 dy dx = \frac{6}{7} \int_0^1 \frac{(x+y)^3}{3} \Big|_0^{1-x} dx = \frac{2}{7} \int_0^1 (1^3 - x^3) dx = \frac{13}{4}.$$

b) Find the marginal densities of X and Y .

$$\text{Solution: } f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_0^1 \frac{6}{7}(x+y)^2 dy = \frac{6}{7} \frac{(x+y)^3}{3} \Big|_0^1 = \frac{2}{7} [(x+1)^3 - x^3] = \frac{2}{7}(3x^2 + 3x + 1). \text{ Similarly, } f_Y(y) = \frac{2}{7}(3y^2 + 3y + 1).$$

c) Are X and Y independent? Why or why not?

Solution: No, because $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$.

5. When taping a television commercial, the probability is 0.30 that a certain actor will get his lines straight on any one take. What is the probability that he will get his lines straight for the first time on the sixth take?

Solution: Geometric with $p = .3$. $P(X = 6) = (.3)(.7)^5 = .0504$.

6. Suppose that in a certain city the number of muggings can be approximated by a Poisson process with $\lambda = 4$ per month.

a) Find the probability of there being 48 muggings in a year.
bigskip

Solution: Let X be the number of muggings in a year. Then $P(X = k) = \frac{48^k}{k!} e^{-48}$ so $P(X = 48) = \frac{48^{48}}{48!} e^{-48} = .0575$.

b) What is the probability of 3 muggings in one week?

Let Y be the number of muggings in a week. Then there are .9230 ($= 48/52$) per week on average, so $P(Y = k) = \frac{(.923)^k}{k!} e^{-.923}$, so $P(Y = 3) = \frac{(.923)^3}{3!} e^{-.923} = .0521$.

I have chosen to round the answers off to four decimal places.

7. The lifetime of a transistor radio is T years, where T is an exponential variable with parameter $\lambda = .5$. What is the probability that the radio will last at least 5 years? Given that it lasts 10 years, what is the probability that it will last at least 5 more years beyond that?

Solution: The cdf for T is $F_T(t) = P(T < t) = 1 - e^{-(.5)t}$. Thus

$$P(T > 5) = 1 - P(T \leq 5) = e^{-(.5)5} = .0821.$$

By the 'lack of memory' property, the answer to the second question is the same as the first.

8. Let X be uniform on $[0, 1]$.

a) Find the density function of X^3 and use this to compute $E(X^3)$.

b) Use the Law of the Unconscious Statistician to find $E(X^3)$.

Solution: Let $Y = X^3$. We have $F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) = \int_0^{y^{1/3}} dx = y^{1/3}$. Then differentiating we get $f_Y(y) = F'_Y(y) = (1/3)y^{-2/3}$, $0 \leq y \leq 1$.

Using this, $E(Y) = \int_0^1 y(1/3)y^{-2/3} dy = 1/4$.

By the LUS, $E(X^3) = \int_0^1 x^3 dx = 1/4$.

9. It is raining cats and dogs. The number of animals N that fall in a given area during a given interval of time is Poisson with parameter λ . Each animal that falls has probability p_c of being a cat, and probability p_d of being a dog, where $p_c + p_d = 1$. Find the average number of cats falling in the given area during the given interval of time.

Solution: Let X be the number of cats that fall. We compute $E(X)$ by conditioning on N :

$$\begin{aligned} E(X) &= \sum_{n=0}^{\infty} E(X|N=n)P(N=n) = \sum_{n=0}^{\infty} np_c \frac{\lambda^n e^{-\lambda}}{n!} = \sum_{n=1}^{\infty} p_c \frac{\lambda^n e^{-\lambda}}{(n-1)!} = \\ & \lambda p_c \sum_{n=1}^{\infty} \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} = \lambda p_c. \end{aligned}$$

10. Let X be gamma distributed with parameters α and λ . Compute $E(1/X)$ for those values of α and λ for which it exists.

Solution:

$$E(1/X) = \int_0^{\infty} \frac{1}{x} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-2} e^{-\lambda x} dx$$

This integral converges as long as $\alpha - 2 > 0$ or $\alpha > 2$ and we get

$$\int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-2} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-2} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha-1)}{\lambda^{\alpha-1}} = \frac{\lambda}{\alpha-1}.$$

11. Show that $\Gamma(1/2) = \sqrt{\pi}$.

Solution: By definition, $\Gamma(1/2) = \int_0^{\infty} x^{1/2-1} e^{-x} dx = \int_0^{\infty} x^{-1/2} e^{-x} dx$. Make the substitution $u^2 = x$ to get $2 \int_0^{\infty} e^{-u^2} du$. This is an integral that we have previously encountered (compare it to the standard normal density) and the answer comes out to $\Gamma(1/2) = \sqrt{\pi}$.