1. Let $G$ be a simply connected topological group, and let $H$ be a discrete normal subgroup. Prove that $\pi_1(G/H, 1) \cong H$. \textbf{(Hint: Show that $G \to G/H$ is a covering space.)}

\textbf{Hint:} The main part of the problem is to show that $p : G \to G/H$ is a covering space. The rest follows from everything we have done with liftings of paths in covering space. Since $H$ is discrete, there is an open set $W$ in $G$ containing 1 such that $W \cap H = \{1\}$. Consider the map $G \times G \to G$ given by $(g_1, g_2) \to g_1g_2^{-1}$. Since $G$ is a topological group this map is continuous, so there is an open set $V$ in $G$ containing 1 such that $VV^{-1} \subset W$. Define $U = p(V)$. Show that $U$ is evenly covered.

For example to show that $U$ is an open set in $G/H$, by definition of the quotient topology we need to show that $p^{-1}p(V)$ is open. But $p^{-1}p(V) = \bigcup_{g \in V} Hg = \bigcup_{h \in H} hV$ and each $hV$ is open because it is a translate of $V$ (translation is a homeomorphism in a topological group).

2. Let $G_i$ be a collection of more than one non-trivial group. Prove that their free product is non-abelian, contains elements of infinite order, and that its center is trivial.

3. Let $G$, $H$, $G'$, and $H'$ be cyclic groups of orders $m$, $n$, $m'$, and $n'$ respectively. If $G * H$ is isomorphic to $G' * H'$ then $m = m'$ and $n = n'$ or else $m = n'$ and $n = m'$.

4. Let $M$ and $N$ be $n$-dimensional manifolds, where $n > 2$. Let $M \# N$ be their connected sum. Show that $\pi_1(M \# N) = \pi_1(M) * \pi_1(N)$. 

\[\text{MATH 795 Algebraic Topology – Problem Set Three with a hint} \]
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