Topology I – Problem Set One Spring 2008

1) Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a SES (short exact sequence) of Abelian groups and suppose C is free. Then $B \cong A \oplus C$.

2) Show that the two definitions given of reduced homology are naturally equivalent. (Recall that the first definition is the kernel of $H_n(X) \to H_n(\{P\})$ and the second definition is the homology of the chain complex

$$\ldots \to S_n(X) \to S_{n-1}(X) \to \ldots \to S_0(X) \xrightarrow{\theta} \mathbf{Z} \to 0$$

where θ is defined by $\theta(\sum m_i x_i) = \sum m_i$.)

3) Let $B \subset A \subset X$ be spaces. Show that there is a LES

$$\ldots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \to \ldots$$

4) Let X be the subspace of $\mathbf{R}^2$ consisting of the graph of $y = \sin(1/x)$, $0 < x \leq 1$, together with the interval $\{(0, y) \mid 0 \leq y \leq 1\}$. Compute the homology groups of X.

5) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact then show that f is surjective if and only if h is injective.

6) If

$$\cdots \to C_{n+1} \to A_n \xrightarrow{h_n} B_n \to C_n \to A_{n-1} \xrightarrow{h_{n-1}} B_{n-1} \to C_{n-1} \to \cdots$$

is exact and every third arrow $h_n : A_n \to B_n$ is an isomorphism, then $C_n = 0$ for all n.

7)

a) If

$$0 \to A \to B \to C \to 0$$

is a SES of abelian groups, then rank B = rank A + rank C (hint: extend a maximal independent subset of A to a maximal independent subsest of B).

b) If

$$0 \to A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to 0$$

is an exact sequence of finitely generated abelian groups, then

$$\Sigma_{i=0}^{n}(-1)^i \text{rank} A_i = 0.$$