

Topology I – Problem Set One – Solutions Spring 2008

1) Let

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

be a SES (short exact sequence) of Abelian groups and suppose C is free. Then $B \cong A \oplus C$.

Solution: Since C is free it has a basis $\{c_\alpha\}$. For each α , let $b_\alpha \in B$ such that $g(b_\alpha) = c_\alpha$, which you can do since g is onto. Since C is free the function $c_\alpha \rightarrow b_\alpha$ extends linearly to give a homomorphism $\phi : C \rightarrow B$ such that $g \circ \phi = Id_C$.

Now define a homomorphism $\mu : A \oplus C \rightarrow B$ by $\mu(a, c) = f(a) + \phi(c)$.

μ is onto:

Let $b \in B$. Then let $b' = b - \phi(g(b))$. We have $g(b') = g(b - \phi(g(b))) = g(b) - g(\phi(g(b))) = g(b) - g(b) = 0$ so $b' \in \ker g = A$, in other words $b' = f(a)$ for some $a \in A$. Then $\mu(a, g(b)) = f(a) + \phi(g(b)) = b' + \phi(g(b)) = b$.

μ is one-to-one: Suppose $\mu(a, c) = 0$. Then $f(a) = -\phi(c)$ so $0 = g(f(a)) = g(-\phi(c)) = -c$, hence $c = 0$. Then, $f(a) = -\phi(c) = -\phi(0) = 0$ and since f is one-to-one it follows that $a = 0$, so $(a, c) = 0$.

2) Show that the two definitions given of reduced homology are naturally equivalent. (Recall that the first definition is the kernel of $H_n(X) \rightarrow H_n(\{P\})$ and the second definition is the homology of the chain complex

$$\dots \rightarrow S_n(X) \rightarrow S_{n-1}(X) \rightarrow \dots \rightarrow S_0(X) \xrightarrow{\theta} \mathbf{Z} \rightarrow 0$$

where θ is defined by $\theta(\sum m_i x_i) = \sum m_i$.)

Solution: Actually, we have *three* different definitions of reduced homology – let's check that they all give the same thing. In the following, A is assumed to be non-empty (we don't define reduced homology for a non-empty space):

- Definition One: Let $f : A \rightarrow \{P\}$ be the unique map to a one-point space, and define $\tilde{H}_n(A) = \ker H_n(f)$.
- Definition Two: Let P be a point in A and define $\tilde{H}_n(A) = H_n(A, \{P\})$.
- Definition Three: Define by $\theta : S_0(A) \rightarrow \mathbf{Z}$ by $\theta(\sum m_i x_i) = \sum m_i$, then define $\tilde{H}_0(A) = \ker \theta / \text{im } \partial_1$.

The composite

$$\{P\} \xrightarrow{i} A \xrightarrow{f} \{P\}$$

is the identity on the space $\{P\}$ so the composite

$$H_0(\{P\}) \xrightarrow{H_0(i)} H_0(A) \xrightarrow{H_0(f)} H_0(\{P\})$$

is the identity, which gives a splitting

$$H_0(A) \cong \text{im } H_0(i) \oplus \ker H_0(f) \cong \text{im } H_0(i) \oplus \text{coker } H_0(i).$$

Consider the LES of the pair $(A, \{P\})$:

$$\dots \rightarrow H_0(\{P\}) \xrightarrow{H_0(i)} H_0(A) \rightarrow H_0(A, \{P\}) \rightarrow 0.$$

Putting these together gives an isomorphism $H_0(A, \{P\}) \cong \text{coker } H_0(i) \cong \ker H_0(f)$, which gives the equivalence of One and Two.

Now consider the induced map $S_0(A) \xrightarrow{f\#} S_0(\{P\})$. Since $S_0(\{P\}) \cong \mathbf{Z}$, the map $f\#$ is the same as θ . Also, $S_0(\{P\}) \cong H_0(\{P\})$, so the kernel of $H_0(f)$ is the same as the kernel of $f\#$ modulo the image of ∂_1 , which is Definition three.

3) Let $B \subset A \subset X$ be spaces. Show that there is a LES

$$\dots \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \rightarrow \dots$$

Solution: By the third isomorphism theorem of algebra, there is a SES of chain complexes:

$$0 \rightarrow S_*(A)/S_*(B) \rightarrow S_*(X)/S_*(B) \rightarrow S_*(X)/S_*(A) \rightarrow 0.$$

Now apply the snake lemma to this SES to get the desired LES.

4) Let X be the subspace of \mathbf{R}^2 consisting of the graph of $y = \sin(1/x)$, $0 < x \leq 1$, together with the interval $\{(0, y) \mid 0 \leq y \leq 1\}$. Compute the homology groups of X .

Solution: Actually, I meant to say the interval $\{(0, y) \mid -1 \leq y \leq 1\}$ but it doesn't matter as far as the result is concerned. If we call

$$A = \{(x, y) \mid y = \sin(1/x), 0 < x \leq 1\}$$

and

$$B = \{(0, y) \mid 0 \leq y \leq 1\}$$

then $X = A \cup B$ is a decomposition of X into its two path components. Furthermore, A and B are each contractible, so $H_0(X) = \mathbf{Z} \oplus \mathbf{Z}$ and $H_i(X) = 0$ for $i > 0$.

5) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact then show that f is surjective if and only if h is injective.

Solution: When possible, avoid dealing with individual elements. It is much more elegant, and generalizable, to prove things like this by referring to kernels, images, and cokernels, rather than individual elements. (The concepts of 'kernel', 'cokernel', 'exact', etc. generalize to categories in which the objects are not necessarily sets of elements.)

f is surjective means $\text{im } f = B$ iff $\ker g = B$ iff $g = 0$ iff $\text{im } g = 0$ iff $\ker h = 0$ which is what it means for h to be injective.

6) If

$$\cdots \rightarrow C_{n+1} \rightarrow A_n \xrightarrow{h_n} B_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{h_{n-1}} B_{n-1} \rightarrow C_{n-1} \rightarrow \cdots$$

is exact and every third arrow $h_n : A_n \rightarrow B_n$ is an isomorphism, then $C_n = 0$ for all n .

Solution: I have labeled the other maps in the sequence. C_n maps onto $\text{im } \partial_n = \ker h_{n-1}$ and the kernel of this is just $\text{im } i_n$ which is the same as $B_n / \ker i_n = B_n / \text{im } h_n = \text{coker } h_n$. Thus we get a SES

$$0 \rightarrow \text{coker } h_n \rightarrow C_n \rightarrow \ker h_{n-1} \rightarrow 0.$$

So far we haven't used anything about h_n being an isomorphism, and in fact it is very useful to note that whenever you have a LES, you can always 'extract' a sequence of SES's in the above fashion.

Now, if h_i happens to be an isomorphism for all i , then $\ker h_{n-1} = 0 = \text{coker } h_n$ and we're done.

7)

a) If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a SES of abelian groups, then $\text{rank } B = \text{rank } A + \text{rank } C$ (hint: extend a maximal independent subset of A to a maximal independent subset of B).

Solution:

Let's assume, for the sake of simplicity, that everything has finite rank, although the same argument works for arbitrary rank, one just has to think about cardinal arithmetic.

Let $\{a_1, a_2, \dots, a_k\}$ be a maximal linearly independent subset of A . Then $\text{rank } A = k$ by definition. If this set is not already a maximal linearly independent subset of B then let $b_1 \in B$ be such that $\{a_1, a_2, \dots, a_k, b_1\}$ is linearly independent in B . If this is not maximal linearly independent then there must be a $b_2 \in B$ such that $\{a_1, a_2, \dots, a_k, b_1, b_2\}$ is linearly independent in B . Continuing in this fashion we get $\{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_r\}$ maximal linearly independent in B . (The process terminates after a finite stage because B is assumed to have finite rank. In the general case one would use Zorn's lemma.) Then $\text{rank } B = k + r$.

Mapping this set to C , the a_i 's go to zero, and let c_j denote the image of b_j . I claim that $\{c_1, c_2, \dots, c_r\}$ is a maximal independent subset of C , which immediately implies the result.

The set $\{c_1, c_2, \dots, c_r\}$ is independent: Suppose that there was a linear dependence relation $u_1c_1 + \dots + u_rc_r = 0$. Then $u_1b_1 + \dots + u_rb_r$ would map to zero in C , hence $a = u_1b_1 + \dots + u_rb_r$ would be in A . Since the b_j 's are independent, a can't be torsion. Hence, a must be in the span of the a_i 's. This would give a linear dependence relation between $\{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_r\}$, contradicting its independence.

$\{c_1, c_2, \dots, c_r\}$ is maximal: Suppose there was an element $c' \in C$ such that $\{c_1, c_2, \dots, c_r, c'\}$ is independent. Then there would be an element $b' \in B$ which maps to c' since $B \rightarrow C$ is onto. The set $\{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_r, b'\}$ couldn't be independent by maximality of $\{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_r\}$, so there would be a linear dependence relation among the elements in the set $\{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_r, b'\}$. This would map to a dependence relation among the elements in $\{c_1, c_2, \dots, c_r, c'\}$ contradicting our assumption of independence of this set.

b) If

$$0 \rightarrow A_n \rightarrow A_{n-1} \rightarrow \dots \rightarrow A_1 \rightarrow A_0 \rightarrow 0$$

is an exact sequence of finitely generated abelian groups, then

$$\sum_{i=0}^n (-1)^i \text{rank } A_i = 0.$$

Solution: Let d_i denote the map $A_i \rightarrow A_{i-1}$. We have SES's

$$0 \rightarrow \ker d_i \rightarrow A_i \rightarrow \operatorname{im} d_i \rightarrow 0$$

for $i = 0, 1, \dots, n$.

By part a) , $\operatorname{rank} A_i = \operatorname{rank} \ker d_i + \operatorname{rank} \operatorname{im} d_i$ and by exactness this is the same as $\operatorname{rank} A_i = \operatorname{rank} \ker d_i + \operatorname{rank} \ker d_{i-1}$ (we can just regard $d_{-1} = 0$). Thus we get

$$\sum_{i=0}^n (-1)^i \operatorname{rank} A_i = \sum_{i=0}^n (-1)^i (\operatorname{rank} \ker d_i + \operatorname{rank} \ker d_{i-1})$$

and this collapses to $\pm \operatorname{rank} \ker d_n = 0$.