Algrebraic Topology – Problem Set Two Spring 2008

1. Let $X = X_1 \cup X_2$ be an open cover. Let

$$\partial : H_i(X) \to H_{i-1}(X_1 \cap X_2)$$

be the connecting homomorphism in the Mayer-Vietoris sequence. Show that ∂ satisfies the following 'formula': If x is a cycle representing $\bar{x}$ in $H_i(X)$, subdivide so that x is homologous to $x_1 + x_2$, where x_i is a chain in X_i. Then $\partial(\bar{x}) = \overline{\partial(x_1)}$.

2. Let T^* be a torus with a small open disk removed. Let $C \subset T^*$ be the boundary circle. Show that the inclusion $C \to T^*$ induces 0 in H_1.

3. Define the *supsension* of a space X, denoted ΣX, to be the following quotient space of the cylinder $X \times I$:

$$\Sigma X = X \times I/\sim$$

where the equivalence relation is given by : $(x, 0) \sim (x', 0)$ for all $x, x' \in X$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

a) Show that ΣS^n is homeomorphic to S^{n+1}.

b) Prove the suspension theorem:

$$H_{n+1}(\Sigma X) \cong H_n(X) \quad \forall n \geq 1$$

(Hint: the proof is just a generalization of the computation of the homology groups of the spheres).

4. Show that for any integer k, and any $n \geq 1$, there exists a map $f : S^n \to S^n$ with degree k (hint: If $f : X \to Y$ is a map, define $\Sigma f : \Sigma X \to \Sigma Y$ by $(\Sigma f)(x, t) = (f(x), t)$ and use induction on n).

5. Let A, B be subsets of S^n, $n \geq 2$. Show

a) If A and B are closed, disjoint, and neither separates S^n, then $A \cup B$ does not separate S^n.

b) If A, B are connected, open, and $A \cup B = S^n$, then $A \cap B$ is connected.