Algrebraic Topology – Problem Set Two Solutions
Spring 2008

1. Let $X = X_1 \cup X_2$ be an open cover. Let

$$\partial : H_i(X) \to H_{i-1}(X_1 \cap X_2)$$

be the connecting homomorphism in the Mayer-Vietoris sequence. Show that ∂ satisfies the following 'formula': If x is a cycle representing $\bar{x}$ in $H_i(X)$, subdivide so that x is homologous to $x_1 + x_2$, where x_i is a chain in X_i. Then $\partial(\bar{x}) = \overline{\partial(x_1)}$.

Solution: Actually, using the definition of the M-V connecting homomorphism given in the notes, the formula should read

$$\partial(\bar{x}) = -\overline{\partial(x_1)}.$$

Refer to the following diagram and recall that the definition of the connecting homomorphism in the M-V sequence is $\partial \circ (h_*)^{-1} \circ l_*$.

$$\begin{tikzcd}
H_i(X_1 \cap X_2) \arrow[r, "i_{2*}"] \arrow[d, "i_{1*}"'] & H_i(X_2) \arrow[r, "k_*"] \arrow[d, "j_{2*}"'] & H_i(X_2, X_1 \cap X_2) \arrow[r, "\partial"] \arrow[d, "h_*"'] & H_{i-1}(X_1 \cap X_2) \arrow[d, "i_{1*}"'] \\
H_i(X_1) \arrow[r, "j_{1*}"] & H_i(X) \arrow[r, "l_*"] & H_i(X, X_1) \arrow[r] & H_{i-1}(X_1)
\end{tikzcd}$$

Note that x_2 is a cycle mod X_1 since $0 = \partial x = \partial x_1 + \partial x_2$ so $\partial x_2 = -\partial x_1$, and we have $l_*(\bar{x}) = \overline{x_2}$. Similarly x_2 is a cycle in X_2 mod $X_1 \cap X_2$ for the same reason and $h_*(\overline{x_2}) = \overline{x_2}$. Finally, using the definition of ∂, we get $\partial\overline{x_2} = -\overline{x_1}$.

We could just as well have defined the M-V connecting homomorphism by interchanging the roles of X_1 and X_2 and then the formula would have read

$$\partial(\bar{x}) = \overline{\partial(x_1)}.$$

2. Let T^* be a torus with a small open disk removed. Let $C \subset T^*$ be the boundary circle. Show that the inclusion $C \to T^*$ induces 0 in H_1.

Solution: There are various ways to see this. If we represent the torus as the quotient space of a square obtained by identifying opposite sides, then let C be in the middle of the square. We can think of C as a 1-cycle and C is homotopic to the boundary of the square, which in the quotient is a figure eight. If we denote the 1-dimensional homology class represented by the top/bottom as a, and the class rep'd by the sides as b, then the generator of the homology of C goes to $aba^{-1}b^{-1}$ which in homology is 0 since homology is abelian.

3. Define the *supsension* of a space X, denoted ΣX, to be the following quotient space of the cylinder $X \times I$:

$$\Sigma X = X \times I/\sim$$

where the equivalence relation is given by : $(x, 0) \sim (x', 0)$ for all $x, x' \in X$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

a) Show that ΣS^n is homeomorphic to S^{n+1}.

Solution: First of all, since I is homeomorphic to the interval $[-1, 1]$, we get a homeomorphic result if we define the suspension by taking $X \times [-1, 1]$ and identify the top and the bottom each to a point. Thinking of $S^n \times [-1, 1]$ and S^{n+1} both as subspaces of R^{n+2}, define $h : S^n \times [-1, 1] \rightarrow S^{n+1}$ by $h(x, t) = ((1-|t|)x, t)/||((1-|t|)x, t)||$. This send $S^n \times \{1\}$ to the north pole of S^{n+1}, and $S^n \times \{-1\}$ to the south pole. This induces a continuous bijection $\bar{h} : \Sigma S^n \rightarrow S^{n+1}$. Since the source is compact and the target is Hausdorff, $\bar{h}$ is a homeomorphism.

b) Prove the suspension theorem:

$$H_{n+1}(\Sigma X) \cong H_n(X) \quad \forall n \geq 1$$

(Hint: the proof is just a generalization of the computation of the homology groups of the spheres).

Solution: Just mimick the proof of the computation of the homology of a sphere, replacing the northern hemisphere by the top half of the suspension, the southern hemisphere by the bottom half, and the equator by X.

4. Show that for any integer k, and any $n \geq 1$, there exists a map $f : S^n \rightarrow S^n$ with degree k (hint: If $f : X \rightarrow Y$ is a map, define $\Sigma f : \Sigma X \rightarrow \Sigma Y$ by $(\Sigma f)(x, t) = (f(x), t)$ and use induction on n).

Solution: Start with S^1: the map $f : S^1 \rightarrow S^1$ defined by $f(e^{i\theta}) = e^{i\theta k}$ induces multiplication by k in $H_1(S^1)$ hence has degree k. There are various ways to prove this. For example, divide the circle into k arcs $x_j = \{e^{i\theta} | \quad \frac{2\pi j}{k} \leq \theta \leq \frac{2\pi(j+1)}{k}\}$, $0 \leq j \leq k-1$. Think of each arc x_j as a 1-simplex. Then $x_0 + x_1 + \cdots + x_{k-1}$ is a cycle which represents a generator of $H_1(S^1)$. You can check this using the Mayer-Vietoris isomorphism $\partial : H_1(S^1) \cong \tilde{H}_0(S^0)$ and the formula from problem 1.

The map f sends each x_j to a 1-cycle representing the generator of $H_1(S^1)$, so f sends $x_0 + x_1 + \cdots + x_{k-1}$ to k times the generator.

Now proceed inductively, using the fact that $f : S^{n-1} \to S^{n-1}$ and $\Sigma f : S^n \to S^n$ are compatible with the Mayer-Vietoris isomorphism $H_n(S^n) \cong H_{n-1}(S^{n-1})$.

5. Let A, B be subsets of S^n, $n \geq 2$. Show

a) If A and B are closed, disjoint, and neither separates S^n, then $A \cup B$ does not separate S^n.

Solution: The complements $S^n - A$ and $S^n - B$ are open and their union is all of S^n so we can apply Mayer-Vietoris to get

$$\cdots \to H_1(S^n) \to \tilde{H}_0(S^n - (A \cup B)) \to \tilde{H}_0(S^n - A) \oplus \tilde{H}_0(S^n - B) \to \cdots$$

By hypothesis, $\tilde{H}_0(S^n - A) = \tilde{H}_0(S^n - B) = 0$ and $H_1(S^n) = 0$ since $n \geq 2$, so $\tilde{H}_0(S^n - (A \cup B)) = 0$ which means $A \cup B$ doesn't separate S^n.

b) If A, B are connected, open, and $A \cup B = S^n$, then $A \cap B$ is connected.

Solution: Apply M-V to the decomposition $A \cup B = S^n$:

$$\cdots \to H_1(S^n) \to \tilde{H}_0(A \cap B) \to \tilde{H}_0(A) \oplus \tilde{H}_0(B) \to \cdots$$

By hypothesis, $\tilde{H}_0(A) = \tilde{H}_0(B) = 0$ and again $H_1(S^n) = 0$, so $\tilde{H}_0(A \cap B) = 0$ which means $A \cap B$ is connected.