Algrebraic Topology – Problem Set Three Spring 2008

1. Let X be a CW complex of dimension n and X' a CW complex of dimension m, where X and X' are homeomorphic. Prove that $n = m$, i.e. the dimension of a CW complex only depends on the topological type, and not the particular CW structure.

HINT: Suppose that m and n are both finite. A separate argument will rule out the case where one is finite and the other is infinite. First show that for an an n-dimensional CW complex X, each n-cell e is open in X. Now assume $n > m$, and let e be an n-cell in X. Then e corresponds under the homeomorphism to an open set in X'. Every cell in X' has dimension smaller than n. Use all this to contradict invariance of domain.

2. Let X be a finite CW complex of dimension n and define the *Euler characteristic* as follows:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \mathrm{rank}(C_i(X))$$

where $C_i(X)$ is the i^{th} cellular chain group.
a) Show that

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \mathrm{rank} H_i(X).$$

b) Let X and Y be finite CW complexes. Show that $X \times Y$ has the structure of a finite CW complex with an (open) $n+m$ dimensional cell $e \times e'$ for each n dimensional cell e in X and each m dimensional cell e' of Y.

c) Show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

d) Suppose $\tilde{X} \xrightarrow{p} X$ is a covering space with k sheets where X is a finite CW complex and k is finite. Show that $\tilde{X}$ has the structure of a finite CW complex with p a cellular map and $\chi(\tilde{X}) = k\chi(X)$.

Construction: Let p and q be relatively prime integers. Think of S^3 as all pairs $(z_0, z_1) \in \mathbf{C}^2$ such that $|z_0|^2 + |z_1|^2 = 1$. Let $\zeta = e^{2\pi i/p}$ be a primitive p^{th} root of unity. Define $h : S^3 \to S^3$ by

$$h(z_0, z_1) = (\zeta z_0, \zeta^q z_1).$$

(The map h is a homeomorphism of S^3 whose p^{th} power is the identity, and can be thought of as an action of the group of $\mathbf{Z}/p$ on S^3.) Define an equivalence relation on S^3 by $(z_0, z_1) \sim (z_0', z_1')$ if there exists an integer m with $h^m(z_0, z_1) = (z_0', z_1')$. The space $S^3/\sim$ is the orbit space of the $\mathbf{Z}/p$ action, and is called a *Lens space* and is denoted by $L(p, q)$.

3. Show that $L(p, q)$ is a compact Hausdorff space.

4. Show that:

a) $L(1,1) = S^3$.

b) $L(2,1) = \mathbf{R}P^3$.

c) If $q \cong q' \mod p$, then $L(p,q) = L(p,q')$.

5. a) Show that S^3 has a CW decomposition with p cells in each dimension ≤ 3 which are permuted by the $\mathbf{Z}/p$ action, and which leads to a CW decomposition of $L(p,q)$ with one cell in every dimension ≤ 3.

HINT: Recall that if $z = \rho e^{i\theta}$ is a complex number, $\rho > 0$, $0 \leq \theta \leq 2\pi$, then define $\arg z = \theta$, the *argument* of z.

Define the following cells in S^3, , for $r = 0, 1, \ldots, p-1$,

$$
\begin{aligned}
e_r^0 &= \{(z_0, 0) \in S^3 | \arg(z_0) = 2\pi r/p\}, \\
e_r^1 &= \{(z_0, 0) \in S^3 | 2\pi r/p < \arg(z_0) < 2\pi(r+1)/p\}, \\
e_r^2 &= \{(z_0, z_1) \in S^3 | \arg(z_1) = 2\pi r/p\}, \\
e_r^3 &= \{(z_0, z_1) \in S^3 | 2\pi r/p < \arg(z_1) < 2\pi(r+1)/p\}.
\end{aligned}
$$

b) Use a) to compute the homology groups of $L(p,q)$.

HINT:

The above CW decomposition for S^3 gives a cellular chain complex $C_*(S^3)$ with

$$
\begin{aligned}
d(e_r^1) &= e_r^0 - e_{r+1}^0 \\
d(e_r^2) &= \Sigma_{i=0}^{p-1} e_i^1 \\
d(e_r^3) &= e_r^2 - e_{r+1}^2.
\end{aligned}
$$

(The subscript is taken mod p in the first and third formula.)

This leads to a cellular chain complex for $L(p,q)$.