

Algebraic Topology – Problem Set Three Spring 2008

1. Let X be a CW complex of dimension n and X' a CW complex of dimension m , where X and X' are homeomorphic. Prove that $n = m$, i.e. the dimension of a CW complex only depends on the topological type, and not the particular CW structure.

2. Let X be a finite CW complex of dimension n and define the *Euler characteristic* as follows:

$$\chi(X) = \sum_{i=0}^n (-1)^i \text{rank}(C_i(X))$$

where $C_i(X)$ is the i^{th} cellular chain group.

a) Show that

$$\chi(X) = \sum_{i=0}^n (-1)^i \text{rank} H_i(X).$$

b) Let X and Y be finite CW complexes. Show that $X \times Y$ has the structure of a finite CW complex with an (open) $n + m$ dimensional cell $e \times e'$ for each n dimensional cell e in X and each m dimensional cell e' of Y .

c) Show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

d) Suppose $\tilde{X} \xrightarrow{p} X$ is a covering space with k sheets where X is a finite CW complex and k is finite. Show that \tilde{X} has the structure of a finite CW complex with p a cellular map and $\chi(\tilde{X}) = k\chi(X)$.

Construction: Let p and q be relatively prime integers. Think of S^3 as all pairs $(z_0, z_1) \in \mathbf{C}^2$ such that $|z_0|^2 + |z_1|^2 = 1$. Let $\zeta = e^{2\pi i/p}$ be a primitive p^{th} root of unity. Define $h : S^3 \rightarrow S^3$ by

$$h(z_0, z_1) = (\zeta z_0, \zeta^q z_1).$$

(The map h is a homeomorphism of S^3 whose p^{th} power is the identity, and can be thought of as an action of the group of \mathbf{Z}/p on S^3 .) Define an equivalence relation on S^3 by $(z_0, z_1) \sim (z'_0, z'_1)$ if there exists an integer m with $h^m(z_0, z_1) = (z'_0, z'_1)$. The space S^3/\sim is the orbit space of the \mathbf{Z}/p action, and is called a *lens space* and is denoted by $L(p, q)$.

3. Show that $L(p, q)$ is a compact Hausdorff space.

4. Show that:

a) $L(1, 1) = S^3$.

b) $L(2, 1) = \mathbf{R}P^3$.

c) If $q \cong q' \pmod{p}$, then $L(p, q) = L(p, q')$.

5. a) Show that S^3 has a CW decomposition with p cells in each dimension ≤ 3 which are permuted by the \mathbf{Z}/p action, and which leads to a CW decomposition of $L(p, q)$ with one cell in every dimension ≤ 3 .

b) Use a) to compute the homology groups of $L(p, q)$.