Topology I – Problem Set Four Spring 2008

1) An abelian group A is called *flat* if tensoring with A preserves exact sequences. Show that an abelian group A is flat if and only if it is torsion free.

2) a) If A and B are abelian groups (not neccessarily f.g.) and $T(A)$ and $T(B)$ denote the torsion subgroups, show that

$$\operatorname{Tor}(A,B) = \operatorname{Tor}(T(A),T(B)).$$

b) Show that $m\operatorname{Tor}(A,B) = 0$ if $mT(A) = 0$.

c) Show that if A is a torsion group then $A \cong \operatorname{Tor}(A,\mathbf{Q}/\mathbf{Z})$ and that in general $\operatorname{Tor}(A,\mathbf{Q}/\mathbf{Z})$ embeds naturally as a subgroup of A. Identify this subgroup.

3) Let A and B be abelian groups and define an *extension of A by B* to be a SES

$$0 \to B \to E \to A \to 0.$$

Let there be two such extensions and say they are *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \\
0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Show that this is an equivalence relation on the set of extensions of A by B and show that the set of equivalence classes is in one-to-one correspondence with the group $Ext(A,B)$ (hence the name).

4) Compute the homology and cohomology groups of

a) $\mathbf{R}P^3 \times \mathbf{R}P^4$.

b) $\mathbf{R}P^5 \times L(3,1)$.

c) $T \times K$ (a torus times a Klein bottle).

5) Compute the following:

a) $H_*(K \times \mathbf{R}P^n; R)$, where R is $\mathbf{Z}$ and $\mathbf{Z}/2$.

b) $H^*(\mathbf{C}P^n; \mathbf{Z}/k)$, where k is any integer.

c) $H_*(\mathbf{R}P^2 \# \mathbf{R}P^2 \# \mathbf{R}P^2; G)$ where $\#$ denotes connected sum, and G is a finitely generated abelian group.

d) $H_*(T \times L(p, 1); G)$ where p is a prime and G is a f.g. abelian group.

6) a) If A_* is chain equivalent to A'_*, and C_* is chain equivalent to C'_*, then show that $A_* \otimes C_*$ is chain equivalent to $A'_* \otimes C'_*$.

b) If F_* is a chain complex of free abelian groups, prove that F_* has zero homology if and only if F_* has a contracting homotopy (a contracting homotopy for a chain complex is a chain homotopy between the identity chain map from the chain complex to itself and the zero chain map).

7) Show that $\mathbf{R}P^3 \times \mathbf{R}P^2$ and $(\mathbf{R}P^2 \vee S^3) \times \mathbf{R}P^2$ have the same homology and cohomology groups and the same fundamental group.

8) If $f : T \to K$ is any map from a torus to a Klein bottle, show that

$$f_* : H_2(T; \mathbf{Z}/2) \to H_2(K; \mathbf{Z}/2)$$

is 0.