Topology I – Problem Set Five Spring 2008

Note: It is a theorem that a compact manifold has finitely generated homology modules. You may use this fact freely.

1) a) Assume that X is a space with finitely generated homology groups. Let T_q denote the torsion subgroup of $H_q(X)$ and let F_q denote the quotient $H_q(X)/T_q$. Show that $H^q(X) \cong F_q \bigoplus T_{q-1}$.

b) Let M be a compact, orientable n-dimensional manifold (hence has f.g. homology groups). Let β_q denote the rank of $H_q(X)$. The integer β_q is called the qth Betti number of M. Show that $\beta_q = \beta_{n-q}$. Is there a similar formula involving T_q?

2) a) Let M be a compact, orientable, odd dimensional manifold. Then $\chi(M) = 0$.

b) Let M be an compact, orientable, even dimensional manifold and suppose the dimension is not divisible by four. Then $\chi(M)$ is even.

3) Show that the Lens space $L(p, q)$ is a compact, orientable 3-dimensional manifold. Compute $H^*(L(p, q); \mathbf{Z}/p)$ *as a ring* (i.e. compute the cup product as well).

4) Show that there is no orientation reversing self homotopy equivalence of $\mathbf{C}P^{2n}$.