Let $X$ be a space. A subspace $e$ of $X$ homeomorphic to $D^n - S^{n-1}$ is called an open $n$-cell. For $n = 0$, we interpret $S^{-1}$ to be the empty set, so an ‘open’ zero cell is a point. Let $C$ be a collection of disjoint open cells $e$ in $X$ of various dimensions. Let $X^k$ denote the union of all cells in $C$ of dimension $\leq k$. $X^k$ is called the $k$-skeleton of $X$. We have a tower

$$\emptyset = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^{k-1} \subseteq X^k \subseteq \cdots X$$

**Definition**

A CW complex is a pair $(X, C)$ where $X$ is a Hausdorff space, $C$ is a collection of open cells in $X$ such that for each $e$ in $C$, there is a map

$$\phi_e : (D^k, S^{k-1}) \to (e \cup X^{k-1}, X^{k-1})$$
where $k$ is the dimension of $e$, and the following conditions hold.

1. The cells in $C$ are disjoint and $X$ is their union.
2. Each map $\phi_e$ is a relative homeomorphism.
3. The closure $\bar{e}$ of each cell in $C$ is contained in the union of finitely many cells in $C$. (Closure finiteness.)
4. A set in $X$ is closed if and only if its intersection with $\bar{e}$ is closed for every cell $e$ in $C$. (The topology on $X$ is called the weak topology relative to the collection of cells if this is true.)

The name ‘CW’ is an abbreviation of the terminology for conditions (3) and (4). The concept was introduced by J. H. C. Whitehead.
If there is such a pair for $X$, we call $X$ a CW complex without explicitly mentioning the family of cells and maps. Note that if $C$ is finite, then conditions (3) and (4) are automatic. However, there are important CW complexes which are not finite. If $X = X^k$ for some $k$, the smallest such $k$ is called the dimension of $X$. It is the largest dimension of any cell in $C$. The general idea is that a CW complex is a space that can be built up inductively by successively adjoining new cells in each dimension. For many purposes they form the most interesting class of topological spaces to study.
Example

Every simplicial complex $K$ is a CW complex. Let $X = |K|$ and let $C$ be the family of all $e = \sigma - \hat{\sigma}$ for $\sigma$ a simplex in $K$. For each $k$-simplex, there is a homeomorphism $\phi_\sigma : D^k \to \sigma$ and that provides a relative homeomorphism

$$\phi_e : (D^k, S^{k-1}) \to (\sigma, \hat{\sigma}) \hookrightarrow (e \cup X^{k-1}, X^{k-1}).$$

Because of the conditions for a simplicial complex, the rules for a CW complex hold.
Example

$\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$ are all finite CW complexes. Consider the family $\mathbb{R}P^n$. There is a natural injection of $\mathbb{R}P^n$ in $\mathbb{R}P^{n+1}$ such that the former space is the $n$-skeleton of the latter and there is one more open cell of dimension $n$. We have then an ascending chain of topological spaces

$$\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \cdots \subset \mathbb{R}P^n \subset \cdots$$

and we may form the union, which is denoted $\mathbb{R}P^\infty$. This set has a cell structure with one cell in each dimension, and we may make it a topological space using the weak topology relative to this family of cells. The result is an infinite CW complex. Similar constructions apply in the other two cases to form $\mathbb{C}P^\infty$ and $\mathbb{H}P^\infty$. For $\mathbb{C}P^\infty$ there is one cell in each even dimension and for $\mathbb{H}P^\infty$ one cell in each dimension divisible by 4.
Example

There are two interesting CW structures to put on $S^n$. The first is very simple. Let $p_n$ denote the point $(0, \ldots, 1)$ (the north pole). Then $S^n - \{p_n\}$ is an open $n$-cell and $\{p_n\}$ is an open 0-cell. These are the only cells. The second is related to our construction of $\mathbb{R}P^n$. There are two cells in each dimension. In dimension $n$ the open upper hemisphere is one open cell and the lower open hemispherie is the other. Their common boundary is a closed cell homeomorphic to $S^{n-1}$. Repeat this for $S^{n-1}$ and continue iteratively down to $k = 0$. 
Example
As discussed in the previous section, $S^m \times S^n$ has a CW structure with one cell each in dimensions $m+n$, $m$, $n$, and 0. If $m = n$, there are two cells in dimension $m$.

Definition
A CW complex is called *regular* if each characteristic map provides a homeomorphism $D^k \to \bar{e}$, and $\bar{e} - e$ is a finite union of open cells of dimension less than $k$ rather than just being contained in such a union.
**Note:** Let $X$ be a CW complex and let $e$ be an open $k$-cell in $X$. Since $\phi_e(D^k)$ is compact and $X$ is Hausdorff, it is closed. Since by (2), $\phi_e(D^k - S^{k-1}) = e$, it is not hard to check that $\phi_e(D^k) = \bar{e}$ and $\phi_e(S^{k-1}) = \bar{e} - e$. The latter is contained, by assumption, in $X^{k-1}$ so by condition (3) in the definition, it is contained in a finite union of cells in $C$ of dimension $< k$. Also, since $\bar{e}$ is compact, our proposition on the recognition of adjunction spaces shows that $\bar{e} = D^k \sqcup_{f_e} (\bar{e} - e)$. So we can say that $e \cup X^{k-1}$ is the space obtained by attaching the $k$-cell $e$ to the $(k - 1)$-skeleton. We also have (the proof is an exercise)

**Proposition**

Let $(X, C)$ be a CW complex. Let $C'$ be a subset of $C$ with the property that for every cell $e$ in $C'$, $\bar{e}$ is contained in a finite union of cells in $C'$. Then the union $X'$ of the cells in $C'$ is a closed subspace of $X$, and $(X', C')$ is a CW complex.

Note that it follows from this that the $k$-sketeton of a CW complex is closed and a sub CW complex.
Now we can start counting cells with homology.

**Proposition**

Let $e$ be a $k$-cell. $H_*(D^k, S^{k-1}) \cong H_*(\overline{e}, \overline{e} - e)$. In particular, $H_i(\overline{e}, \overline{e} - e) = \mathbb{Z}$ for $i = k$ and it is zero otherwise.

**Start of Proof.**

The proof is essentially a variation on the proof we gave earlier to study the homology of adjunction spaces. Let $E^k = D^k - S^{k-1}$, $s = \overline{e} - e$. Consider the diagram

$$
\begin{align*}
H_i(D^k, S^{k-1}) & \xrightarrow{1} H_i(D^k, D^k - \{0\}) \\
\downarrow 2 & \hspace{2cm} \downarrow 3 \\
H_i(\overline{e}, s) & \xrightarrow{4} H_i(\overline{e}, \overline{e} - \{p\})
\end{align*}
$$
Continuation of Proof.
where the morphisms 2 and 3 come from $\phi = \phi_e$ and $p = \phi(0)$. 1 and 4 are isomorphisms. For 1, consider the long exact sequences of the pairs $(D^k, S^{k-1})$ and $(D^k - \{0\}, S^{k-1})$, and the morphisms between making appropriate diagrams commute. (Draw those diagrams if you are not sure of the argument.) $H_i(S^{k-1}) \rightarrow H_i(D^k - \{0\})$ is an isomorphism because the space on the left is a deformation retract of the space on the right. Of course the identity homomorphism $H_i(D^k) \rightarrow H_i(D^k)$ is an isomorphism. Now apply to five lemma to conclude that 1 is an isomorphism. A similar argument works for 4. ($s$ is a deformation retract of $\bar{e} - \{p\}$ because $\bar{e}$ is an appropriate adjunction space.) It follows that we need only prove that 3 is an isomorphism. \qed
Finish proof.
Consider next the diagram

\[
\begin{array}{c}
H_i(D^k, D^k - \{0\}) \overset{5}{\longrightarrow} H_i(E^k, E^k - \{0\}) \\
\downarrow{3} \quad \downarrow{6} \\
H_i(\bar{e}, \bar{e} - \{p\}) \overset{7}{\longrightarrow} H_i(e, e - \{p\})
\end{array}
\]

The map 5 is an isomorphism because we may excise \( S^{k-1} \) and 7 is an isomorphism because we may excise \( s \). Finally, 6 is an isomorphism because \( E^k \to e \) is a homeomorphism. \( \square \)
Now that we know how to isolate one cell homologically, we want to apply this wholesale to the collection of all open $k$ cells.

For each $k$ cell $e$ in $X$, let $\phi_e : D^k_e \to X$ be a characteristic map for $\overline{e}$ where we label $D^k$ to keep the domains separate. The collection of these maps gives us a map

$$
\phi : \bigsqcup_{e \in C^k} D^k_e \to X
$$

which carries $\bigsqcup_e S^k_{e^{-1}}$ into $X^{k-1}$, where $C^k$ denotes the set of $k$-cells. Denote by $f$ the restriction of $\phi$ to that subspace. So

$$
f : \bigsqcup_e S^k_{e^{-1}} \to X^{k-1}.
$$
Proposition

\[
\left( \bigcup_{e \in C} D^k_e \right) \sqcup_f X^{k-1} \simeq X^k
\]

Proof.
The proof is a point set topological argument which mimics the proof in the case of a single cell. The details are spelled out in the book. \qed
Theorem

Let $X$ be a CW complex. There is an isomorphism

$$
\bigoplus_{e \in C^k} H_i(D^k_e, S^{k-1}_e) \to H_i(X^k, X^{k-1}).
$$

In particular, $H_k(X^k, X^{k-1})$ is free on a basis in one-to-one correspondence with the set of open $k$ cells, and $H_i(X^k, X^{k-1}) = 0$ for $i \neq k$.

Start of proof.

The (relative) singular homology groups of a disjoint union are certainly the direct sum of the homology groups of the factors. To prove the theorem, we just mimic the proof in the case of one $k$-cell. In particular consider the diagram

\[\]

\[\]
\[(\bigsqcup_{e} D_{e}^k, \bigsqcup S_{e}^{k-1}) \rightarrow (\bigsqcup D_{e}^k, \bigsqcup D_{e}^k - \{0_{e}\})\]

\[(X^k, X^{k-1}) \rightarrow (X^k, X^k - \bigcup_{e}\{p_{e}\})\]

\[\leftarrow (\bigsqcup E_{e}^k, \bigsqcup E_{e}^k - \{0_{e}\})\]

\[\leftarrow (\bigcup_{e} e, \bigcup (e - \{p_{e}\}))\]
The only point worth mentioning about the argument is that $X^{k-1}$ is a deformation retract of $X^k - \{p_e \mid e \in C^k\}$. The argument depends as before on understanding the product of a quotient space with $I$. 

\[
X^{k-1} \rightarrow X^k - \{p_e \mid e \in C^k\} \rightarrow X^k
\]