CHAPTER 8

Cell Complexes

1. Introduction

We have seen examples of 'triangulations' of surfaces for which the rule that two simplexes intersect in at most an edge fails. For example, the decomposition indicated in the diagram below of a torus into two triangles fails on that ground. However, if we ignore that fact and calculate the homology of the associated chain complex (using the obvious boundary), we will still get the right answer for the homology of a 2-torus. Even better, there is no particular reason to restrict our attention to triangles. We could consider the torus as a single '2-cell' ϕ as indicated below with its boundary consisting of two '1-cells' σ, τ, which meet in one '0-cell' ν. Moreover the diagram suggests the following formulas for the 'boundary':

$$\partial_2\phi = \sigma + \tau - \sigma - \tau = 0$$
$$\partial_1\sigma = \partial_1\tau = \nu - \nu = 0.$$

From this it is easy to compute the homology: $H_0 = \mathbf{Z}\nu, H_1 = \mathbf{Z}\sigma \oplus \mathbf{Z}\tau$, and $H_2 = \mathbf{Z}\phi$. A decomposition of this kind (not yet defined precisely) is called a cellular decomposition, and the resulting structure is called a cell complex. The diagram below indicates how this looks with the torus imbedded in $\mathbf{R}^3$ in the usual way.

Indicated below is a calculation of the homology of $\mathbf{R}P^2$ by similar reasoning. Note that it is clear why there is an element of order two in H_1.

Here is a similar calculation for S^2 where we have one 2-cell ϕ, no 1-cells, and one 0-cell ν.

This same reasoning could be applied to S^n so as to visualize it as a cell complex with one n-cell and one 0-cell.

1.1. $\mathbf{C}P^n$. The definition given previously for $\mathbf{R}P^n$, real projective n-space, may be mimicked for any field F. Namely, consider the vector space F^{n+1}, and define an equivalence relation in $F^{n+1} - \{0\}$

$$x \sim y \Leftrightarrow \exists c \in F^* \quad \text{such that } y = cx,$$

and let FP^n be the quotient space of this relation. Thus FP^n consists of the set of *lines* or 1 dimensional linear subspaces in F^{n+1} suitably topologized. If $x = (x_0, \dots, x_n)$, the components $x_0, \dots, x_n$ are called *homogeneous coordinates* of the corresponding point of FP^n. As above, different sets of homogeneous coordinates for the same point differ by a constant, non-zero multiplier. Note also that FP^0 consists of a single point.

Let $F = \mathbf{C}$. Then the *complex projective space* $\mathbf{C}P^n$ is the quotient space of $\mathbf{C}^{n+1} - 0 \simeq \mathbf{R}^{2n+2} - 0$. We may also view it as a quotient of S^{2n+1} as follows. First note that by multiplying by an appropriate positive real number, we may assume that the homogeneous coordinates $(x_0, x_1, \dots, x_n)$ of a point in $\mathbf{C}P^n$ satisfy

$$\sum_{i=0}^{n} |x_i|^2 = 1,$$

so the corresponding point in $C^{n+1} - 0$ lies in S^{2n+1}. Furthermore, two such points, $(x'_0, \dots, x'_n)$ and $(x_0, \dots, x_n)$, will represent the same point in $\mathbf{C}P^n$ if and only if $x'_i = cx_i, i = 0, \dots, n$ with $|c| = 1$, i.e., $c \in S^1$. Hence, we can identify $\mathbf{C}P^n$ as the orbit space of the action of the group S^1 on S^{2n+1} defined through complex coordinates by

$$c(x_0, \dots, x_n) = (cx_0, \dots, cx_n).$$

The student should verify that the map $p : S^{2n+1} \to \mathbf{C}P^n$ is indeed a quotient map. It follows that $\mathbf{C}P^n$ is compact. We also need to know that it's Hausdorff.

PROPOSITION 8.1. *$\mathbf{C}P^n$ is compact Hausdorff.*

PROOF. It suffices to prove that $p : S^{2n+1} \to \mathbf{C}P^n$ is a closed map. Then by Proposition 3.9 we know that $\mathbf{C}P^n$ is compact Hausdorff.

To show that p is closed consider a closed set $A \subset S^{2n+1}$ and look at the diagram

$$\begin{array}{ccc} S^1 \times S^{2n+1} & \longrightarrow & S^{2n+1} \\ \uparrow & \nearrow & \\ S^1 \times A & & \end{array}$$

The horizontal map is the action of S^1 on S^{2n+1}, which is continuous. Since $S^1 \times A$ is compact, the image of $S^1 \times A$ under this action is compact, hence closed. But this image is just $p^{-1}(p(A))$, which is what we needed to show was closed. □

We shall describe a cellular decomposition of $\mathbf{C}P^n$. First note that $\mathbf{C}P^{k-1}$ may be imbedded in $\mathbf{C}P^k$ through the map defined using homogeneous coordinates by

$$(x_0, x_1, \ldots, x_{k-1}) \mapsto (x_0, x_1, \ldots, x_{k-1}, 0).$$

With these imbeddings, we obtain a tower

$$\mathbf{C}P^n \supset \mathbf{C}P^{n-1} \supset \cdots \supset \mathbf{C}P^1 \supset \mathbf{C}P^0.$$

We shall show that each of the subspaces $\mathbf{C}P^k - \mathbf{C}P^{k-1}, k = 1, \ldots, n$ is homeomorphic to an open $2k$-ball in $\mathbf{R}^{2k}$. To prove this, assume as above that the homogeneous coordinates $(x_0, x_1, \ldots, x_k)$ of a point in $\mathbf{C}P^k$ are chosen so $\sum_{i=0}^k |x_i|^2 = 1$. Let $x_k = r_k e^{i\theta_k}$ where $0 \le r_k = |x_k| \le 1$. By dividing through by $e^{i\theta_k}$ (which has absolute value 1). we may arrange for $x_k = r_k$ to be real and non=negative without changing the fact that $\sum_i |x_i|^2 = 1$. Then,

$$0 \le x_k = \sqrt{1 - \sum_{i=0}^{k-1} |x_i|^2} \le 1.$$

Let D^{2k} denote the closed $2k$-ball in $\mathbf{R}^{2k}$ defined by

$$\sum_{i=1}^{2k} y_i^2 \le 1.$$

Define $f_k : D^{2k} \to \mathbf{C}P^k$ by

$$f_k(y_1, \ldots, y_{2k}) = (x_0, \ldots, x_k)$$

where $x_0 = y_1 + iy_2,\ x_1 = y_3 + iy_4,\ \ldots,\ x_{k-1} = y_{2k-1} + iy_{2k}$ and $x_k = \sqrt{1 - |y|^2}$. The coordinates on the left are ordinary cartesian coordinates, and the coordinates on the right are homogeneous coordinates. f_k is clearly continuous and, by the above discussion f_k is onto. Since D^{2k} is compact and $\mathbf{C}P^k$ is Hausdorff, it follows that f_k is a closed map. We shall show that it is one-to-one on the open ball $D^{2k} - S^{2k-1}$, and it follows easily from this that it a homeomorphism on the open ball.

Suppose $f_k(y') = f_k(y)$, i.e.,

$$(x'_0, \ldots, x'_k) = c(x_0, \ldots, x_k) \qquad c \in S^1.$$

Suppose $|y'| < 1$. Then, $x'_k = \sqrt{1 - |y'|^2} > 0$, so it follows that c is real and positive, hence $c = 1$, which means $|y| < 1$ and $y' = y$. Note that this argument shows a little more. Namely, if two points of D^{2k} map to the same point of $\mathbf{C}P^k$, then they must both be on the boundary S^{2k-1} of D^{2k}.

We now investigate the map f_k on the boundary S^{2k-1}. First note that $|y| = 1$ holds if and only if $x_k = 0$, i.e., if and only $f_k(y) \in \mathbf{C}P^{k-1}$. Thus, f_k does map $D^{2k} - S^{2k-1}$ homeomorphically onto $\mathbf{C}P^k - \mathbf{C}P^{k-1}$ as claimed. Moreover, it is easy to see that the restriction of f_k to S^{2k-1} is just the quotient map described above taking S^{2k-1} onto $\mathbf{C}P^{k-1}$. (Ignore the last coordinate x_k which is zero.)

The above discussion shows us how to view $\mathbf{C}P^n$ as a cell complex (but note that we haven't yet defined that concept precisely.) First take a point σ_0 to be viewed as $\mathbf{C}P^0$. Adjoin to this a disk D^2 by identifying its boundary to that point. This yields $\mathbf{C}P^2$ which we see is homemorphic to S^2. Call this 'cell' σ_2. Now attach D^4 to this by mapping its boundary to σ_2 as indicated above; call the result σ_4. In this way we get a sequence of cells

$$\sigma_0 \subset \sigma_2 \subset \cdots \subset \sigma_{2n}$$

each of which is the quotient of a closed ball of the appropriate dimension modulo the equivalence relation described above on the boundary of the ball. In what follows we shall develop these ideas somewhat further, and show that the homology of such a cell complex may be computed by taking as chain group the free abelian group generated by the cells, and defining an appropriate boundary homomorphism. The definition of that boundary homomorphism is a bit tricky, but in the present case, since there are no cells in odd dimensions, the boundary homomorphism should turn out to be zero. That is, we should end up with the following chain complex for $\mathbf{C}P^n$:

$$\begin{aligned} C_{2i} &= \mathbf{Z} && 0 \le i \le n, \\ C_{2i-1} &= 0 && 1 \le i \le n, \quad \text{and} \\ \partial_k &= 0 && \text{all } k. \end{aligned}$$

Hence, *after we have finished justifying the above claims* we shall have proved the following assertion:

THEOREM 8.2. *The singular homology groups of $\mathbf{C}P^n$ are given by*

$$H_k(\mathbf{C}P^n) = \begin{cases} \mathbf{Z} & \text{if } k = 2i, 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

2. Adjunction Spaces

We now look into the idea of 'adjoining' one space to another through a map. We will use this to build up cell complexes by adjoining one cell at a time.

Let X, Y be spaces and suppose $f : A \to Y$ is a map with domain A a *subspace* of X. In the *disjoint union* $X \sqcup Y$ consider the equivalence relation generated by the basic relations $a \sim f(a)$ for $a \in A$. The quotient space $X \sqcup Y/\sim$ is called the adjunction space obtained by attaching X to Y through f. It is denoted $X \sqcup_f Y$. f is called the *attaching map*.

EXAMPLE 8.3. Let $f : A \to \{P\}$ be a map to a point. Then $X \sqcup_f \{P\} \simeq X/A$.

EXAMPLE 8.4. Let $X = D^2$ be the closed disk in $\mathbf{R}^2$ and let Y also be the closed disk. Let $A = S^1$ and let $f : A \to Y$ be the degree 2 map of S^1 onto the boundary of Y which is also S^1. Then $X \sqcup_f Y$ is a 2-sphere with antipodal points on its equator identified. Can you further describe this space?

Note that there is a slight technical problem in the definition. Since X and Y could in principle have points in common, the disjoint union $X \sqcup Y$ must be defined by taking spaces homeomorphic to X and Y but which are disjoint and then forming their union. That means that assertions like $a \sim f(a)$ don't technically make sense in $X \sqcup Y$. However, if one is careful, one may identify X and Y with the corresponding subspaces of $X \sqcup Y$. You should examine the above examples with these remarks in mind.

Let $\rho_Y : Y \to X \sqcup_f Y$ be the composite map

$$Y \xrightarrow{\iota} X \sqcup Y \xrightarrow{\rho} X \sqcup_f Y$$

where ι is inclusion into the disjoint union, and ρ is projection onto the quotient space. Define ρ_X analagously.

Proposition 8.5. *(i) ρ_Y maps Y homeomorpically onto a subspace of $X \sqcup_f Y$.*

(ii) If A is closed then $\rho_Y(Y)$ is closed in $X \sqcup_f Y$.

(iii) If A is closed, then ρ_X is a homeomorphism of $X - A$ onto $X \sqcup_f Y - \rho_Y(Y)$.

Because of (i), we may identify Y with its image in $X \sqcup_f Y$. This is a slight abuse of terminology.

The map $\rho_X : X \to X \sqcup_f Y$ is called the *characteristic map* of the adjunction space. Because relations of the form $a \sim f(a)$ will imply realtions of the form $a_1 \sim a_2$ where $f(a_1) = f(a_2)$, it won't generally be true that X can be identified with a subspace of $X \sqcup_f Y$. Nevertheless, (iii) says that $X - A$ can be so identified. In general, a map of pairs $h : (X, A) \to (X', A')$ is called a *relative homeomorphism* if its restriction to $X - A$ is a homeomorphism onto $X' - A'$. Thus, if A is closed, ρ_X provides a relative homeomorphism of (X, A) with $(X \sqcup_f Y, Y)$.

Proof. (i) Note first that two distinct elements of Y are never equivalent, so ρ_Y is certainly one-to-one. Let U be an open set in Y. Then $f^{-1}(U)$ is open in A, so $f^{-1}(U) = A \cap V$ for some open set V in X. $V \sqcup U$ is open in $X \sqcup Y$, and it is not hard to see that it is a union of equivalence classes of the relation $\sim$. That implies that its image $\rho(V \sqcup U)$ is open in $X \sqcup_f Y$. However, $\rho_Y(U) = \rho_Y(Y) \cap \rho(V \sqcup U)$, so $\rho_Y(U)$ is open in $\rho_Y(Y)$.

(ii) Exercise.

(iii) ρ_X is certainly one-to-one on $X - A$. Since $X - A$ is open in X, any open set U in $X - A$ is open in X. However, any subset of $X - A$ is a union of (singleton) equivalence classes, so it follows that

$$\rho(U \sqcup \emptyset) = \rho_X(U)$$

is open in in $X \sqcup_f Y$. However, this is contained in the open set $X \sqcup_f Y - Y$ of $X \sqcup_f Y$, so it is open in that. The fact that $\rho_X(X - A) = X \sqcup_f Y - Y$ is obvious. □

We now turn our attention to a special case of fundamental importance, namely when $X = D^n$ and $A = S^{n-1}$. Given a map $S^{n-1} \to Y$, the resulting adjunction space $D^n \sqcup_f Y$ is referred to as the space obtained by attaching, or adjoining, an n-cell to Y.

The following result allows us to recognize when a subspace of a space is the result of adjoining an n-cell.

PROPOSITION 8.6. *Let Y be a compact Hausdorff space, e, S disjoint subspaces with S closed. Suppose there is a map*

$$\phi : (D^n, S^{n-1}) \to (e \cup S, S)$$

which is a relative homeomorphism. Then $D^n \sqcup_f S \simeq e \cup S$ where $f = \phi|S^{n-1}$.

PROOF. Since $D^n \sqcup S$ is compact and Y is Hausdorff, it suffices by Proposition 3.6 to show that for the map $D^n \sqcup S \to e \cup S$, the preimages of points are the equivalence classes of the relation in $D^n \sqcup S$ generated by $x \sim f(x), x \in S^{n-1}$. This is clear since ϕ is one-to-one on $D^n - S^{n-1}$. $\square$

EXAMPLE 8.7. Let $X = D^{2n}, A = S^{2n-1}$, and let $f : S^{2n-1} \to \mathbf{C}P^{n-1}$ be the map described in the previous section. Then $D^{2n} \sqcup_f S^{2n-1} \simeq \mathbf{C}P^n$. For the discussion in the previous section establishes that the hypothesis of Proposition 8.6 is satisfied.

We now want to study the effect on homology of adjoining an n-cell D^n to a space Y through an attaching map $f : S^{n-1} \to Y$. Take U to be the open set in $D^n \sqcup_f Y - Y$ which is homeomorphic to the open cell $D^n - S^{n-1}$ through the characteristic map ϕ. Take $V = D^n \sqcup_f Y - \{\phi(0)\}$. Then $D^n \sqcup_f Y = U \cup V$, and $U \cap V$ is homeomorphic to an open ball less its center, so it has S^{n-1} as a deformation retract.

LEMMA 8.8. *Y is a deformation retract of V.*

PROOF. Define a retraction $r : V \to Y$ by

$$r(z) = \begin{cases} z & \text{if } z \in Y, \\ \phi(z/|z|) & \text{if } z \in D^n - \{0\}. \end{cases}$$

To show that r is deformation retraction, we define a homotopy $F : V \times I \to V$ as follows:

$$F(z,t) = \begin{cases} z & \text{if } z \in Y, \\ \phi((1-t)z + tz/|z|) & \text{if } z \in D^n - \{0\}. \end{cases}$$

Usually, we just assert that it is obvious that a map is continuous, but in this case, because of what happens on the boundary of D^n, it is not so obvious. Consider the following diagram

$$\begin{array}{ccc} ((D^n - \{0\}) \sqcup Y) \times I & \xrightarrow{\tilde{F}} & (D^n - \{0\}) \sqcup Y \\ \Big\downarrow{\scriptstyle \rho\times\mathrm{Id}} & & \Big\downarrow{\scriptstyle \rho} \\ V \times I & \xrightarrow{F} & V \end{array}$$

where $\tilde{F}$ is defined in the obvious way on each component of the disjoint sum. It is easy to check that the diagram commutes and that the vertical map on the right (induced from the quotient map $D^n \sqcup Y \to D^n \sqcup_f Y$) is a quotient map. If we knew that the vertical map on the left were a quotient map, then it would follow that F is continuous (check this!). Unfortunately the product of a quotient map with the the identity map of a space needn't be a quotient map, but it's true in this case because I is a nice space. We quote the following result about quotient spaces and products. A proof can be found, for example, in Munkres, *Elements of Algebraic Topology*, Theorem 20.1.

Proposition 8.9. *Let $\rho : X \to Y$ be a quotient map. Suppose Z is any locally compact Hausdorff space. Then $\rho \times \mathrm{Id} : X \times Z \to Y \times Z$ is a quotient map.*

By the above remarks, this completes the proof of Lemma 8.8. □

Suppose $n > 0$. The conditions for a Mayer–Vietoris sequence apply to $U \cup V$, so we have a long exact sequence

$$\dots \to \tilde{H}_i(S^{n-1}) \to \tilde{H}_i(Y) \to \tilde{H}_i(D^n \sqcup_f Y) \to \tilde{H}_{i-1}(S^{n-1}) \to \dots$$

For $i \neq n, n-1$, this yields

$$\tilde{H}_i(Y) \cong \tilde{H}_i(D^n \sqcup_f Y).$$

For $i \neq 0$, the $\sim$'s are not needed, and in fact, it is not hard to see that they are not needed for $i = 0$. Thus, we have essentially proved the following

Theorem 8.10. *Let $n > 0$ and let $f : S^{n-1} \to Y$ be a map. Then, for $i \neq n, n-1$,*

$$H_i(Y) \cong H_i(D^n \sqcup_f Y).$$

For $i = n, n-1$, we have an exact sequence

$$0 \to H_n(Y) \to H_n(D^n \sqcup_f Y) \to \mathbf{Z} \to H_{n-1}(Y) \to H_{n-1}(D^n \sqcup_f Y) \to 0.$$

We can now determine the homology groups of complex projective spaces without explicit use of 'cellular chains', but of course that idea is implicit in the argument. We repeat the statement in the previous section.

THEOREM 8.11. *The singular homology groups of* $\mathbf{C}P^n$ *are given by*

$$H_k(\mathbf{C}P^n) = \begin{cases} \mathbf{Z} & \text{if } k = 2i, 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Use $\mathbf{C}P^n = D^{2n} \sqcup_f \mathbf{C}P^{n-1}$. The corollary is true for $n = 0$. By the above discussion $H_i(\mathbf{C}P^n) \cong H_i(\mathbf{C}P^{n-1})$ for $i \ne 2n, 2n-1$. For $i = 2n, 2n-1$, we have

$$0 \to H_{2n}(\mathbf{C}P^{n-1}) \to H_{2n}(\mathbf{C}P^n) \to \mathbf{Z} \to H_{2n-1}(\mathbf{C}P^{n-1} \to H_{2n-1}(\mathbf{C}P^n) \to 0.$$

Since $H_{2n}(\mathbf{C}P^{n-1}) = H_{2n-1}(\mathbf{C}P^{n-1}) = 0$, it follows that $H_{2n-1}(\mathbf{C}P^n) = 0$ and $H_{2n}(\mathbf{C}P^n) = \mathbf{Z}$. □

EXAMPLE 8.12 (Products of Spheres). For $m, n > 0$,

$$S^m \times S^n \simeq D^{m+n} \sqcup_f (S^m \vee S^n)$$

for an appropriate attaching map f.

Model D^m by I^m. Then D^{m+n} is modelled by $I^{m+n} \simeq I^m \times I^n$. It is not hard to check the formula

$$\partial(I^m \times I^n) = \partial I^m \times I^n \cup I^m \cup \partial I^n$$

which is a set theoretic version of the product formula from calculus. Choose points $p_m \in S^m$ and $p_n \in S^n$, say the 'north poles' of each. Let $f_m : I^m \to S^m$ be a map which takes the interior of I^m onto $S^m - \{p_m\}$ and ∂I^m onto p_m. Then $f_m \times f_n$ maps I^{m+n} onto $S^m \times S^n$ and ∂I^{m+n} onto

$$(\{p_m\} \times S^n) \cup (S^m \times \{p_n\}) \simeq S^m \vee S^n.$$

(Note that $(\{p_m\} \times S^n) \cap (S^m \times \{p_n\}) = \{(p_m, p_n)\}$.) Thus, we have a mapping of pairs

$$f_m \times g_m : (I^{m+n}, \partial I^{m+n}) \to (S^m \times S^n, S^m \vee S^n)$$

as required by Proposition 8.6.

COROLLARY 8.13. *Let* $m, n > 0$. *If* $m < n$, *then*

$$H_i(S^m \times S^n) = \begin{cases} \mathbf{Z} & \text{if } i = 0, \\ \mathbf{Z} & \text{if } i = m, n, \\ \mathbf{Z} & \text{if } i = m+n, \\ 0 & \text{otherwise.} \end{cases}$$

If $m = n$, *then the only difference is that* $H_m(S^m \times S^m) = \mathbf{Z} \oplus \mathbf{Z}$.

Proof. Assume $m < n$. By a straightforward application of the Mayer–Vietoris sequence, we see that $H_i(S^m \vee S^n) = \mathbf{Z}$ if $i = 0, m, n$ and is zero otherwise.

Assume $m \neq 1$. Then, $H_i(S^m \times S^n) = \mathbf{Z}$ for $= 0, m, n$, it is zero otherwise except for the cases $i = m+n, m+n-1$. These cases are determined by considering

$$0 \to H_{m+n}(S^m \vee S^n) = 0 \to H_{m+n}(S^m \times S^n) \\ \to \mathbf{Z} \to H_{m+n-1}(S^m \vee S^n) = 0 \to H_{m+n-1}(S^m \times S^n) \to 0.$$

We see that $H_{m+n}(S^m \times S^n) = \mathbf{Z}$ and $H_{m+n-1} = 0$. The remaining cases $m = 1 < n$ and $m = n$ are left as excercises. □

Example 8.14 (Real Projective spaces). The diagram below illustrates the construction of $\mathbf{R}P^n$ by a scheme similar to that described previously for $\mathbf{C}P^n$.

We conclude

Proposition 8.15. *$\mathbf{R}P^n \simeq D^n \sqcup_f \mathbf{R}P^{n-1}$ for the usual (attaching) map $f : S^{n-1} \to \mathbf{R}P^{n-1}$.*

For example, $\mathbf{R}P^1 = D^2 \sqcup_f \mathbf{R}P^0 = S^1$.

Note that we cannot use the method used for $\mathbf{C}P^n$ and in the previous example to compute $H_*(\mathbf{R}P^n)$. The reason is that there are cells in every dimension. (You should try using the argument to see what goes wrong.)

Example 8.16 (Quaternionic Projective Spaces). It is known that there are precisely three real division algebras, $\mathbf{R}$, $\mathbf{C}$, and the quaternion algebra $\mathbf{H}$. The quaternion algebra is not commutative, but every non-zero element is invertible. We remind you of how it is defined. $\mathbf{H} = \mathbf{R} \times \mathbf{R}^3 = \mathbf{R}^4$ as a real vector space. Think of elements of $\mathbf{H}$ as pairs $(a, \mathbf{v})$ where $a \in \mathbf{R}$ and $\mathbf{v}$ is a 3-dimensional vector. The product in $\mathbf{H}$ is defined by

$$(a, \mathbf{u})(b, \mathbf{v}) = (ab - \mathbf{u} \cdot \mathbf{v}, a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v}).$$

Any element in $\mathbf{H}$ may be written uniquely $a + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -1 \\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k} \\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i} \\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}. \end{aligned}$$

These rules together with the distributive law determine the product.

Define $|(a, \mathbf{v})| = \sqrt{a^2 + |\mathbf{v}|^2}$ which is of course just the usual norm in $\mathbf{R}^4$. Then in addition to the usual rules, we have

$$|xy| = |x||y| \qquad x, y \in \mathbf{H}.$$

The set of non-zero quaternions forms a group under multiplication, and the set $S^3 = \{x \mid |x| = 1\}$ is a subgroup. (In fact, S^1 and S^3 are the only spheres which have group structures making them topological groups.)

Quaternionic projective space $\mathbf{H}P^n$ is defined to be the set of 1-dimensional $\mathbf{H}$-subspaces of $\mathbf{H}^{n+1}$. Let $(x_0, \dots, x_n) \in \mathbf{H}^{n+1} - \{0\} = \mathbf{R}^{4n+4} - \{0\}$ be homogenous quaternionic coordinates representing a point in $\mathbf{H}P^n$. By dividing by $\sqrt{\sum_i |x_i|^2}$ we may assume this point lies in S^{4n+3}. In fact, two points in S^{4n+3} represent the same point if and only if they differ by a quaternion mutiple x of norm 1, i.e., if and only if they are in the same orbit of the action of S^3 on S^{4n+3} given by $x(x_0, \dots, x_n) = (xx_0, \dots, xx_n)$. Then, reasoning as before, we have

$$\mathbf{H}P^n \simeq D^{4n} \sqcup_f \mathbf{H}P^{n-1}.$$

Corollary 8.17.

$$H_k(\mathbf{H}P^n) = \begin{cases} \mathbf{Z} & \text{if } k = 4i, 0 \le i \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Reason as in the case of $\mathbf{C}P^n$. □

Note. There is one other similar example. Namely, there is an 8 dimensional real algebra called the Cayley numbers which is a division algebra in the sense that every non-zero element is invertible. It is not an associative algebra. In any case, one can define projective n-'space' over the Cayley numbers and determine its homology groups. In approximately 1960, Adams showed that this is as far as one can go. There are no other real, possibly non-associative algebras in which every non-zero element is invertible. The proof uses arguments from algebraic topology although the result is purely algebraic.

3. CW Complexes

Let X be a space. A subspace e of X homeomorphic to $D^n - S^{n-1}$ is called an *open n-cell*. For $n = 0$, we interpret S^{-1} to be the empty set, so an 'open' zero cell is a point.

Let $\mathcal{C}$ be a collection of *disjoint* open cells e in X of various dimensions. Let X^k denote the union of all cells in $\mathcal{C}$ of dimension $\leq k$. X^k is called the k-skeleton of X. We have a tower

$$\emptyset = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^{k-1} \subset X^k \subseteq \ldots X$$

A *CW complex* is a pair $(X, \mathcal{C})$ where X is a Hausdorff space, $\mathcal{C}$ is a collection of open cells in X such that for each e in $\mathcal{C}$, there is a map

$$\phi_e : (D^k, S^{k-1}) \to (e \cup X^{k-1}, X^{k-1})$$

where k is the dimension of e, and the following rules hold.

(1) The cells in $\mathcal{C}$ are disjoint and X is their union.

(2) Each map ϕ_e is a relative homeomorphism.

(3) The closure $\overline{e}$ of each cell in $\mathcal{C}$ is contained in the union of finitely many cells in $\mathcal{C}$. (Closure finiteness.)

(4) A set in X is closed if and only if its interesection with $\overline{e}$ is closed for every cell e in $\mathcal{C}$. (The topology on X is called the weak topology relative to the collection of cells if this is true.) The name 'CW' is an abbreviation of the terminology for conditions (3) and (4). The concept was introduced by J. H. C. Whitehead.

If there is such a pair for X, we call X a CW complex without explicitly mentioning the family of cells and maps. Note that if $\mathcal{C}$ is finite, then conditions (3) and (4) are automatic. However, there are important CW complexes which are not finite. If $X = X^k$ for some k, the smallest such k is called the dimension of X. It is the largest dimension of any cell in $\mathcal{C}$.

The general idea is that a CW complex is a space that can be built up inductively by successively adjoining new cells in each dimension. For many purposes they form the most interesting class of topological spaces to study.

Example 8.18. Every simplicial complex K is a CW complex. Let $X = |K|$ and let $\mathcal{C}$ be the family of all $e = \sigma - \dot{\sigma}$ for σ a simplex in K. For each k-simplex, there is a homeomorphism $\phi_\sigma : D^k \to \sigma$ and that provides a relative homeomorphism

$$\phi_e : (D^k, S^{k-1}) \to (\sigma, \dot{\sigma}) \hookrightarrow (e \cup X^{k-1}, X^{k-1}).$$

Because of the conditions for a simplicial complex, the rules for a CW complex hold.

EXAMPLE 8.19. $\mathbf{R}P^n, \mathbf{C}P^n$, and $\mathbf{H}P^n$ are all finite CW complexes. Consider the family $\mathbf{R}P^n$. There is a natural injection of $\mathbf{R}P^n$ in $\mathbf{R}P^{n+1}$ such that the former space is the n-skeleton of the latter and there is one more open cell of dimension n. We have then an ascending chain of topological spaces

$$\mathbf{R}P^0 \subset \mathbf{R}P^1 \subset \mathbf{R}P^2 \subset \cdots \subset \mathbf{R}P^n \subset \dots$$

and we may form the union, which is denoted $\mathbf{R}P^\infty$. This set has a cell structure with one cell in each dimension, and we may make it a topological space using the weak topology relative to this family of cells. The result is an infinite CW complex.

Similar constructions apply in the other two cases to form $\mathbf{C}P^\infty$ and $\mathbf{H}P^\infty$. For $\mathbf{C}P^\infty$ there is one cell in each even dimension and for $\mathbf{H}P^\infty$ one cell in each dimension divisible by 4.

EXAMPLE 8.20. There are two interesting CW structures to put on S^n.

The first is very simple. Let p_n denote the point $(0, \dots, 1)$ (the north pole). Then $S^n - \{p_n\}$ is an open n-cell and $\{p_n\}$ is an open 0-cell. These are the only cells.

The second is related to our construction of $\mathbf{R}P^n$. There are two cells in each dimension. In dimension n the open upper hemisphere is one open cell and the lower open hemsiphere is the other. Their common boundary is a closed cell homeomorphic to S^{n-1}. Repeat this for S^{n-1} and continue iteratively down to $k = 0$.

EXAMPLE 8.21. As discussed in the previous section, $S^m \times S^n$ has a CW structure with one cell each in dimensions $m+n, m, n$, and 0. If $m = n$, there are two cells in dimension m.

A CW complex is called regular if each characteristic map provides a homeomorphism $D^k \to \overline{e}$, and $\overline{e} - e$ is a finite union of open cells of dimension less than k rather than just being contained in such a union.

4. The Homology of CW complexes

Let X be a CW complex and let e be an open k-cell in X. $\phi_e(D^k)$ is compact, so, since X is Hausdorff, it is closed. Since by (2), $\phi_e(D^k - S^{k-1}) = e$, it is not hard to check that $\phi_e(D^k) = \overline{e}$ and $\phi_e(S^{k-1}) = \overline{e} - e$. The latter is contained, by assumption, in X^{k-1} so by condition (3) in the definition, it is contained in a finite union of cells in $\mathcal{C}$ of dimension $< k$. Also, since $\overline{e}$ is compact, Proposition 8.6 on recognizing adjunction spaces shows that $\overline{e} = D^k \sqcup_{f_e} (\overline{e} - e)$.

PROPOSITION 8.22. *Let $(X, \mathcal{C})$ be a CW complex. Let $\mathcal{C}'$ be a subset of $\mathcal{C}$ with the property that for every cell e in $\mathcal{C}'$, $\overline{e}$ is contained in a finite union of cells in $\mathcal{C}'$. Then the union X' of the cells in $\mathcal{C}'$ is a closed subspace of X, and $(X', \mathcal{C}')$ is a CW complex.*

We call $(X', \mathcal{C}')$ a sub-CW-complex, or just a subcomplex. Note that it follows from this that the k-sketeton of a CW complex is closed and is a subcomplex.

PROOF. Exercise. □

PROPOSITION 8.23. $H_*(D^k, S^{k-1}) \cong H_*(\overline{e}, \overline{e} - e)$. *In particular, $H_i(\overline{e}, \overline{e} - e) = \mathbf{Z}$ for $i = k$ and it is zero otherwise.*

This lemma gives us a way to start counting cells through homology.

PROOF. The second statement follows from what we know about the relative homology of (D^k, S^{k-1}). (Use the long exact homology sequence.)

To demonstrate the isomorphism, use the argument we applied previously to study the homology of adjunction spaces. Let $E^k = D^k - S^{k-1}$, $s = \overline{e} - e$. Consider the diagram, in which we want to prove that the left hand vertical map is an isomorphism.

$$\begin{array}{ccccc}
H_i(D^k, S^{k-1}) & \xrightarrow{1} & H_i(D^k, D^k - \{0\}) & \xleftarrow{5} & H_i(E^k, E^k - \{0\}) \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 6} \\
H_i(\overline{e}, s) & \xrightarrow{4} & H_i(\overline{e}, \overline{e} - \{p\}) & \xleftarrow{7} & H_i(e, e - \{p\})
\end{array}$$

where the morphisms 2 and 3 come from $\phi = \phi_e$ and $p = \phi(0)$. 1 and 4 are isomorphisms. For 1, consider the long exact sequences of the pairs $(D^k, S^{k-1}$ and $(D^k - \{0\}, S^{k-1})$, and the morphisms between making appropriate diagrams commute. (Draw those diagrams if you are not sure of the argument.) $H_i(S^{k-1}) \to H_i(D^k - \{0\})$ is an isomorphism because the space on the left is a deformation retract of the space on the right. Of course the identity homomorphism $H_i(D^k) \to H_i(D^k)$ is an isomorphism. Now apply to five lemma to conclude that 1 is an isomorphism. A similar argument works for 4. (s is a deformation retract of $\overline{e} - \{p\}$ because $\overline{e}$ is an appropriate adjunction space.) It follows that we need only prove that 3 is an isomorphism.

The map 5 is an isomorphism because we may excise S^{k-1} and 7 is an isomorphism because we may excise s. Finally, 6 is an isomorphism because $E^k \to e$ is a homeomorphism. $\square$

Now that we know how to isolate one cell homologically, we want to apply this wholesale to the collection of all open k cells.

For each k cell e in X, let $\phi_e : D^k_e \to X$ be a characteristic map for $\overline{e}$ where we label D^k to keep the domains separate. The collection of these maps gives us a map

$$\phi : \bigsqcup_{e\in\mathcal{C}^k} D^k_e \to X$$

which carries $\sqcup_e S^{k-1}_e$ into X^{k-1}. Denote by f the restriction of ϕ to that subspace.

Proposition 8.24. $(\bigsqcup_{e\in\mathcal{C}^k} D^k_e) \sqcup_f X^{k-1} \simeq X^k$

Proof. Let ρ denote the quotient map of $\bigsqcup_e D^k_e \sqcup X^{k-1}$ onto $(\bigsqcup_e D^k_e) \sqcup_f X^{k-1}$, and let Φ be the obvious map from the former disjoint union into X. By seeing what Φ does on equivalence classes, we conclude that there is a unique map $\tilde{\phi}$ from the adjunction space into X such that $\Phi = \tilde{\phi} \circ \rho$. Moreover, this map is certainly one-to-one and onto X^k. Hence, it suffices to show that it is a closed map. Take a saturated closed set in $(\bigcup_e D^k_e) \sqcup X^{k-1}$, i.e., one consisting of equivalence classes. Such a set is necessarily a disjoint union of closed sets of the form $\sqcup_e Y_e \sqcup Z$ where $f^{-1}(Z) = \sqcup_e (Y_e \cap S^{k-1}_e)$.

Because X^k is a CW complex, it suffices to prove that the image under Φ of this set intersects the closure of every cell in X^k in a closed set. Since no matter where we consider it, Z is closed in X^{k-1} which itself is closed, it follows that the intersections for cells of dimension less than k are all closed. Fix an open k cell e'. It remains to show that $\Phi(\sqcup_e Y_e)$ intersects $\overline{e}'$ in a closed set. However, $\Phi(Y_e) \cap \overline{e}'$ is already contained in a finite union of cells of dimension less that k for $e \neq e'$, so we need only consider the case $e = e'$. However, in that case we already know that $\phi_e(Y_k)$ is closed in $\overline{e}$ because ϕ_e is a quotient map. $\square$

THEOREM 8.25. *Let X be a CW complex. $\tilde{\phi}$ induces and isomorphism*

$$\bigoplus_{e\in\mathcal{C}^k} H_i(D_e^k, S_e^{k-1}) \to H_i(X^k, X^{k-1}).$$

In particular, $H_k(X^k, X^{k-1})$ is free on a basis in one-to-one correspondence with the set of open k cells, and $H_i(X^k, X^{k-1}) = 0$ for $i \neq k$.

PROOF. The (relative) singular homology of a disjoint union is certainly the direct sum of the homologies of the factors. To prove the theorem, we just mimic the proof in the case of one k-cell. In particular consider the diagram

$$\begin{array}{ccccc}
(\sqcup_e D_e^k, \sqcup S_e^{k-1}) & \longrightarrow & (\sqcup D_e^k, \sqcup D_e^k - \{0_e\}) & \longleftarrow & (\sqcup E_e^k, \sqcup E_e^k - \{0_e\}) \\
\downarrow & & \downarrow & & \downarrow \\
(X^k, X^{k-1}) & \longrightarrow & (X^k, X^k - \cup_e\{p_e\}) & \longleftarrow & (\cup_e e, \cup(e - \{p_e\}))
\end{array}$$

The only point worth mentioning about the argument is that X^{k-1} is a deformation retract of $X^k - \{p_e \,|\, e \in \mathcal{C}^k\}$. The argument depends as before on understanding the product of a quotient space with I.

There have been quite a few details omitted from this proof, which you might try to verify for yourself. In so doing, you will have to give names to some maps and untangle some identifications implicit in the above discussion. □

Let $(X, \mathcal{C})$ be a CW complex. Define $C_k(X) = H_k(X^k, X^{k-1})$. By the above result, it is free with basis the set of open cells of dimension k. We shall define a boundary morphism $\partial_k : C_k(X) \to C_{k-1}(X)$ such that the homology of the complex $C_*(X)$ is the singular homology of X. (This terminology seems to conflict with the notation for simplicial complexes, but since the latter theory may be subsumed under the theory of CW complexes, we don't have to worry about that.)

The boundary map is the connecting homomorphism in the homology sequence of the triple (X^k, X^{k-1}, X^{k-2})

$$H_k(X^{k-1}, X^{k-2}) \to H_k(X^k, X^{k-2}) \to H_k(X^k, X^{k-1}) \xrightarrow{\partial_k} H_{k-1}(X^{k-1}, X^{k-2}) \to \dots$$

PROPOSITION 8.26. $\partial_k \circ \partial_{k+1} = 0$.

PROOF. The following commutative diagram arises from the obvious map of triples $(X^k, X^{k-1}, X^{k-2}) \to (X^{k+1}, X^{k-1}, X^{k-2})$

$$\begin{array}{ccccc} H_{k+1}(X^{k+1}, X^k) & \xrightarrow{\partial_{k+1}} & H_k(X^k, X^{k-1}) & \xrightarrow{i_k} & H_k(X^{k+1}, X^{k-1}) \\ & & \downarrow{\scriptstyle \partial_k} & \swarrow{\scriptstyle \partial'_k} & \\ & & H_{k-1}(X^{k-1}, X^{k-2}) & & \end{array}$$

Since the composite homomorphism across the top is trivial, the result follows. □

THEOREM 8.27. *Let X be a CW complex. Then $H_*(C_*(X)) \cong H_*(X)$*

PROOF. The top row of the diagram (CD) above extends to the right with

$$H_k(X^k, X^{k-1}) \xrightarrow{i_k} H_k(X^{k+1}, X^{k-1}) \to H_k(X^{k+1}, X^k) = 0$$

so i_k is an epimorphism. Similarly, the vertical column on the right may be extended upward

$$\begin{array}{c} H_k(X^{k-1}, X^{k-2}) = 0 \\ \downarrow \\ H_k(X^{k+1}, X^{k-2}) \\ \downarrow{\scriptstyle j_k} \\ H_k(X^{k+1}, X^{k-1}) \end{array}$$

so j_k is a monomorphism. A bit of diagram chasing should convince you that i_k maps $H_k(X^{k+1}, X^{k-2})$ (monomorphically) onto $j_k(\ker \partial_k) \cong \ker \partial_k / \operatorname{Im} d_{k+1} = H_k(C_*(X))$. Hence,

$$H_k(X^{k+1}, X^{k-2}) \cong H_k(C_*(X)).$$

It remains to prove that the left hand side of the above isomorphism is isomorphic to $H_k(X)$. First note that because $H_k(X^{k-i}, X^{k-i-1}) = H_{k-1}(X^{k-i}, X^{k-i-1}) = 0$ for $i \geq 2$, it follows that

$$H_k(X^{k+1}, X^{k-2}) \cong H_k(X^{k+1}, X^{k-3}) \cong \ldots \cong H_k(X^{k+1}, X^{-1}) = H_k(X^{k+1}).$$

Similarly, $H_{k+1}(X^{n+1}, X^n) = H_k(X^{n+1}, X^n) = 0$ for $n \geq k+2$ so $H_k(X^n) \to H_k(X^{n+1})$ is an isomorphism in that range. Hence,

$$H_k(X^{k+1}) \to H_k(X^n)$$

is an isomorphism for $n \geq k+1$. If X is a finite CW complex or even one of bounded dimension, then we are done.

In the general case, we need a further argument. Let u be a cycle representing an element of $H_k(X)$. Since only finitely many singular simplices occur in u, it comes from a cycle in some *compact* subspace of X. However, any compact subset of a CW complex lies in some n skeleton so we may assume u is actually a singular cycle representing an element in $H_k(X^n)$ for some n and clearly we loose nothing by assuming $n \geq k+1$. This together with the above argument shows $H_k(X^{k+1}) \to H_k(X)$ is onto. Suppose now that u is a cycle representing an element of $H_k(X^{k+1})$ which maps to zero in $H_k(X)$. Then $u = \partial_{k+1} v$ for some singular $k+1$-chain in X. As above, we may assume u, v are singular chains in some X^n for some $n \geq k+1$. That means u represents zero in $H_k(X^n)$. Since $H_k(X^{k+1}) \to H_k(X^n)$ is an isomorphism, it represents zero in $H_k(X^{k+1})$. That completes the proof.

Note the above isomorphisms are natural in the sense that a cellular map between CW complexes will yield appropriate commutative diagrams. This follows because everything is made up from natural homomorphisms in homology diagrams. The way the CW structure enters is in the characterization of the filtration of the space by k skeltons. A cellular map will carry one such filtration into the other. □

We may use the above result to calculate $H_k(\mathbf{R}P^n)$ as follows. First consider the CW complex on S^n described above with two open cells e_k, f_k in each dimension $k = 0, 1 \dots, n$. With this decomposition, the k skeleton of S^n may be identified with S^k for $k = 0, \dots, n$. By the above theory, in that range,

$$C_k(S^n) = H_k(S^k, S^{k-1}) \cong \mathbf{Z}e_k \oplus \mathbf{Z}f_k.$$

There is one slight subtlety here. The copies of $\mathbf{Z}$ are obtained from the characteristic maps

$$(D^k, S^{k-1}) \to (\overline{e}_k, \overline{e}_k - e_k)$$
$$(D^k, S^{k-1}) \to (\overline{f}_k, \overline{f}_k - f_k)$$

so they are the images of

$$H_k(\overline{e}_k, \overline{e}_k - e_k) \to H_k(S^k, S^{k-1})$$
$$H_k(\overline{f}_k, \overline{f}_k - f_k) \to H_k(S^k, S^{k-1})$$

respectively. However, the generators of these summands are only uniquely determined modulo sign.

In any case, this decomposition is natural with respect to cellular maps. Consider in particular the antipode map $a_k : S^k \to S^k$. Clearly, this interchanges the two cells, and we may assume in the representation of $C_k(X)$ that $f_k = a_k(e_k), a_k(f_k) = e_k$.

Consider next the boundary homomorphism $\partial_k : C_k(S^n) \to C_{k-1}(S^n)$ This has to commute with the antipode map by naturality. Start in dimension 0. Choose a map $\pi : S^n \to \{P\}$ to a point. We may choose $e_0 \in H_0(S^0, S^{-1}) = H_0(S^0)$ so that $\pi_*(e_0)$ is a specific generator of $H_0(\{P\}) \cong \mathbf{Z}$, and clearly $\pi_*(f_0) = \pi_*(a_{0*}(e_0)$ is also that generator. Hence, in $H_0(S^n)$, we must have $e_0 \sim f_0$ so $e_0 - f_0$ must be a boundary in $C_0(S^n)$. Let $\partial_1 e_1 = xe_0 - yf_0$. Then $xe_0 \sim yf_0 \sim ye_0$ implies that $(x-y)e_0 \sim 0$. Since $H_0(S^n) = \mathbf{Z}$, it has no elements of finite order, so $x - y = 0$, i.e., $x = y$. Hence, $\partial_1 e_0 = x(e_0 - f_0)$ and applying the antipode map, we see $\partial_1 f_1 = x(f_0 - e_0)$. Hence, $\partial_1(ue_1 + vf_1) = x(u-v)(e_0 - f_0)$. Thus the only way that $e_0 - f_0$ could be a boundary is if $x = \pm 1$. By changing the signs of both e_1 and f_1 if necessary, we may assume $\partial_1 e_1 = e_0 - f_0, \partial_1 f_1 = f_0 - e_0$.

Consider next dimension 1. $e_1 + f_1$ is a cycle. However, $H_1(S^n) = 0$ (at least if $n > 1$.) It follows that $e_1 + f_1$ is a boundary. Let $\partial_2 e_2 = xe_1 + yf_1$. Subtracting off $y(e_1 + f_1)$ shows that $(x-y)e_1$ is a boundary, hence a cycle, but that is false unless $x = y$. Reasoning as above, we can see that $x = \pm 1$ and again we may assume it is 1. Thus, $\partial_2 e_2 = \partial_2 f_2 = e_1 + f_1$.

This argument may be iterated to determine all the ∂_k for $k = 1, 2, \dots, n$. We conclude that

$$\partial_k e_k = e_{k-1} + (-1)^k f_{k-1}$$
$$\partial_k f_k = f_{k-1} + (-1)^k e_{k-1}.$$

Now consider $\mathbf{R}P^n$ with the CW structure discussed previously, one open cell $\bar{e}_k$ in each dimension $k = 0, \dots, n$. Using the cellular map $S^n \to \mathbf{R}P^n$, we may write $C_k(\mathbf{R}P^n) = \mathbf{Z}c_k$ where c_k is the image of e_k (and also of f_k). (See the Exercises.) Then by naturality, we find that for $k = 1, \dots, n$,

$$\begin{aligned} \partial_k c_k &= (1 + (-1)^k) c_{k-1} \\ &= \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2c_{k-1} & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

We have now established the following

THEOREM 8.28.

$$H_k(\mathbf{R}P^n) = \begin{cases} \mathbf{Z} & \text{if } k = 0, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbf{Z} & \text{if } k = n \text{ and } k \text{ is odd,} \\ 0 & \text{otherwise} \end{cases}$$

For example, $H_1(\mathbf{R}P^3) = \mathbf{Z}/2\mathbf{Z}, H_2(\mathbf{R}P^3) = 0, H_3(\mathbf{R}P^3) = \mathbf{Z}$, and $H_1(\mathbf{R}P^4) = \mathbf{Z}/2\mathbf{Z}, H_2(\mathbf{R}P^4) = 0, H_3(\mathbf{R}P^4) = \mathbf{Z}/2\mathbf{Z}, H_4(\mathbf{R}P^4) = 0$.