CHAPTER 11

Manifolds and Poincaré duality

1. Manifolds

The homology $H_*(M)$ of a manifold M often exhibits an interesting symmetry. Here are some examples.

$$M = S^1 \times S^1 \times S^1: \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}, H_2 = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}, H_3 = \mathbf{Z}$$
$$M = S^2 \times S^3: \quad H_0 = \mathbf{Z}, H_1 = 0, H_2 = \mathbf{Z}, H_3 = \mathbf{Z}, H_4 = 0, H_5 = \mathbf{Z}$$
$$M = \mathbf{R}P^3: \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z}/2\mathbf{Z}, H_2 = \mathbf{Z}$$

Note that the symmetry $H_i \cong H_{n-1}$ is complete in the first two cases, and is complete in the third case if we just count ranks. Also, $H_n(M) = \mathbf{Z}$ if n is the dimension of the manifold in these cases. It turns out that all these manifolds are orientable in a sense to be made precise below. (In fact orientability is tantamount to $H_n(M) = \mathbf{Z}$, and as we shall say an orientation may be thought of as a choice of generator for this group.) The example of the Klein bottle illustrates what can happen for a non-orientable manifold.

$$M = K: \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, H_2 = 0.$$

(You should work that out for yourself.) However, the universal coefficient theorem shows that

$$H_0(K; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}, H_1(K; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, H_2(K; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$$

which has the same kind of symmetry. Such a manifold would be called $\mathbf{Z}/2\mathbf{Z}$-orientable. Finally, a space which is not a manifold will not generally exhibit such symmetries.

$$X = S^2 \vee S^3: \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z}, H_2 = \mathbf{Z}, H_3 = \mathbf{Z}.$$

1.1. Orientability of Manifolds. The naive notion of orientability is easy to understand in the context of finite simplicial complexes. (The manifold in this case would be compact.) Roughly, we assume each n-simplex of the manifold is given an order which specifies an orientation for that simplex. Moreover, we assume the orientations of the simplices can be chosen coherently so that induced orientations on common faces cancel. In this case, the sum of all the n-simplices will be a cycle which represents a homology class generating $H_n(M)$. To define orientability in the context of singular homology is more involved. We shall do this relative to a coefficient ring R so we may encompass things like the example of the Klein bottle with $R = \mathbf{Z}/2\mathbf{Z}$. (In that case, if we used a simplicial decomposition, the sum of all the simplices would be a cycle modulo 2 because when the same face showed up twice in the boundary, cancellation would occur because $1 + 1 = 0$ in $\mathbf{Z}/2\mathbf{Z}$.)

In all that follows let M be a *not necessarily connected n-manifold.* If $x \in M$, abbreviate $M - x = M - \{x\}$.

Lemma 11.1. *Let $x \in M$.*

$$H_i(M, M - x; R) = \begin{cases} R & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Choose an open neighborhood U of x which is homeomorphic to an n-ball. $M - U$ is closed and contained in $M - x$ which is open. Hence, excising $M - U$ yields

$$H_i(U, U - x; R) \cong H_(M, M - x).$$

However, since $U - x$ has S^{n-1} as a retract, the long exact sequence for reduced homology shows that

$$H_i(M, M - x) \cong \tilde{H}_{i-1}(S^{n-1}; R) = R \ (i = n) \quad \text{or 0 otherwise.}$$

(Why do we still get a long exact sequence for relative homology with coefficients? Check this for yourself.) □

A local R-orientation at a point x in an n-manifold M is a choice μ_x of generator for $H_n(M, M - x; R)$ (as R-module). For $R = \mathbf{Z}$ there would be two possible choices, for $\mathbf{R} = \mathbf{Z}/2\mathbf{Z}$ only one possible choice, and in general, the number of possible choices would depend on the number of units in the ring R. You should think of μ_x as a generic choice for orientations of spheres in a euclidian neighborhood of x centered at x. Such a sphere may be considered (up to homeomorphism) to be the boundary of a simplex, and specifying an orientation for the simplex (as an ordering of its vertices) is the same thing as specifying a generator for the homology of its boundary.

Introduce the following notation. If $L \subset K \subset M$, let $\rho_{K,L}$ denote the functorial homomoprhism $H_*(M, M-K; R) \to H_*(M, M-L; R)$. If K is fixed, just use ρ_L. Call this homomorphism 'restriction'.

We shall say a that a choice of local R-orientations for each point $x \in M$ is continuous if for each point of M, there is a neighborhood N of x and an element $\mu_N \in H_n(M, M-N; R)$ such that $\rho_y(\mu_N) = \mu_y$ for each $y \in N$. We shall say that M is R-orientable if there is a continuous choice of local R-orientations for each point.

EXAMPLE 11.2. Let $M = S^n$ and choose a generator $\mu \in H_n(S^n; R)$. Let $\mu_x = \rho_x(\mu)$ where

$$\rho_x = \rho_{S^n,x} : H_n(S^n; R) = H_n(S^n, S^n - S^n; R) \to H_n(S^n, S^n - x; R).$$

Take $N = S^n$ for every point. Note that the long exact sequence for the pair $(S^n, \{x\})$ shows that ρ_x is an isomorphism. Hence, μ_x is in fact a generator of $H_n(S^n, S^n - x; R)$ for every $x \in S^n$.

It is not hard to see that if M is R-orientable, then any open subset U of M inherits an R-orientation. Hence, $\mathbf{R}^n$ which is an open subset of S^n is also R-orientable.

Since $H_n(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$, it is not too hard to see that $\mathbf{R}P^n$ is $\mathbf{Z}/2\mathbf{Z}$-orientable.

Our aim is to show—at least in the case of a compact R-orientable manifold—that $H_n(M; R) = R\mu$ for some μ and $\rho_x(\mu) = \mu_x$ for each $x \in M$. If M is not compact, we show instead that this is true for $H_n(M, M-K; R)$ for every compact subset K of M. (If M is compact, this would include $K = M, M - K = \emptyset$.) First we need a technical lemma.

LEMMA 11.3. *Let M be an n-manifold and K a compact subspace.*

- $H_i(M, M-K; R) = 0$ *for* $i > n$.
- $\alpha \in H_n(M, M-K; R)$ *is zero if and only if* $\rho_x(\alpha) = 0$ *for each* $x \in K$.

We shall abbreviate the second statement by saying that the elements of K *detect* the degree n homology of $(M, M-K)$.

PROOF. **Step 1**. Let $M = \mathbf{R}^n$ and let K be a compact convex subspace.

Let $x \in K$. Enclose K in a large closed ball B centered at x. Then there is a retraction of $R^n - x$ onto $S = \partial B$ and its restriction to $R^n - K$ and $R^n - B$ are also retractions. It follows that the restrictions in

$$H_i(R^n, S) \cong H_i(R^n, R^n - B; R) \cong \\ H_i(R^n, R^n - K; R) \cong H_i(R^n, R^n - x; R)$$

are all isomorphisms, which proves the second statement. The isomorphism

$$H_i(R^n, S; R) \cong H_{i-1}(S; R)$$

proves the first statement.

Step 2. Suppose that M is arbitrary, and the lemma has been proved for compact subsets K_1, K_2 and $K_1 \cap K_2$. Let $K = K_1 \cup K_2$. Then $M - K = (M - K_1) \cap (M - K_2)$, $M - K_1 \cap K_2 = (M - K_1) \cup (M - K_2)$, and there is a *relative* Mayer–Vietoris sequence

$$\to H_i(M, M - K; R) \to H_i(M, M - K_1; R) \oplus H_i(M, M - K_2; R) \to \\ H_i(M, M - K_1 \cap K_2; R) \to H_{i-1}(M, M - K; R) \to \dots$$

It is easy to establish the first statement for K from this. Also,

$$H_n(M, M - K; R) \to H_n(M, M - K_1; R) \oplus H_n(M, M - K_2; R)$$

is a monomorphism. Suppose $\rho_{K,x}(\alpha) = 0$ for each $x \in K = K_1 \cup K_2$. It follows that $\rho_{K,K_1}(\alpha) = 0$ and similarly for K_2. Hence, $\alpha = 0$.

Step 3. $M = R^n$ and K is a finite union of convex compact subspaces. Use steps 1 and 2.

Step 4. Let $M = \mathbf{R}^n$ and suppose K is any compact subspace. To prove the result in this case, let $\alpha \in H_i(M, M - K; R)$. Let $a \in S_i(M)$ be a chain with boundary $\partial a \in S_i(M - K)$ which represents α. Choose a covering of K by closed balls $B_i, i = 1, r$ which are disjoint from $|\partial a|$. (This is possible since both sets are compact.)

Then $\partial a \in S_i(M - B)$ where $B = B_1 \cup \dots \cup B_r$, so a represents and element $\alpha' \in H_i(M, M - B; R)$ which restricts to $\alpha \in H_i(M, M - K; R)$. If $i > n$, then by Step 3, $\alpha' = 0$, so $\alpha = 0$. Since α could have been anything, this shows $H_i(M, M - K; R) = 0$ for $i > n$. Suppose $i = n$. Suppose $\rho_{K,x}(\alpha) = 0$ for each $x \in K$. Then $\rho_{B,x}(\alpha') = 0$ for each $x \in K$. If we can show that the same is true for each $x \in B$, then it will follow from Step 3 that $\alpha' = 0$ and so $\alpha = 0$. To see this, first note that we may assume that each B_i intersects K non-trivially or else we could have left it out. If $x \in B_i$ then there is a $y \in K \cap B_i$ and we have isomorphisms

$$H_n(M, M - y; R) \leftarrow H_n(M, M - B_i) \rightarrow H_n(M, M - x; R).$$

It follows that if $\rho_{B,y}(\alpha') = \rho_{B,B_i}(\rho_{B_i,y}(\alpha') = 0$, then $\rho_{B,x}(\alpha') = 0$.

Step 5. Suppose M is arbitrary and K is contained in an open euclidean neighborhood U homeomorphic to $\mathbf{R}^n$. Excising $M - U$ (which is contained in the interior of $M - K$) we get

$$H_i(U, U - K; R) \cong H_i(M, M - K; R).$$

We can now apply Step 4.

Step 6. The general case. We may assume $K = K_1 \cup K_2 \cup \cdots \cup K_r$ where each K_i is as in Step 5. The same would be true of each intersection of K_i with the union of those that preceded it. Hence, we may apply Step 2 and Step 5.

□

THEOREM 11.4. *Let M be an R-orientable n-manifold with local orientations $\mu_x, x \in M$. Let K be any compact subspace. There exists a unique element*

$$\mu_K \in H_n(M, M - K; R) \quad \textit{such that } \rho_x(\mu_K) = \mu_x$$

for each $x \in K$. In particular, if M is compact itself, there is a unique element $\mu_M \in H_n(M; R)$ such that $\rho_x(\mu_M) = \mu_x$ for each $x \in M$.

In case M is compact, μ_M is called the fundamental class of M. As we shall see later, if M is also connected, then μ_M is a generator of $H_n(M; R)$ as an R-module. If M is not connected, then $H_n(M; R)$ will turn out to be a direct sum of copies of R, one for each component of M. μ_M will be a sum of generators of this direct sum, one for each component. This will correspond to choosing a collection of orientations in the usual sense for the components of M.

PROOF. By the lemma above, such an element μ_K is unique. To show existence, argue as follows.

Step 1. According to the definition of continuity of a choice of local orientations, there is a neighborhood N with the right property. If K is contained in such an N, restricting μ_N to an element $\mu_K \in H_n(M, M - K; R)$ will work.

Step 2. Suppose $K = K_1 \cup K_2$, both compact, and the theorem has been verified for K_1 and K_2. From the lemma, the relative Mayer–Vietoris sequence yields the exact sequence

$$0 \to H_n(M, M - K; R) \to H_n(M, M - K_1; R) \oplus H_n(M, M - K_1; R) \\ \to H_n(M, M - K_1 \cap K_2; R) \to 0.$$

As usual, the homomorphism on the right maps (μ_{K_1}, μ_{K_2}) to

$$\rho_{K_1, K_1 \cap K_2}(\mu_{K_1} - \rho K_2, K_1 \cap K_2(\mu_{K_2}) \in H_n(M, M - K_1 \cap K_2; R).$$

Further restricting these elements to $H_n(M, M - x; R)$ for $x \in K_1 \cap K_2$ yields zero, so by the lemma, the above difference is zero. Hence, by the exactness of the sequence, there is a unique $\mu_K \in H_n(M, M - K; R)$ such that $\rho_{K,K_1}(\mu_K) = \mu_{K_1}$ and $\rho_{K,K_2}(\mu_K) = \mu_{K_2}$, and it is easy to check that μ_K has the right property.

Step 3 Let K be an arbitrary compact subspace. By the continuity condition, we may cover K by neighborhoods N with appropriate elements μ_N. By taking smaller neighborhoods if necessary, we may assume that each N is compact. Since the sets $\overset{\circ}{\to} N$ cover K, we may pick out finitely many $N_i, i = 1 \dots, r$ which cover K. Let $K_i = N_i \cap K$, and now apply steps (i) and (ii) and induction. □

2. Poincaré Duality

Assume in what follows that R is a ring with reasonable properties as described above, e.g., $R = \mathbf{Z}$ or R is a field.

The Poincaré Duality Theorem asserts that if M is a compact R-oriented n-manifold, then

$$H^r(M; R) \cong H_{n-r}(M; R).$$

(Then, we may use the universal coefficient theorem to relate the homology in degrees r and $r - 1$ to the homology in degree $n - r$. Think about it!) In order to describe the isomorphism we need another kind of product which relates cohomology and homology.

2.1. Cap Products. First we need to establish some notation. As usual, let R be a commutative ring (usually $R = \mathbf{Z}$ or R is a field), and let $S_*(X; R) = S_*(X) \otimes R$ as usual. This is a chain complex which is free over R, and we have a natural isomorphsim

$$\mathrm{Hom}_R(S_*(X; R), R) \cong \mathrm{Hom}(S_*(X), R) = S^*(X; R)$$

provided by $f \in \mathrm{Hom}_R(S_*(X) \otimes R, R) \mapsto f' \in \mathrm{Hom}(S_*(X), R)$ where $f'(\sigma) = f(\sigma \otimes 1)$. Moreover, $c \otimes f \mapsto f(c)$ defines a homomorphism

(called a pairing)

$$S_q(X;R) \otimes_R S^q(X;R) \to R$$

for each q. Note the order of the factors on the left. At this point the order is arbitrary, but it will be important in what follows. (Compare this with the definition of the morphism $H^n(X;R) \to \mathrm{Hom}(H_n(X),R)$.) Define an extension of this pairing as follows. Let $q \le n$. For $c \in S_n(X;R), f \in H^q(X;R)$ define $c \cap f \in H_{n-q}(X;R)$ by

$$\sigma \cap f = f(\sigma \circ [e_0,\ldots,e_q])\sigma \circ [e_q,\ldots,e_n]$$

for σ a singular n-simplex in X and extending by linearity. (There is a slight abuse of notation here. We are identifying σ with the element $\sigma \otimes 1 \in S_n(X;R) = S_n(X) \otimes R$. These elements form an R-basis for $S_n(X;R)$.) Note that if $q = n$, this is just the evaluation pairing described above. This cap product may also be described abstractly but this requires some fiddling with signs. See the Exercises.

Proposition 11.5. *The cap product satisfies the following rules.*

(1) $c \cap (f \cup g) = (c \cap f) \cap g$. *Also* $c \cap e = c$ *where* $e \in S^0(X;R)$ *is defined by* $e(\sigma) = 1$ *for each singular 0-simplex* σ.
(2) $(-1)^q \partial(c \cap f) = (\partial c) \cap f - c \cap \delta f$.
(3) *Let* $j : X \to X'$ *be a map,* $c \in S_*(X;R), f' \in S^*(X';R)$. *Then* $j_*(c \cap j^*(f')) = j_*(c) \cap f'$.

Notes: The first statement may be interpreted as saying that $S_*(X;R)$ is a right module over the ring $S^*(X;R)$. To make sense of this, we let $c \cap f = 0$ if $q = \deg f > n = \deg c$. Note that e is the identity element of $S^*(X;R)$. The distributive laws for $c \cap f$—required for a module—are automatic since it is a pairing. A consequence of the second statement is that if f is a cocycle and c is a cycle, then $c \cap f$ is also a cycle and its homology class depends only on the classes of c and f respectively. Hence, it defines a pairing between cohomology and homology

$$H_n(X;R) \otimes H^q(X;R) \to H_{n-q}(X;R)$$

denoted $\bar{c} \cap \overline{f} = f(c)$. This pairing satisfies the rules asserted in the first statement.

Proof. The statement about e is immediate from the formula.

To prove the associativity assertion, proceed as follows. Let σ be a singular n-simplex. Let $q_1 = \deg f, q_2 = \deg g$, let $q = q_1 + q + 2$, and let $r = n - q$. By linearity, it suffices to prove the formula if $c = \sigma$ is a singular n-simplex.

$$\sigma \cap (f \cup g)$$
$$= (f \cup g)(\sigma \circ [e_0, \dots, e_q])\sigma \circ [e_q, \dots, e_n]$$
$$= f(\sigma \circ [e_0, \dots, e_q] \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_0, \dots, e_q] \circ [e_{q_1}, \dots, e_q])\sigma \circ [e_q, \dots, e_n]$$
$$= f(\sigma \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_{q_1}, \dots, e_q])\sigma \circ [e_q, \dots, e_n].$$

On the other hand,

$$\sigma \cap f$$
$$= f(\sigma \circ [e_0, \dots, e_{q_1}])\sigma \circ [e_{q_1}, \dots, e_n]$$
$$(\sigma \cap f) \cap g$$
$$= f(\sigma \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_{q_1}, \dots, e_n] \circ [e_0, \dots, e_{q_2}])\sigma \circ [e_{q_1}, \dots, e_n] \circ [e_{q_2}, \dots, e_q]$$
$$= f(\sigma \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_{q_1}, \dots, e_q]])\sigma \circ [e_q, \dots, e_n].$$

To prove the boundary formula, argue as follows. First note that any r-chain $c' \in S_r(X; R)$ is completely determined if we know $h(c')$ for every q cycle $h \in S^r(X; R)$. Now

$$\begin{aligned} h(\partial(c \cap f)) &= (\delta h)(c \cap f) = (c \cap f) \cap \delta h \\ &= c \cap (f \cup \delta h) = c \cap ((-1)^q(\delta(f \cup h) - \delta f \cup h)) \\ &= (-1)^q(\delta(f \cup h)(c) - c \cap (\delta f \cup h)) \\ &= (-1)^q((f \cup h)(\partial c) - (c \cap \delta f) \cap h) \\ &= (-1)^1((\partial c) \cap f - c \cap \delta f) \cap h \\ &= h((-1)^q(\partial c \cap f - c \cap \delta f)) \end{aligned}$$

as required.

Note: If you use a more abstract definition of the cap product, then one introduces a suitable sign in the cap product. With that sign, the proof comes down to the assertion that the defining morphism is just a chain homomorphism.

The last formula is left as an exercise for the student. □

2.2. Poincaré Duality.

THEOREM 11.6. *Let M be a compact oriented R-manifold with fundamental class $\mu_M \in H_n(M; R)$. Then the homomorphism*

$$H^q(M; R) \to H_{n-q}(M; R)$$

defined by $a \mapsto \mu_M \cap a$ is an isomorphism.

Note that if $R = \mathbf{Z}$, then by the universal coefficient theorem for cohomology, we get a (non-natural) isomorphism

$$\mathrm{Hom}(H_q(M), \mathbf{Z}) \oplus \mathrm{Ext}(H_{q-1}(M), \mathbf{Z}) \cong H_{n-q}(M)$$

In particular, if $H_*(M)$ is free, then

$$H_q(M) \cong \mathrm{Hom}(H_q(M), Z) \cong H_{n-1}(M)$$

as we observed previously. If R is a field, then

$$H_q(M; R) \cong \mathrm{Hom}_R(H_q(M; R), R) \cong H_{n-1}(M; R).$$

We will prove the Poincaré Duality Theorem later, but first we give some important applications.

3. Applications of Poincaré Duality

3.1. Cohomology Rings of Projective Spaces.

THEOREM 11.7. (1) $H^*(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}[X]/(X^{n+1})$ *where* $\deg X = 2$.
(2) $H^*(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}[X]/(X^{n+1})$ *where* $\deg X = 1$.
(3) $H^*(\mathbf{H}P^n; \mathbf{Z}) \cong \mathbf{Z}[X]/(X^{n+1})$ *where* $\deg X = 4$.

PROOF. We shall do the case of $\mathbf{C}P^n$.

We know the result for $n = 1$ since $\mathbf{C}P^1 \simeq S^2$. Assume the result is true for $1, \dots, n-1$. In our calculation of $H_*(\mathbf{C}P^n)$, we in effect showed that $H_k(CP^{n-1}) \to H_k(\mathbf{C}P^n)$ is an isomorphism for $0 \leq k \leq 2n-2$. (This is also immediate from the calculation of the homology using the cellular decomposition.) Since $H_k(\mathbf{C}P^m)$ is trivial is odd degrees, the universal coefficient theorem for cohomology shows us that the induced homomorphism

$$H^{2k}(\mathbf{C}P^n; \mathbf{Z}) = \mathrm{Hom}(H_{2k}(\mathbf{C}P^n); \mathbf{Z}) \xrightarrow{\rho^{2k}} H^{2k}(\mathbf{C}P^{n-1}; \mathbf{Z}) = \mathrm{Hom}(H_{2k}(\mathbf{C}P^{n-1}); \mathbf{Z})$$

is an isomorphism. Choose X a generator of $H^2(\mathbf{C}P^n; \mathbf{Z})$. $X' = \rho^2(X)$ generates $H^2(\mathbf{C}P^{n-1}; \mathbf{Z})$, so by induction, $(X')^{n-1}$ generates $H^{2n-2}(\mathbf{C}P^{n-1}; \mathbf{Z})$. However, $X'^{n-1} = \rho(X)^{n-1} = \rho(X^{n-1}$, so X^{n-1} generates $H^{2n-2}(\mathbf{C}P^n; \mathbf{Z})$. Now apply the Poincaré Duality isomorphism

$$a \in H^{n-1}(\mathbf{C}P^n; \mathbf{Z}) \mapsto \mu \cap a \in H_2(\mathbf{C}P^n).$$

It follows that $\mu \cap X^{n-1}$ generates $H_2(X)$. Hence, the evaluation morphism must yield

$$\mu \cap X^{n-1} \mapsto (\mu \cap X^{n-1}) \cap X = \pm 1$$

Hence,

$$\mu \cap X^n = \mu \cap (X^{n-1} \cup X) = \pm 1$$

so X^n generates $H^{2n}(\mathbf{C}P^n; \mathbf{Z})$.

The argument for $\mathbf{H}P^n$ is essentially the same. The argument for $\mathbf{R}P^n$ is basically the same except that homology and cohomology have coefficients in $\mathbf{Z}/2\mathbf{Z}$. You should write out the argument in that case to make sure you understand it. □

You may recall the following result proved in a special case by covering space theory.

Theorem 11.8. *preserving map $f : S^n \to S^m$ with $0 \le m < n$.*

The case we did earlier was $m = 1, n > 1$. The argument was that if we have such a map $f : S^n \to S_1$, it would induce a map $\overline{f} : \mathbf{R}P^n \to \mathbf{R}P^1 = S^1$ which would by the antipode property take a nontrivial loop in $\mathbf{R}P^n$ into a nontrivial loop in $\mathbf{R}P^1$. This would contradict what we know about the fundamental groups of those spaces.

The Borsuk–Ulam Theorem had many interesting consequences which you should now review along with the theorem.

Proof. Assume $1 < m < n$. As in the previous case, we get an induced map and a diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^m \\ \downarrow & & \downarrow \\ \mathbf{R}P^n & \xrightarrow{\overline{f}} & \mathbf{R}P^m \end{array}$$

By covering space theory, the fundamental groups of both projective spaces are $\mathbf{Z}/2\mathbf{Z}$ and by the antipode property, the induced map of fundamental groups is an isomorphism. Since $H_1(X) \cong \pi_1/[\pi_1, \pi_1]$, it follows that the first homology group of both projective spaces is $\mathbf{Z}/2\mathbf{Z}$ and the induced morphism is an isomorphism. Thus, the universal coefficient theorem for cohomology allows us to conclude that

$$\overline{f}^* : H^1(\mathbf{R}P^m; \mathbf{Z}/2\mathbf{Z}) \to H^1(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z})$$

is an isomorphism, so a generator X_m of the former goes to a generator X_n of the latter. Hence,

$$0 = \overline{f}^*({X_m}^n) = (\overline{f}^*(X_m))^n = (X_n)^n$$

which contradicts our calculation of $H^*(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z})$.

□

As mentioned previously, there is a non-associative division algebra of dimension 8 called the Cayley numbers. Also, n-dimensional projective space ofer the Cayley numbers may be defined in the usual way. It is a compact $8n$-dimensional manifold. The above argument shows that its cohomology ring is a truncated polynomial ring on a generator of degree 8.

3.2. Aside on the Hopf Invariant. Earlier in this course, we discussed the homotopy groups $\pi_n(X, x_0)$ of a space X with a base point x_0. We mentioned that these are abelian for $n > 1$, and that $\pi_m = 0$ for $X = X^n$ and $0 < m < n$. In this section we shall show that $\pi_m(S^n) \neq 0$ for $m = 2n - 1$. The cases $n = 2, 4, 8$ use the above calculations of cohomology rings for projective spaces, so that is why we include this material here.

Consider a map $f : S^{2n-1} \to S^n$. Since $S^{2n-1} = \partial D^{2n}$, we may form the adjunction space $D^{2n} \sqcup_f S^n$. We showed earlier that

$$\begin{aligned} H_{2n}(D^{2n} \sqcup_f S^n) &= \mathbf{Z} \\ H_n(D^{2n} \sqcup_f S^n) &= H_n(S^n) = \mathbf{Z} \\ H_0(D^{2n} \sqcup_f S^n) &= \mathbf{Z} \\ H_k(D^{2n} \sqcup_f S^n) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Hence, the universal coefficient theorem for cohomology shows that

$$\begin{aligned} H^{2n}(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= \mathbf{Z} \\ H^n(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= \mathbf{Z} \\ H^0(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= \mathbf{Z} \\ H^k(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Let a_k be a generator of $H^k(D^{2n} \sqcup_f S^n; \mathbf{Z})$ for $k = n, 2n$. Then $a_n \cup a_n = H(f) a_{2n} \in H^{2n}(D^{2n} \sqcup_f S^n; \mathbf{Z})$ where $H(f)$ is an integer called the *Hopf invariant* of the map f. Note that $H(f)$ depends on the generators a_n, a_{2n}, but that choosing different generators will at worst change its sign. Since all the constructions we have made are invariant under homotopies, $H(f)$ is an invariant of the homotopy class of the map $f : S^{2n-1} \to S^n$, i.e., of the element of $\pi_{2n-1}(S^n)$ that it defines.

Consider the special case $n = 2$ and $f : S^3 \to S^2$ is the attaching map for the identification $\mathbf{C}P^2 = D^4 \sqcup_f \mathbf{C}P^1$. ($f$ is the quotient map where $S^2 = \mathbf{C}P^1$ is identified as the orbit space of the action of S^1 on S^3 discussed previously.) Since $X \cup X$ generates $H^4(\mathbf{C}P^2; \mathbf{Z})$ the Hopf invariant of the attaching map is ± 1.

We have not discussed the group operation in π_m for $m > 1$, but it is possible to show that $f \to H(f)$ defines a group homomorphism $\pi_{2n-1}(S^n) \to \mathbf{Z}$. Hence, the above argument shows that this homomorphism is onto. Hence, $\pi_3(S^2)$ has a direct summand isomorphic to $\mathbf{Z}$.

A similar argument works for quaternionic projective space. In that case $\mathbf{H}P^1 = D^4 \sqcup_g \mathbf{H}P^0$ which is a 4-cell adjoined to a point, i.e., $\mathbf{H}P^1 \simeq S^4$. Hence, the attaching map f in $\mathbf{H}P^2 = D^8 \sqcup_f \mathbf{H}P^1$ has hopf invariant ± 1.

In fact the above argument will work for any finite dimensional real division algebra (associative or not). In particular, it also works for the Cayley numbers, so there is a map $f : S^{15} \to S^8$ of Hopf invariant 1. Moreover, one can use the theory of the Hopf invariant to help show that there are no other finite dimensional real division algebras.

For example, it is not hard to see that if n is odd, any map $f : S^{2n-1} \to S^n$ has $H(f) = 0$. This follows immediately from the fact that in the odd case $a_n \cup a_n = -a_n \cup a_n = 0$.

If n is even, there is always a map of Hopf invariant 2. For consider the identification

$$S^n \times S^n = D^{2n} \sqcup_g (S^n \vee S^n).$$

Define the 'folding map' $S^n \vee S^n \to S^n$ by sending each component identically to itself. Let $f : S^{2n-1} \to S^n$ be the composition of $g : S^{2n-1} \to S^n \vee S^n$ with the folding map. Let X be the quotient space of $D^{2n} \sqcup_g S^n \vee S^n$ under the relation which identifies points of $S^n \vee S^n$ which fold to the same point. Then we get a commutative diagram

$$\begin{array}{ccc}
S^{2n-1} & \xrightarrow{\;g\;} & D^{2n} \sqcup_g S^n \vee S^n = S^n \times S^n \\
{\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\
S^{2n-1} & \xrightarrow{\;f\;} & X
\end{array}$$

where ϕ is the quotient map (induced by folding). It is not hard to see that this presents X as $D^{2n} \sqcup_f S^n$, and $\phi^* : H^{2n}(D^{2n} \sqcup S^n; \mathbf{Z}) \to H^{2n}(S^n \times S^n; \mathbf{Z})$ is an isomorphism. Also, if c_n generates $H^n(D^{2n}(\sqcup_f S^n) = H^n(S^n; \mathbf{Z})$, then $\phi^*(c_n) = a_n + b_n$ for appropriate generators of $H^n(S^n \times S^n; \mathbf{Z}) = H^n(S^n \vee S^n; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. Since in $H^*(S^n \times S^n; \mathbf{Z})$, we have ${a_n}^2 = {b_n}^2 = 0$, we have $(a_n + b_n)^2 = 2a_n b_n$. (Since n is even, a_n, b_n commute.) But $a_n b_n$ generates $H^{2n}(S^n \times S^n; \mathbf{Z})$ so it follows that $c_n \cup c_n$ is twice a generator. Hence, its Hopf invariant is ± 2.

Note that the above argument shows that $\pi_{2n-1}(S^n) \to \mathbf{Z}$ at worst goes onto the subgroup of even integers if n is even. Since this is also isomorphic to $\mathbf{Z}$, we see that $\mathbf{Z}$ is a direct summand of that homotopy group for n even. Using cohomology operations called the Steenrod squares, which are a generalization of the cup product, one can show that a nessary condition for the homomorphism $\pi_{2n-1}(S^n) \to \mathbf{Z}$ to be onto, i.e. for there to be an element of Hopf invariant 1, is for n to be a power of 2. In 1960, J. F. Adams settled the question by showing that the only cases in which there exist elements of Hopf invariant 1 are the ones described above, i.e. when $n = 2, 4, or\, 8$, but we leave that fact for you to explore in another course or by independent reading.

4. Cohomology with Compact Supports

In order to prove the Poincaré Duality theorem for compact manifolds, we shall need to use an argument which reduces it to constituents of a covering by open Euclidean neighborhoods. Such sets, unfortunately, are not compact manifolds, so we must extend the theorem beyond the realm of compact manifolds in order to prove it. However, in the non-compact case, we saw that we had to deal with fundamental classes $\mu_K \in H_n(M, M - K; R)$ with K compact rather than one fundamental class $\mu_M \in H_n(M; R)$. The basic result, as we stated it earlier, is not true for an individual K, but as we shall see, it is true 'in the limit'.

4.1. Direct Limits. Let I be a partially ordered set. The example which shall be most important for us is the set of compact subsets of a space X ordered by inclusion. A partially ordered set may be considered a category with the objects the elements of the set and the morphisms the pairs (j, i) such that $j \geq i$ in the ordering. We consider i to be the source of the morphism and j its target. Thus, for each pair j, i of elements in I, either $\mathrm{Hom}(j, i)$ is empty or $\mathrm{Hom}(j, i)$ consists of the single element (j, i). If $k \geq j, j \geq i$, we let $(k, j) \circ (j, i) = (k, i)$. It is easy to check that this composition satisfies the requirements for a category.

Let R be a commutative ring and let F be a functor from I to the category of R-modules. Thus, we are given an R-module $F(i)$ for each $i \in I$ and a module homomorphism $\phi_{i,j} : F(i) \to F(j)$ for each $j \geq i$ such that

$$\phi_{i,i} = \mathrm{Id} : F(i) \to F(i)$$

$$\phi_{k,j} \circ \phi_{j,i} \qquad \text{for } k \geq j \geq i.$$

A direct limit of such a functor consists of an R module M and module homomorphisms $\psi_i : F(i) \to M$ such that for $j \geq i$, $\psi_i = \psi_j \circ \phi_{j,i}$

and M and the collection of ϕ_i is universal for this property, i.e., given another R-module M' and module homomorphisms $\psi'_i : F(i) \to M'$ such that for $j \geq i$, $\psi'_i = \psi'_j \circ \phi_{j,i}$ then there is a unique module homomorphism $\rho : M \to M'$ such that $\psi'_i = \rho \circ \psi_i$ for each i in I.

The defintion of direct limit may be extended to an arbitrary functor from one category to another. We leave it to the student to fill in the details of the definition.

A direct limit of a functor F is unique up to unique isomorphism. This follows immediately from the universal mapping property by some diagram chasing. We leave it for the student to verify as an exercise. We use the notation $\varinjlim F$ to denote 'the' direct limit. A direct limit always exists. To see this consider first the case where the ordering of I is empty, i.e., $i \geq i$ for $i \in I$ are the only orderings. In this case, the direct limit is just the direct sum $\oplus_{i\in I} F(i)$ and $\eta_i : F(i) \to \bigoplus$ be the inclusion of the summand $FIi)$ in the direct sum. This collection of homomorphisms presents $\bigoplus$ as a direct limit. Assume now that I is any partially ordered set. In $\oplus_i F(i)$, consider the subgroup T generated by all elements of the form $\eta_j(\phi_{j,i}(x_i)) - \eta_i(x_i)$ for $x_i \in F(i)$ and $j \geq i$. Let $M = (\oplus_i F(i))/T$ and let $\psi_i : F(i) \to M$ be the homomorphism which composes the factor morphisms with η_i. We leave it to the student to check that M and the morphisms ψ_i yield a direct limit.

Let I be a partially ordered set. We say that I is *directed* if each pair $i, j \in I$ has an upper bound.

We say a sequence $F' \to F \to F''$ of functors on a partially ordered set I is exact if $F'(i) \to F(i) \to F''(i)$ is exact for each $i \in I$.

PROPOSITION 11.9. *Let I be a directed partially ordered set. If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of functors on I,* *then*

$$0 \to \lim_{\to} F' \to \lim_{\to} F \to \lim_{\to} F'' \to 0$$

is exact.

PROOF. We let the student verify that the sequence of limits is right exact. That does not even require that the set I be directed.

To show that it is exact on the left, i.e., that

$$\lim_{\to} F' \to \lim_{\to} F$$

is a monomorphism, we must use the directed property. Let

$$M' = \lim_{\to} F', \qquad M = \lim_{\to} F.$$

First note the following facts which hold in the directed case.

1. Any element $x \in M$ is of the form $\psi_i(x_i)$ for some $x_i \in F(i)$.

For, by the above construction, any element in M is of the form $x = \sum_j \psi_j(x_j)$ where all but a finite number of terms in the sum are zero. By the directed property we can choose an $i \geq j$ for all such j. Since $\psi_i(\phi_{i,j}(x_j)) = \phi_j(x_j)$ in M, it follows that

$$x = \psi_i(\sum_i \phi_{i,j}(x_j))$$

as claimed.

2. If $\psi(x_i) = 0$ for $x_i \in F(i)$, then there is an $l \geq i$ such that $\phi_{l,i}(x_i) = 0 \in F(l)$.

This is left as an exercise for the student. (Hint: By adding on appropriate elements and replacing x_i by $\phi_{i',i}(x_i)$, one can assume in $\bigoplus_j F(j)$ that $x_i = \sum_{i,j}(\phi_{i,j}(x_j) - x_j)$ where $i \geq j$ for each non-zero term in the sum.)

Now let $x' = \psi'_i(x'_i) \in M'$ and suppose $x'_i \mapsto x_i \in F(i)$ with $\psi_i(x_i) = 0 \in M$. Then, by (2), there is a $k \geq i$ such that $\phi_{k,i}(x_i) \to 0 \in F(k)$. Hence, $\phi_{k,i}(x_i) = 0$, so $\psi_i(x_i) = \psi_k(\phi_{i,i}(x_i)) = 0$. □

Let F be a functor from a directed set I to the category of R-modules. A subset I' is said to be *cofinal* Note that a cofinal set is also directed. if for each $i \in I$, there is a $i' \in I'$ with $i' \geq i$. Let F' be the restriction of F to I'. Using the homomorphisms

$$\psi_{i'} : F'(i') = F(i) \to M = \lim_{\to} F \qquad i' \in I'$$

we get $\Psi : M' \lim_{\to} F' \to M$ making the appropriate diagrams commute, i.e., $\Psi \circ \psi'_i = \psi_{i'}$ for each $i' \in I'$.

PROPOSITION 11.10. *With the above notation, if I is directed and I' is cofinal in I, then Ψ is an isomorphism.*

PROOF. Ψ is an epimorphism. For, let $x = \psi_i(x_i) \in M$ with $x_i \in F(i)$. Let $i' \geq i$ with $i' \in I'$. Then,

$$\Psi(\psi'_{i'}(\phi_{i',i}(x_i)) = \psi_{i'}(\phi_{i',i}(x_i)) = x.$$

Ψ is a monomorphism. For, let $x' = \psi'_{i'}(x_{i'}) \in M'$ with $x_{i'} \in F(i')$, and suppose $\Psi(x') = 0$. Calculating as above shows that $\psi_{i'}(x_{i'}) = 0$, but this is the same as saying $\psi_{i'}(x_{i'}) = 0$. $\square$

Let M be an n-manifold, and consider the family $\mathcal{K}$ of compact subsets K of M ordered by inclusion. This is a directed set since if K, L are compact, so is $K \cup L$. If $K \subseteq L$, then $M - K \supseteq M - L$ so in cohomology we get an induced homomorphism

$$\rho^q_{L,K} : H^q(M, M - K; R) \to H^q(M, M - L; R).$$

In this way, $H^q(M, M - K; R)$ is a functor from $\mathcal{K}$ to the category of R-modules. We define the *cohomology with compact supports* by

$$H^q_c(M; R) = \varinjlim H^q(M, M - K; R).$$

This may also be defined in a slightly different way. Let

$$S^q_c(M; R) = \varinjlim S^q(M, M - K; R).$$

Note that if $K \subseteq L$, then we actually have

$$S^q(M, M - K; R) \subseteq S^q(M, M - L; R) \subset S^q(M; R)$$

and $S^q_c(M; R)$ is just the union of all the $S^q(M, M - K; R)$. It consists of all q-cochains f which *vanish outside some compact subset*, where the compact set would in general depend on the cochain. This explains the reason for the name 'cohomology with compact support'.

Define $\delta^q : S^q_c(M; R) \to S^{q+1}(M; R)$ is the obvious way. Then $S^*(M; R)$ is a cochain complex, and we may take its homology. Since direct limits on directed sets preserve exact sequences, it is easy to check that taking cocycles, coboundaries, and cohomology commutes with the direct limit. Hence, $H^q_c(M; R)$ is just the q-dimensional homology of the complex $S^*(M; R)$.

5. Proof of Poincaré Duality

Let M be an R-oriented n-manifold. For each compact subspace K of M, we may define a homomorphism

$$S_n(M, M - K; R) \otimes_R S^q(M, M - K; R) \to S_{n-q}(M; R)$$

by

$$c \otimes f \mapsto c \cap f.$$

This makes sense because if σ is a singular n-simplex $S_n(M-K;R)$, its front q-face $\sigma \circ [e_0, \dots, e_q]$ is also in $S_n(M-K;R)$. Hence, if $f \in S^q(M, M-K;R)$; i.e., f is a q-cochain on M which vanishes on $S_1(M-K)$, then $\sigma \cap f = 0$. Hence, $c \cap f$ depends only on the class of c in $S_n(M, M-K;R) = S_n(M;R)/S_n(M-K;R)$.

Taking homology, we see get a sequence of homomorphisms

$$P^q_K : H^q(M, M-K;R) \to H_{n-1}(M;R)$$

defined by $a \mapsto \mu_K \cap a$, where μ_K is the fundamental class of K. It is not hard to check that if $K \subseteq L$, then

$$P^q_L \circ \rho^1_{L,K} = P^q_K$$

so it follows that there is an induced homomorphism on the direct limit

$$P^q : H^q_c(M;R) \to H_{n-q}(M;R).$$

THEOREM 11.11. *Let M be an R-oriented n-manifold. The homomophisms*

$$P^q : H^q_c(M;R) \to H_{n-q}(M;R).$$

are isomorphisms.

Note that if M is itself compact, then $\{M\}$ is a cofinal subset of $\mathcal{K}$, and of course the direct limit that subset is just $H^q(M, M-M;R) = H^q(M;R)$. Hence, in this case $H^q(M;R) \cong H^q_c(M;R)$ and it is easy to check that P^q is just the homomophisms discussed previously. Hence, we get the statement of Poincaré Duality given previously for M compact.

PROOF. **Step 1**. Assume $M = \mathbf{R}^n$.

Let B be a closed ball in $\mathbf{R}^n$. Then as above since $R^n - B$ has S^{n-1} as a deformation retract,

$$H_q(\mathbf{R}^n, \mathbf{R}^n - B;R) = \begin{cases} R & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, an orientation for $\mathbf{R}^n$ implies a choice of generator $\mu_B \in H_n(\mathbf{R}^n, \mathbf{R}^n - B;R) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - B;R)$ (where $x \in B$). We claim the homomorphism

$$H^q(\mathbf{R}^n, \mathbf{R}^n - B;R) \to H_{n-q}(\mathbf{R}^n)$$

defined by $a \mapsto \mu_B \cap a$ is an isomorphism for $0 \le q \le n$. For, if $q \ne n$, then both sides are zero. For $q = n$, the homomorphism is just the evaluation morphism, and the result follows by the universal coefficient theorem

$$H^n(\mathbf{R}^n, \mathbf{R}^n - b;R) \cong \mathrm{Hom}(H_n(\mathbf{R}^n, \mathbf{R}^n - B), R) \cong \mathrm{Hom}_R(H_n(\mathbf{R}^n, \mathbf{R}^n - B;R), R).$$

since μ_B is a generator of $H_n(\mathbf{R}^n, \mathbf{R}^n - B; R)$ and both sides of the homomorphism are R. (Also, the Tor terms are zero.)

It follows that we have an isomorphism of the direct limit

$$\lim_{\rightarrow} H^q(\mathbf{R}^n, \mathbf{R}^n - B; R) \cong H_{n-q}(\mathbf{R}^n; R)$$

where the limit is taken over all closed balls B of $\mathbf{R}^n$. However, it is easy to see that for $\mathbf{R}^n$, the set of closed balls is cofinal, so we conclude that

$$P : H^q_c(\mathbf{R}^n; R) \to H_{n-1}; R)$$

is an isomorphsim.

Step 2. Suppose $M = U \cup V$ and that the theorem is true for U, V and $U \cap V$. We shall show that it is true for M.

We construct a Mayer–Vietoris sequence for cohomology with compact supports.

Let K, L be compact subsets of U, V respectively. Then there the relative Mayer–Vietoris sequence described previously in Section 1 yields for cohomology

$$\begin{aligned} \to H^q(M, M - K \cap L; R) \to H^q(M, M - K; R) \oplus H^q(M, M - L; R) \\ \to H^q(M, M - K \cup L; R) \to H^{q+1}(M, M - K \cup L; R) \to \dots \end{aligned}$$

Note that this is dual to the relative sequence described previously.

Let $C = M - U \cap V$. Then C is closed and contained in the interior of $M - k \cap L$, so by excision, we have a natural isomorphism

$$H^q(M, M - K \cap L; R) \cong H^q(U \cap V, U \cap V - K \cap L; R).$$

By similar excision arguments we have

$$\begin{aligned} H^q(M, M - K; R) &\cong H^q(U, U - K; R) \\ H^q(M, M - L; R) &\cong H^q(V, V - L; R) \end{aligned}$$

Now consider the following diagram

where the top sequence is obtained by replacing the cohomology groups in the above sequence by their isomorphs under excision, and the bottom sequence is the homology Mayer–Vietoris sequence for the triple $M = U \cup V, U, V$. The vertical arrows are the relative cap product morphisms discussed previously for each of the pairs $(U \cap V, U \cap V - K \cap L)$, $(U, U - K)$, $(V, V - L)$, and $(M, M - K \cup L)$.

Lemma 11.12. *The above diagram commutes.*

We shall not try to prove this Lemma here. Suffice it to say that it involves a lot of diagram chasing and use of properites of the cap product. (See Massey for details.)

Assuming that the diagram commutes, we may take direct limits for the upper row. The set of all $K \cap L$ is cofinal in the set of all compact subsets of $U \cap V$, K and L are arbitrary compact subsets of U and V respectively, and the set of all $K \cup L$ is cofinal in the set of all compact subsets of $M = U \cup V$. If follows that we get a diagram

where the upper line is what we will call the Mayer-Vietoris sequence for cohomology with compact supports. The upper line remains exact since direct limits preserve exactness. We may now derive the desired result by means of the five lemma.

Step 3. Let I be a *linearly ordered set* and let M be the union of an ascending chain of open subsets U_i indexed by $i \in I$, i.e., assume

$$i \leq j \Rightarrow U_i \subseteq U_j.$$

Assume the theorem is true for each U_i. We shall prove the theorem for M.

Note first that $H_r(U_i; R)$ is a functor on the directed set of U_i.

Lemma 11.13. $\varinjlim H_r(U_i; R) = H_r(M; R)$.

Proof. The support of any cycle c in $S_r(M; R)$ is compact and so must be contained in one of the U_i. Using this, it is not hard to check the above lemma. □

Now let K be a compact subset of some U_i. Excision of $M - U_i$ yields an isomophism

$$H^q(M, M - K; R) \cong H^q(U, U - K; R).$$

It follows that the *inverses* of these isomorphisms for K compact in U_i yield an isomorphism of direct limits

$$\varinjlim H^q(U_i, U_i - K; R) \to \varinjlim H^q(M, M - K; R)$$

where both limits are taken over the directed set of compact subset of U_i. The left hand side by definition $H^q_c(U_i; R)$. On the right, since we only have a subfamily (not even cofinal) of the set of compact subsets of M, the best we have is a homomorphism

$$\varinjlim H^q(M, M - K; R) \to H^q_c(M; R).$$

Putting this together, we get the upper row of the following diagram

$$\begin{array}{ccc} H^q_c(U_i; R) & \longrightarrow & H^q_c(M; R) \\ \downarrow & & \downarrow \\ H_{n-1}(U_i; R) & \longrightarrow & H^q_c(M; R) \end{array}$$

where the vertical arrows are the Poincaré Duality homomorphisms. Moreover, it is possible to see with some effort that this diagram commutes.

Now take the limit with respect to the U_i. On the bottom, we get an isomorphism, as mentioned in the above Lemma. The vertical arrow on the left is an isomorphism because by assumption it is a direct limit of isomorphisms. (Think about that!). Hence, it suffices to see that the arrow on top is an isomorphism. To see this use the general principle that *direct limits commute*. (We leave it to the student to state that precisely and to prove it.) On the left, we are first fixing i, then taking the limit with respect to $K \subseteq U_i$ and then taking the limit with respect to i. We could just as well fix a K and consider only those $U_i \supseteq K$. (The set of such is certainly cofinal in the set of all U_i.) Since, as noted above, all the $H^q(U_i, U_i - K; R) \to H^q(M, M - K; R)$ are isomorphisms, it follows that in the limit over all such i, we get an isomorphism. Now take the limit with respect to the set of all compact K in M. The limit is again an isomorphism.

Step 4. We establish the theorem for any open set M in $\mathbf{R}^n$.

First note that if M is convex, then it is homeomorphic to $\mathbf{R}^n$, so the theorem is true by Step 1. More generally, we can cover M by a family of open convex subsets $V_1, V_2, \ldots$. Define

$$\begin{aligned} U_1 &= V_1 \\ U_2 &= V_1 \cup V_2 \\ &\vdots \\ U_i &= U_{n-1} \cup V_n \\ &\vdots \end{aligned}$$

The theorem is true for each U_i by Step 2 and induction. Hence, the theorem is true for M by step 3.

Step 5. M may be covered by a countable collection of Euclidean neighborhoods.

Use the same argument as in Step 4. Note that this takes care of all the usual interesting manifolds.

Step 6. The general case.

We need a slight diversion here about transfinite induction.

Zorn's Lemma.

Let I be a *non* partially ordered set. We say that I is inductively ordered if any linearly ordered subset has an upper bound. Then *Zorn's Lemma* asserts that I has a maximal element, i.e., an element m such that there is no element $i \in I$ with $i > m$. Note that this doesn't say that every element of I is bounded by m. Such an element would be unique and be called the *greatest* element of I. There may be many maximal elements.

Zorn's Lemma is not really a lemma. It can be shown to be logically equivalent to several other assertions in set theory. The most notable of these are the *axiom of choice* and *well ordering principle*. You can learn about this material by studying some logic. Some mathematicians consider the use of these assertions questionable. Indeed, Paul Cohen showed that the axiom of choice in a suitable sense is independent of the other axioms of set theory.

However, we shall use Zorn's Lemma, blinking only slightly in the process.

To complete Step 6, Let $\mathcal{U}$ be the collection of all connectged open sets of M for which the theorem is true, ordered by inclusion. Since the theorem is true for every Euclidean neighborhood of M, the set $\mathcal{U}$ is non-empty. It is also inductively ordered by Step 3. Hence, by Zorn's Lemma, there is a maximal connect open set U of M for which the theorem is true. We claim that $U = M$. Otherwise, let $x \in \partial(M - U)$

and choose an open Euclidean neighborhood V of x. By Step 2, the theorem is true for $U \cup V$, and this contradicts the maximality of U. $\square$