

CHAPTER 5

Manifolds and Surfaces

1. Manifolds and Surfaces

Recall that an n -manifold is a Hausdorff space in which every point has a neighborhood homeomorphic to an open ball in \mathbf{R}^n .

Here are some examples of manifolds. \mathbf{R}^n is certainly an n -manifold. Also, S^n is an n -manifold. D^n is not a manifold, but there is a more general concept of ‘manifold with boundary’ of which D^n is an example. The figure eight space is not a manifold because the point common to the two loops has no neighborhood homeomorphic to an open interval in \mathbf{R} .

Since the product of an n -ball and an m -ball is homeomorphic to an $n + m$ -ball, it follows that the product of an n -manifold with an m -manifold is an $n + m$ -manifold. It follows that any n -torus is an n -manifold.

If $p : \tilde{X} \rightarrow X$ is a covering, then it is not hard to see that X is an n -manifold if and only if \tilde{X} is an n -manifold. It follows that $\mathbf{R}P^n$ is an n -manifold. Note that a manifold is certainly locally path connected. Hence, if \tilde{X} is a connected manifold on which a finite group G of homeomorphisms acts, then $\tilde{X} \rightarrow X = \tilde{X}/G$ is a covering and X is also a manifold.

In general, a quotient space of a manifold need not be a manifold. For example, a figure eight space may be viewed as a quotient space of S^1 . However, many quotient spaces are in fact manifolds. For example, the Klein bottle K is a 2-manifold, as the diagram below indicates.

In fact, if we take any polygon in the plane with an even number of sides and identify sides in pairs we obtain a 2-manifold.

Sometimes one requires that a manifold also be second countable, i.e., that there be a countable set \mathcal{S} of open subsets such that any open set is a union of open sets in \mathcal{S} .

It is the purpose of this chapter to classify all compact, connected 2-manifolds. Such manifolds are called *compact surfaces*. (These are also sometimes called closed surfaces.)

One tool we shall use is the concept of *connected sum*. Let S_1, S_2 be surfaces. Choose subspaces $D_1 \subset S_1, D_2 \subset S_2$ which are homeomorphic to disks. In particular, D_1 and D_2 have boundaries $\partial D_1, \partial D_2$ which are homeomorphic to S^1 and to each other. Suppose we choose a homeomorphism $h : \partial D_1 \rightarrow \partial D_2$. We can use this to define an equivalence relation ' \sim ' on the disjoint union of $S_1 - D_1^\circ$ and $S_2 - D_2^\circ$. All equivalence classes are singleton sets except for pairs of the form $\{x, h(x)\}$ with $x \in \partial D_1$. We denote the quotient space by $S_1 \# S_2$ and call it the connected sum. Notice that it is connected, and it is not too hard to see that it is a compact 2-manifold.

It is in fact possible to show that different choices of D_1, D_2 and h yield homeomorphic spaces, so $S_1 \# S_2$ in fact depends only on S_1 and S_2 up to homeomorphism.

EXAMPLE 5.1. The connected sum of two projective planes $\mathbf{R}P^2 \# \mathbf{R}P^2$ is homeomorphic to a Klein bottle K .

Note that we have chosen in a specially convenient way the disks D_1 and D_2 , but that is allowable since the location of the disks is immaterial.

The connected sum has certain reasonable properties. First, it is associative up to homeomorphism, i.e.,

$$S_1 \# (S_2 \# S_3) \cong (S_1 \# S_2) \# S_3.$$

Moreover, if S is any surface, then

$$S \sharp S^2 \cong S.$$

This says that we may make the set of all homeomorphism classes of surfaces into what is called a monoid. Connected sums provide an associative binary operation with an identity (S^2). Note that the operation is also commutative.

The basic result about classification of surfaces is the following.

THEOREM 5.2. *Let S be a compact surface. Then S is homeomorphic to one of the following:*

- (1) S^2 ,
- (2) *the connected sum of n torii ($n \geq 1$).*
- (3) *the connected sum of n real projective planes ($n \geq 1$).*

We shall outline a proof of this theorem in the next section.

In the rest of this section, we shall outline a proof that this classifies compact surfaces up to homeomorphism. In fact, we shall show that no surface in one class can have the same fundamental group as one in another group and that within the same group the fundamental group depends on n . This shows in fact that these surfaces are not even of the same homotopy type.

We start with a description of a connected sum of torii as the quotient of disk (or polygon) with portions of its boundary identified. For two torii, the diagram below indicates how to form the connected sum.

Iterate this for a connected sum of three torii.

The general result is a disk with the boundary divided into $4n$ segments

$$a_1 b_1 \bar{a}_1 \bar{b}_1 \dots a_n b_n \bar{a}_n \bar{b}_n$$

identified in pairs (or equivalently a $4n$ -gon). The number n is called the *genus* of the surface. The genus may be visualized as follows. A single torus may be thought of as a sphere with a handle attached.

Each time a torus is added, you can think of it as adding another handle.

Hence, the connected sum of n torii can be thought of as a sphere n handles.

A similar analysis works for the connected sum of projective planes. For two, we get a Klein bottle as before.

For three,

Finally, in general, we get a disk or $2n$ -gon with edges identified in pairs

$$a_1 a_1 a_2 a_2 \dots a_n a_n.$$

Consider the possible fundamental groups. For S^2 ,

the fundamental group is trivial. For a connected sum of n torii $T\sharp T\sharp \dots \sharp T$,

we may apply the Seifert-VanKampen Theorem as in the previous chapter. The fundamental group Π is the free group on $2n$ generators $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ modulo the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \dots \alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1} = 1.$$

Note that since the relation is a product of commutators, $\Pi/[\Pi, \Pi]$ is free abelian on $2n$ generators.

To complete the argument, we want to calculate $\Pi/[\Pi, \Pi]$ for the fundamental group of a connected sum of projective planes. Doing this from the diagram is not quite as helpful. We find that the fundamental group Π is the free group on n generators modulo the relation

$$\alpha_1\alpha_1 \dots \alpha_n\alpha_n = 1.$$

It follows that $\Pi/[\Pi, \Pi]$ written additively is free on n generators x_1, x_2, \dots, x_n modulo the relation

$$2x_1 + 2x_2 + \dots + 2x_n = 0.$$

By standard techniques in the theory of abelian groups, it is not hard to show that $\Pi/[\Pi, \Pi] \cong \mathbf{Z}^{n-1} \oplus \mathbf{Z}/2\mathbf{Z}$.

However, we shall use another more geometric approach which exhibits the connected sum of n projective planes differently. Not only does this allow us to calculate $\Pi/[\Pi, \Pi]$ without knowing much about abelian groups, but it also allows us to determine easily the connected sum of any two surfaces.

PROPOSITION 5.3. $\mathbf{R}P^2\sharp\mathbf{R}P^2\sharp\mathbf{R}P^2 \cong T\sharp\mathbf{R}P^2$.

PROOF. Since $\mathbf{R}P^2\sharp\mathbf{R}P^2 \cong K$, it suffices to show that $K\sharp\mathbf{R}P^2 \cong T\sharp\mathbf{R}P^2$. To see this first note that $\mathbf{R}P^2$ contains a subspace isomorphic to a Moebius band.

We shall show that the connected sums of on one hand a Moebius band with a torus and on the other hand a Moebius band with a Klein bottle are homeomorphic. First, the diagram below shows several equivalent ways to describe the first surface.

Next, the following diagram exhibits several ways to visualize a Klein bottle with a disk cut out.

Finally, paste that into a Moebius band with a disk cut out and transform as follows.

□

COROLLARY 5.4. *The connected sum of n projective planes is homeomorphic with the connected sum of a torus with a projective plane if n is odd or with a Klein bottle if n is even.*

PROOF. Apply the above proposition iteratively until you get either a single projective plane (n odd) or two projective planes, i.e., a Klein bottle, (n even). □

We may now calculate the fundamental group.

For $n = 2m + 1$ odd, we see that Π is free on n generators modulo the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\dots\alpha_m\beta_m\alpha_m^{-1}\beta_m^{-1}\gamma\gamma = 1.$$

($2m$ generators from the torii and one from the projective plane.) It follows that $\Pi/[\Pi, \Pi] \cong \mathbf{Z}^{2m} \times \mathbf{Z}/2\mathbf{Z}$.

For $n = 2m$ even, we see that Π is free on n generators modulo the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\dots\alpha_{m-1}\beta_{m-1}\alpha_{m-1}^{-1}\beta_{m-1}^{-1}\gamma\delta\gamma^{-1}\delta = 1.$$

(The last part of the relation comes from the Klein bottle.) Modulo $[\Pi, \Pi]$, this relation becomes

$$\gamma\delta\gamma^{-1}\delta = \gamma\delta\gamma^{-1}\delta^{-1}\delta^2 = \delta^2 = 1.$$

Hence, $\Pi/[\Pi, \Pi] \cong \mathbf{Z}^{2m-2} \times \mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \cong \mathbf{Z}^{2m-1} \times \mathbf{Z}/2\mathbf{Z}$.

We may summarize all this by

THEOREM 5.5. *Let S be a compact surface. If S is homeomorphic to a connected sum of n 2-torii, then*

$$\pi(S)/[\pi(S), \pi(S)] \cong \mathbf{Z}^{2n}.$$

If S is homemorphic to n projective planes, then

$$\pi(S)/[\pi(S), \pi(S)] \cong \mathbf{Z}^{n-1} \times \mathbf{Z}/2\mathbf{Z}.$$

In particular, these surfaces are all distinguished by their fundamental groups.

Note that the surfaces involving a projective plane (explicitly or implicitly as in the case of a Klein bottle) have a factor of $\mathbf{Z}/2\mathbf{Z}$ in the abelianized fundamental group. We call the surfaces without projective plane components *orientable* and those with such components *non-orientable*. We shall explore the issue of orientability later in great detail. For the moment, notice that the orientable surfaces can be viewed as spheres with handles imbedded in \mathbf{R}^3 , and we can assume it is a smooth surface with well defined normals at every point. For such a surface we can distinguish the two ‘sides’ of the surface, and this is one of the meanings of orientation.

For surfaces having a component which is a projective plane, we can pick out a subspace homeomorphic to a Moebius band. It is hoped that the student is familiar with the problem of assigning sides to a Moebius band and the possibility of reversing orientation by going around the band.

2. Outline of the Proof of the Classification Theorem

We shall give a rather sketchy outline of the argument. See Massey for more (but not all) details.

First, we note that every compact surface S can be *triangulated*. This is actually a rather deep theorem not at all easy to prove. It means the following. We can decompose S as a union $\cup_{i=1}^n T_i$ of finitely many curvilinear ‘triangles’ satisfying certain conditions.

That is, each T_i is homeomorphic to an actual closed triangle in \mathbf{R}^2 . Through the homeomorphisms, it makes sense to discuss the ‘vertices’ and the ‘edges’ of the T_i . Then, any two curvilinear triangles T_i and T_j are either disjoint, have a single edge in common, or have a single vertex in common. In addition, each edge is common to exactly two curvilinear triangles, and at each vertex, the curvilinear triangles with that vertex can be arranged in a cycle such that each curvilinear triangle has an edge at that vertex in common with the next curvilinear triangle (including cyclically the last and the first).

Finally, we can arrange the homeomorphisms from the triangles in \mathbf{R}^2 to the curvilinear triangles in the surface so that on edges they result in linear maps from one triangle in \mathbf{R}^2 to another. Such a map either reverses or preserves orientation of the edge.

Given such a triangulation, we may find a model for S as a quotient space of a disk (or polygon) with arcs of its boundary (edges) identified in pairs. To do this, pick one curvilinear triangle, call it T_1 , and map it to a triangle T'_1 in \mathbf{R}^2 as above. Now pick a curvilinear triangle, call it T_2 , which shares an edge e_1 with T_1 , and map it to T'_2 in \mathbf{R}^2 . Note that we can assume that T'_i and T'_2 have a common edge corresponding to e_1 . (This may require a linear transformation of T'_2 which includes a 'flip' in order to get the orientation of the coincident edge right.) Continue this process until we have mapped all the curvilinear triangles onto triangles in a connected polygonal figure P in \mathbf{R}^2 .

Each external edge e' of the figure P corresponds to an edge e in the surface which is the common of edge of exactly two curvilinear triangles. The other triangle must appear somewhere else in P , and one of its external edges e'' also corresponds to e . Hence, there is a linear map identifying e' with e'' through e . It is clear that the resulting polygonal figure P with pairs of external edges identified is homeomorphic to S .

One problem remains. We must show that P is simply connected, i.e., that it is homeomorphic to a disk. This follows by the following argument. We start with a simply connected polygon T'_1 . At each stage we 'glue' a triangle to it on one edge. This process clearly yields a simply connected polygon. (If you have any doubts, you could always use the Seifert-VanKampen Theorem!) Hence, the final result is simply connected. By a suitable mapping, we may assume P is the unit disk with its boundary divided into $2n$ equal arcs equivalent in pairs. Alternately, we may take it to be a regular $2n$ -gon with edges identified in pairs.

To complete the proof of the classification theorem, we show how to apply transformations by ‘cutting and pasting’ to get it in one of the forms discussed in the previous section.

(0) If at any point in the discussion, we are reduced to precisely two ‘edges’, then S is either a S^2 or $\mathbf{R}P^2$.

Assume now that the number of edges is at least four.

(1) If there are adjacent edges $a\bar{a}$, then they may be collapsed.

We assume this reduction is automatically made wherever possible.

(2) We may assume there is only one vertex under equivalence. For, if not all vertices are equivalent, we can always find an inequivalent pair P, Q at opposite ends of some edge a . The diagram below indicates a cutting and pasting operation which eliminates one occurrence of P replacing it by an occurrence of Q .

(3) If an edge a occurs in two places with the same orientation, we may assume these two occurrences are adjacent.

(4) If after applying these transformations, we only have adjacent pairs with the same orientation, then the result is homeomorphic to a connected sum of projective planes. Suppose instead that there is a pair $c \dots \bar{c}$ separated by other edges.

In fact, c and \bar{c} must be split by another such pair d, \bar{d} as indicated above ($\dots c \dots d \dots \bar{c} \dots \bar{d}$). For otherwise, the vertices on one side of the pair c, \bar{c} could not be equivalent to those on the other side. Note also that the intervening pair d, \bar{d} must also have opposite orientations since we have assumed that all pairs with the same orientation are adjacent. The diagram below indicates how to cut and paste so as to replace the c, d pairs with one of the form $a\bar{a}b\bar{b}$.

To complete the proof, just remember that the connected sum of a 2-torus and a projective plane is homeomorphic to the connected sum of three projective planes. Thus, if there is only one adjacent pair, we may iteratively transform torii into pairs of projective planes.

3. Some Remarks about Higher Dimensional Manifolds

We have shown above that the fundamental group of a compact 2-manifold completely determines the homeomorphism class of the manifold. Also, two surfaces are homeomorphic if and only if they have the same homotopy type.

In general, arbitrary spaces can have the same homotopy type without being homeomorphic, e.g., all contractible spaces have the same homotopy type. Another example would be an open cylinder and an open Moebius band (omitting the boundary in each case). S^1 is a deformation retract of either, but they are not homeomorphic. (Why?)

Thus, non-compact surfaces can have the same homotopy type without being homeomorphic.

What about higher dimensional manifolds. It turns out that there are compact three manifolds with the same homotopy type but which are not homeomorphic.

EXAMPLE 5.6 (Lens Spaces). Let p and q be relatively prime positive integers. Let G_p be the subgroup of \mathbf{C}^* generated by $\zeta = \zeta_p = e^{2\pi i/p}$. G_p is cyclic group of order p . Define an action of G_p on S^3 as follows. Realize S^3 as the subspace $\{(z, w) \in \mathbf{C}^2 \mid |z|^2 + |w|^2 = 1\}$ of \mathbf{C}^2 (i.e., \mathbf{R}^4). Let $\zeta^n(z, w) = (\zeta^n z, \zeta^n w)$. The orbit space $L(p, q) = S^3/G_p$ is a compact three manifold with fundamental group $\mathbf{Z}/p\mathbf{Z}$. It is possible to find pairs (p, q) and (p, q') such that $L(p, q)$ and $L(p, q')$ are homotopy equivalent but not homeomorphic. This is quite difficult and we shall not discuss it further.

The classification of compact n -manifolds for $n \geq 3$ is extremely difficult. A first step would be the *Poincaré Conjecture* which asserts that any compact simply connected 3-manifold is homeomorphic to S^3 . (It is possible to show that any such 3-manifold has the same homotopy type as S^3 .) This is one of the famous as yet unsolved conjectures of mathematics. However, for $n > 3$, the corresponding question, while difficult, has been answered. In 1960, Smale showed that for $n > 4$, any compact n -manifold with the same homotopy type as S^n is homeomorphic to S^n . In 1984, Michael Freedman proved the same result for $n = 4$. (For $n > 3$, being simply connected alone does not imply that a compact n -manifold is homotopy equivalent to S^n . For example, $S^2 \times S^2$ is simply connected. We shall see later in this course that it is not homotopy equivalent to S^4 .)

4. An Introduction to Knot Theory

A knot is defined to be a subspace of \mathbf{R}^3 which is homeomorphic to S^1 .

It is clear intuitively that some knots are basically the same as others, but it is not obvious how to make this precise. (Try making trefoil knot out of string and then manipulate it into a trefoil knot with the opposite orientation!) One idea that comes to mind is to say that

two knots are equivalent if one can be continuously deformed into the other. However, since a knot has no thickness, you can ‘deform’ any knot into a simple unknotted circle by ‘pulling it tight’ as indicated below.

A sensible definition of equivalence requires some concepts not available to us now, so we shall not go into the matter further. However, some thought suggests that the properties of the *complement* of the knot in \mathbf{R}^3 should play a significant role, since it is how the knot is imbedded in \mathbf{R}^3 that is crucial. With this in mind, we shall call the fundamental group $\pi(\mathbf{R}^3 - K)$ the knot group of the knot K . It is plausible that any transformation which converts a knot K_1 to an ‘equivalent’ knot K_2 —however that should be defined—will not change the knot group up to isomorphism. (Note that it is not obvious that $\mathbf{R}^3 - K$ is path connected. For example, if K were just an image of S^1 rather than a homeomorph of S^1 .)

Any knot K is a compact subset of \mathbf{R}^3 so, by an exercise, we know that the inclusion $\mathbf{R}^3 - K \rightarrow S^3 - K$ induces an isomorphism of fundamental groups $\pi(\mathbf{R}^3 - K) \cong \pi(S^3 - K)$. It is often more convenient to use the latter group.

The simplest knot—really an ‘unknot’—is a circle which we may think of as the unit circle in the x_1, x_2 -plane. We shall show that its knot group is \mathbf{Z} . To this end, consider K to be imbedded in a closed solid torus A of ‘large’ radius $\sqrt{2}$ and ‘small’ radius 1, centered on the x_3 -axis. (The reason for this particular choice of A will eventually be clear.)

The boundary of A is a 2-torus T . Let B be the closure in S^3 of the complement of A .

LEMMA 5.7. *B is homeomorphic to a solid torus with T corresponding to its boundary.*

We shall defer the proof of this lemma, but the diagram below hints at why it might be true.

It is clear that B is a deformation retract of $S^3 - K$. (Just project the inside of $A - K$ onto the boundary T of A .) However, since B is a solid torus, its ‘center circle’ is a deformation retract. It follows that $S^3 - K$ has the same homotopy type as a circle, so its fundamental group is \mathbf{Z} .

More interesting are the so-called torus knots. Let $n > m > 1$ be a pair of relatively prime integers. Let $p : \mathbf{R}^2 \rightarrow T$ be the the universal covering the torus T . In \mathbf{R}^2 , consider the the linear path $h : I \rightarrow \mathbf{R}^2$ starting at $(0, 0)$ and ending at (n, m) . (Note that since $(m, n) = 1$, it doesn’t pass through a lattice point in between the two ends.) Then $p \circ h$ is a loop in T , and viewing T as imbedded in \mathbf{R}^3 in the usual way, the image K of $p \circ h$ is a knot in \mathbf{R}^3 . K is called a *torus knot* of type (m, n) .

Using the $T \simeq S^1 \times S^1$, we may project on either factor.

It is clear from the diagram above, that the projection of $p \circ h$ on the first factor has degree n and on the second factor has degree m . This is a precise way to say that the knot K goes around the torus n times in one direction and m times in the other.

We propose to determine the fundamental group $\pi(S^3 - K)$ for a torus knot and to show that we may determine the integers m, n from it. We shall see that we can recover the type $n > m > 1$ from the knot group. This gives us a way to distinguish one torus knot from another. However, unlike the case of compact surfaces, the fundamental group made abelian $\Pi/[\Pi, \Pi]$ will not suffice. (In fact, it is generally true for a knot group that $\Pi/[\Pi, \Pi] \cong \mathbf{Z}$, so the group made abelian is of no use whatsoever in distinguishing one knots from another. We shall not try to prove this general fact now, although you might try to do it for torus knots.)

Our method will be to use the Seifert-VanKampen Theorem. The analysis above suggests that the decomposition

$$S^3 - K = A - K \cup B - K$$

might work. Unfortunately, the sets on the right are not open. To get around this difficulty, we need to ‘fatten’ them slightly, but this must be done very carefully to be of any use.

First form a very thin open tube N of radius ϵ about K .

$S^3 - N$ is a deformation retract of $S^3 - K$, so we may work with it instead. Next, let U be the open solid torus obtained by letting A ‘grow’ by a small amount, say $\epsilon/2$ but not including the bounding torus. Note that $U - N \cap U$ is homeomorphic to an open solid torus. It looks like U with a groove cut in its surface following the knot K . $U - U \cap N$ has as deformation retract its center circle, so $\pi(U - U \cap N) \cong \mathbf{Z}$ with that central circle as generator. The retraction may be accomplished by first including $U - U \cap N$ in U and then retracting to the central circle. We may also imbed T in U and retract to the center circle and in so doing, one of the two generators of $\pi(T)$ retracts to what we may identify as a generator α of $\pi(U - U \cap N)$. Similarly, let V be the open solid torus obtained by letting B encroach inward on A by $\epsilon/2$. As above $V - V \cap N$ is also homeomorphic to a solid torus, and $\pi(V - V \cap N) \cong \mathbf{Z}$. We may identify a generator β of $\pi(V - V \cap N)$ as the retraction of the *other* generator of $\pi(T)$. Now consider the decomposition

$$S^3 - N = (U - U \cap N) \cup (V - V \cap N).$$

The intersection $U \cap V - U \cap V \cap N$ may be viewed as a band going around T .

Its center is a simple loop which is a deformation retract of $U \cap V - U \cap V \cap N$, so

$$\pi(U \cap V - U \cap V \cap N) \cong \mathbf{Z}.$$

Let γ be the generator just described. It is represented by a loop in T with image the knot K shifted over slightly. The commutative diagram

allows us to identify the image of γ in $\pi(U - U \cap N)$. In fact, by the degree argument mentioned above, since it is basically the same as the knot K , it maps either to m times a generator or n times a generator depending how T has been imbedded in \mathbf{R}^3 . Suppose we choose the imbedding so $\gamma \mapsto \alpha^m$. A similar argument shows that under $\pi(U \cap V \cap -U \cap V \cap N) \rightarrow \pi(U - U \cap N)$, $\gamma \mapsto \beta^n$. It follows from the Seifert–VanKampen Theorem that $\pi(S^3 - N)$ is the free product of an infinite cyclic group generated by α with an infinite cyclic group generated by β modulo the normal subgroup generated by all elements of the form $\alpha^{mi}\beta^{-ni}$. It is not too hard to see from this that $\Pi = \pi(S^3 - K) \cong \pi(S^3 - N)$ is isomorphic to the free group on two generators α, β modulo the relation $\alpha^m = \beta^n$.

We shall now show how we can recover $n > m > 1$ from this group. First consider the subgroup C generated by $\alpha^m = \beta^n$. It is clearly a central subgroup since every element in it commutes with both α and β . The quotient group $\bar{\Pi}$ is the free product of a cyclic group of order m generated by $\bar{\alpha}$ and a cyclic group of order n generated by $\bar{\beta}$. (Just check that Π/C has the appropriate universal mapping property.) However, it is generally true that any free product has trivial center. (Exercise.) From this it follows that C is the center of Π . We can recover m, n from Π as follows. First, since $\bar{\Pi} = \Pi/C \cong \mathbf{Z}/m\mathbf{Z} * \mathbf{Z}/n\mathbf{Z}$, it follows that the group made abelian $\bar{\Pi}/[\bar{\Pi}, \bar{\Pi}] \cong \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$. (Again, just use a universal mapping property argument.) Hence, its order is mn . On the other hand, it is true that any element of finite order in $\mathbf{Z}/m\mathbf{Z} * \mathbf{Z}/n\mathbf{Z}$ must be conjugate to an element in $\mathbf{Z}/m\mathbf{Z}$ or to an element in $\mathbf{Z}/n\mathbf{Z}$. (Exercise.) Hence, the maximal order of any element of finite order in $\bar{\Pi}$ is n (since $n > m > 1$). Thus $\bar{\Pi} \cong \Pi/C$ —which is Π modulo its center and so does not depend on any particular presentation of Π —contains enough information for us to recover n and mn , and hence m .

It follows that if K_1 and K_2 are torus knots of types $n_1 > m_1 > 1$ and $n_2 > m_2 > 1$ respectively, then they have the same knot groups if and only if $n_1 = n_2$ and $m_1 = m_2$.

To complete the discussion, we prove the Lemma which asserts that $B = S^2 - A^\circ$ is homeomorphic to a solid torus. To this end we describe stereographic projection from $(0, 0, 0, 1)$ on S^3 to \mathbf{R}^3 imbedded as the hyperplane $x_4 = 0$. Let (x_1, x_2, x_3, x_4) be a point on S^3 ($x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$) and let $(y_1, y_2, y_3, 0)$ be the corresponding point in \mathbf{R}^3 . Then

$$\begin{aligned}x_1 &= 0 + t(y_1 - 0) = ty_1 \\x_2 &= 0 + t(y_2 - 0) = ty_2 \\x_3 &= 0 + t(y_3 - 0) = ty_3 \\x_4 &= 1 + t(0 - 1) = 1 - t.\end{aligned}$$

Put this in the equation of S^3 to obtain

$$1 = t^2(y_1^2 + y_2^2 + y_3^2) + t^2 - 2t + 1.$$

Put $\rho = \sqrt{y_1^2 + y_2^2 + y_3^2}$. Then we get

$$t = \frac{2}{\rho^2 + 1}.$$

Thus

$$\begin{aligned}x_1 &= \frac{2y_1}{\rho^2 + 1} \\x_2 &= \frac{2y_2}{\rho^2 + 1} \\x_3 &= \frac{2y_3}{\rho^2 + 1} \\x_4 &= \frac{\rho^2 - 1}{\rho^2 + 1}.\end{aligned}$$

Now consider the surface obtained by intersecting $x_1^2 + x_2^2 = u^2$ with S^3 . This may also be described by $x_3^2 + x_4^2 = v^2$ where $u^2 + v^2 = 1$. We may assume here that $0 \leq u, v \leq 1$. This surface maps under stereographic projection to

$$u^2(\rho^2 + 1)^2 = 4(y_1^2 + y_2^2).$$

Put $r = \sqrt{y_1^2 + y_2^2}$. Doing some algebra yields the equation

$$\left(r - \frac{1}{u}\right)^2 + y_3^2 = \left(\frac{v}{u}\right)^2.$$

This is a 2-torus in \mathbf{R}^3 with ‘large’ radius $\frac{1}{u}$ and ‘small’ radius $\frac{v}{u}$. $u = 0, v = 1$ is a special case. That means $x_1 = x_2 = 0$ and the locus on S^3 is a circle rather than a surface. The stereographic projection of

this circle less the point $(0, 0, 0, 1)$ is the y_3 -axis. Similarly, $v = 0, u = 1$ is a special case with the stereographic projection being the circle $y_3 = 0, r = 1$. The torii for $\frac{1}{\sqrt{2}} \leq u \leq 1$ fill out the solid torus A . However, the set B (which is the closure of the complement of A in S^3) may be described by $0 \leq u \leq \frac{1}{\sqrt{2}}$ or equivalently $\frac{1}{\sqrt{2}} \leq v \leq 1$. However, interchanging x_1 with x_3 and x_2 with x_4 clearly provides a homeomorphism of A with B . It follows that B is also a solid torus and the common boundary of A and B boundary is given by $u = v = \frac{1}{\sqrt{2}}$. This corresponds to the torus of ‘large’ radius $\sqrt{2}$ and small radius 1, i.e., to T . Note that the generator of $\pi(B)$ corresponds to the ‘other’ generator of $\pi(T)$ as required.