CHAPTER 4

Group Theory and the Seifert–Van Kampen Theorem

1. Some Group Theory

We shall discuss some topics in group theory closely connected with the theory of fundamental groups.

1.1. Finitely Generated Abelian Groups. We start with the followng theorem which you may have seen proved in a course in algebra either at the graduate or undergraduate level. We leave the proof for such a course.

THEOREM 4.1 (Structure Theorem for Finitely Generated Abelian Groups). *A finitely generated abelian group A is isomrphic to a direct sum of cyclic groups.*

Associated with this theorem is a uniqueness result, the statement of which requires some terminology. Note first that in discussing abelian groups one often uses additive notation and terminology and we shall generally, but not always, do that here. In the decomposition asserted in the theorem, some of the factors will be finite cyclic groups and some will be isomorphic to the infinite cyclic group $\mathbf{Z}$. The finite cyclic factors add up to a subgroup $t(A)$ of A which consists of all elements of A of finite order. This is called the torsion subgroup of A. The quotient group $A/t(A) = f(A)$ has no torsion elements, so it is a direct sum of copies of $\mathbf{Z}$ or what we call a *free abelian group*. The theorem then can be read as asserting that the projection $p : A \to f(A)$ has a left inverse $r : f(A) \to A$ whence

$$A = t(A) \oplus r(f(A)) \cong t(A) \oplus f(A).$$

In language you are probably familiar with by now, this is summarized by saying that the short exact sequence

$$0 \to t(A) \to A \to f(A) \to 0$$

splits. (A sequence of groups and homomorphisms is exact if at each stage, the image of the homomorphism into a group equals the kernel of the homomorphism out of it. For a sequence of the above type, exactness at $t(A)$ says the homomorphism $i : t(A) \to A$ is a monomorphism and exactness at $f(A)$ says $p : A \to f(A)$ is an epimorphism. Exactness at A says $A/\operatorname{Im} i \cong f(A)$.) Note that the torsion subgroup $t(A)$ is uniquely determined by A but the complementary subgroup $r(f(A)) \cong f(A)$ depends on the choice of r.

As note above, the group $f(A)$ is a free abelian group, i.e., it is a direct sum of copies of $\mathbf{Z}$. The number of copies is called the *rank* of A. The rank of a free abelian group may be defined using the concept of basis as in the theory of vector spaces over a field. As in that theory one shows that every basis has the same number of elements which is the rank. That gives us one part of the uniqueness statement we want. The discussion of the subgroup $t(A)$ is a bit more complicated. It is necessarily a direct sum of a finite number of finite cyclic groups, but there can be many different ways of doing that depending on the orders of those cyclic groups. For example.

$$\mathbf{Z}/6\mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$$

but

$$\mathbf{Z}/4\mathbf{Z} \ncong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

THEOREM 4.2 (Uniqueness Theorem for Finitely Generated Abelian Groups). *Let A be a finitely generated abelian group and suppose*

$$\begin{aligned} A &\cong \mathbf{Z}/d_1\mathbf{Z} \oplus \mathbf{Z}/d_2\mathbf{Z} \oplus \dots \mathbf{Z}/d_k\mathbf{Z} \oplus M \\ &\cong \mathbf{Z}/d'_1\mathbf{Z} \oplus \mathbf{Z}/d'_2\mathbf{Z} \oplus \dots \mathbf{Z}/d'_{k'}\mathbf{Z} \oplus M' \end{aligned}$$

where M, M' are free, and $d_1|d_2|\dots|d_k$, $d'_1|d'_2|\dots|d'_{k'}$ are integers > 1. Then M and M' have the same rank (so are isomorphic), $k' = k$ and $d'_i = d_i, i = 1, \dots, k$.

As above, we assume this was proved for you (or will be proved for you) in an algebra course.

An alternate approach to the uniqueness statement first divides $t(A)$ up as a direct sum of its Sylow subgroups. Each Sylow group has order p^N for some prime p, and if we break it up as a direct sum of cyclic subgroups, we can arrange the divisibility relations simply by writing the factors in increasing order. This gives an a somwhat different version of the theorem where the orders of the cyclic factors are unique prime powers (whether in order or not), but we eschew simplifications of the form $\mathbf{Z}/12\mathbf{Z} \cong \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$.

One important property of a free abelian group M is that any homomorphism of abelian groups $f : M \to N$ is completely determined by what it does to a basis, and conversely, we may always define a homomorphism by specifying it on a basis and extending it by linearity to

M. (This generalizes the corresponding statment about vector spaces, linear transformations, and bases.) The may be restated as follows. Let X be a basis for M and let $i : X \to M$ be the inclusion map. Then $M, i : X \to M$ has the following *universal mapping property.* Let $j : X \to N$ be any mapping of X to an abelian group N (as sets). Then there exists a unique homomorphism $f : M \to N$ such that $j = f \circ i$.

1.2. Free Groups. Let X be a set. We want to form a group $F(X)$ which is generated by the elements of X and such that there are as 'few' relations among the elements of X as possible consistent with their being elements of a group. We shall outline the construction of such a group here. (A complete treatment of this is done in algebra courses. There is also a discussion in Massey.) First consider all possible sequences—called words—of the form

$$x_1{}^{e_1} x_2{}^{e_2} \dots x_n{}^{e_n}$$

where each $x_i \in X$ and $e_i = \pm 1$. Include as a possibility the empty word which we shall just denote by 1. Then justaposition defines a law of composition on the set of all such words. Define a relation on the set of all such words as follows. Say $w \sim w'$ if w contains a two element subsequence of the form xx^{-1} or $x^{-1}x$ and w' is the word obtained by deleting it *or vice-versa*. Call this an elementary reduction. In general say that $w \sim w'$ if there is a sequence of elementary reductions $w = w_1 \sim w_2 \sim \dots \sim w_k = w'$. It is not hard to check that this is an equivalence relation and that it is consistent with the binary operation. That is, if $w_1 \sim w_1'$ and $w_2 \sim w_2'$ then $w_1 w_2 \sim w_1' w_2'$. An important fact useful in analyzing this relation is that each equivalence class of words contains a unique word of minimal length (containing no xx^{-1} or $x^{-1}x$) called a reduced word.

It follows that the set of equivalence classes $F(X)$ is endowed with a binary operation. With some work, it is possible to see that it is a group, called the free group on the set X. $F(X)$ has the following universal mapping property. Let $i : X \to F(X)$ be the map of sets defined by letting $i(x)$ be the equivalence class of the word x. Let $j : X \to G$ be a set map from X to a group G. Then there is a unique group homomoprhism $f : F(X) \to G$ such that $j = f \circ i$.

Example 4.3. The free group $F(x)$ on one generator x is infinite cyclic. If G is any cyclic group generated say by a of order n, then $x \mapsto a$ defines a homomorphism $f : F \to G$. This is an epimorphism and its kernel is the cyclic subgroup R of F generated by x^n. Clearly, $F/R \cong G$.

Example 4.4. Let $F = F(x, y)$ and let G be the dihedral group of order 8. G is generated by a, b where $a^4 = 1, b^2 = 1$ and $bab^{-1} = a^{-1}$. Then $f : F \to G$ is defined by $x \mapsto a, y \mapsto b$. The kernel of f contains the elements $x^4, y^2, yxy^{-1}y$. Let R be the normal subgroup of F generated by these elements, i.e., the smallest normal subgroup containing them. R consists of all possible products of *conjugates in* F of these three elements and their inverses. By doing some calculations with elements, it is possible to show that $|F/R| = 8$. Since $F \to G$ is clearly onto, it follows that f induces an isomorphism $F/R \cong G$.

Example 4.5. Let $F = F(x, y)$ and let $G = \mathbf{Z} \times \mathbf{Z}$ where this time we write G multiplicatively and denote generating basis elements by a and b. Define $f : F \to G$ by $x \mapsto a, y \mapsto b$. Then it is possible to show that $R = \operatorname{Ker} f$ is the normal subgroup generated by $xyx^{-1}y^{-1}$. Note that the word 'normal' is crucial. Indeed, it can be shown that any subgroup of a free group is free, so R is free, but viewed as a group in its own right it has denumerably many generators.

The above examples illustrate a very general process. For any group G generated say by a set X, we can a free group $F(X')$ where $|X'| = |X|$ and an epimorphism $f : F(X') \to G$. In fact, if one is careful with the notation, there is no point in not taking $X' = X$. If the kernel R of this epimorphism is generated as a normal subgroup by elements $r_1, r_2, \ldots$, we say that we have a *presentation* of G as the group generated by the elements of X subject to the relations $r_1, r_2, \ldots$. This in principle reduces the study of G to the study of F/R. However, in most cases, it is hard to determine when two words determine the same element of the quotient. (It has in fact been proved that there can be no general algorithm which accomplishes this!)

1.3. Free Products. We may generalize the construction of the free group as follows. Let H, K be two groups. We form a group $H * K$ called the free product of H with K as follows. Consider all words of the form

$$g_1 g_2 \cdots g_n$$

where for each i either $g_i \in H$ or $g_i \in K$. Allow the empty word which is denoted by 1. Then concatenation defines a binary operation on the set of all such words. Define an equivalence relation as before. First, say $w \sim w'$ by an elementary reduction if one of the two words contains a subsequence $g_i g_{i+1}$ where both are in H or both are in K and the other has length one less and contains instead the the single element of H or of K which is their product. (If $g_i = {g_{i+1}}^{-1}$ then assume the other has length two less and no corresponding element.) Say two words are equivalent in general if there is a sequence of elementary reductions going from one to the other. It is possible to show that any equivalence class contains a unique reduced word of the form

$$h_1 k_1 h_2 k_2 \dots h_r k_r$$

where each $h_i \in H$, each $k_i \in K$, all the $h_i \neq 1$ except possibly the first, and all the $k_i \neq 1$ except possibly for the last. As above, the equivalence relation is consistent with concatenation and defines a binary operation on the set $H * K$ of equivalence classes which is called the *free product* of K with H.

The free product has the following universal mapping property. Let $i : H \to H * K$ be the map defined by letting $i(h)$ be the equivalence of the word h and similarly for $j : K \to H * K$. Then given any pair of homomorphisms $p : H \to G, q : K \to G$, there is a unique homomorphism $f : H * K \to G$ such that $p = f \circ i, q = f \circ j$.

Namely, we define

$$f(g_1 g_2 \dots g_n) = g_1' g_2' \dots g_n'$$

where $g_i' = p(g_i)$ if $g_i \in H$ and $g_i' = q(g_i)$ if $g_i \in K$. For a reduced word,

$$f(h_1 k_1 \dots h_l k_l) = p(h_1) q(k_1) \dots p(h_l) q(k_l).$$

With some effort, one may show that f sends equivalent words to the same element and that it defines a homomoprhism.

The universal mapping property specifies $H * K$ up to isomorphism. For, suppose P' were any group for which there were maps $i' : H \to P', j' : K \to P'$ satisfying this same universal mapping property. Then there exist homomorphisms $f : H * K = P \to P'$ and $f' : P' \to P = H * K$ with appropriate consistency properties

The homomorphisms $f \circ f'$ and $f' \circ f$ have the appropriate consistency properties for homomorphisms $P' \to P'$ and $P \to P$ as do the identity homomorphisms. Hence, by uniqueness, $f \circ f'$ and $f' \circ f$ are the identities of P' and P respectively, whence f and f' are inverse isomorphisms.

This same construction can be repeated with any number of groups to form the free product $H_1 * H_2 * \cdots * H_l$, which has a universal mapping property you should state for yourself. Indeed,

$$H_1 * (H_2 * \cdots * H_l) \cong H_1 * H_2 * \cdots * H_l.$$

The free product of groups is related to the concept of free group as follows. Let $X = \{x_1, x_2 \ldots, x_n\}$, and let $F_i = F(x_i)$ be the infinite cyclic group which is the free group generated by x_i. Then

$$F(X) \cong F_1 * F_2 * \cdots * F_n.$$

Since all the F_i are isomorphic to $\mathbf{Z}$, we may also write this

$$F(X) \cong \mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z} \quad n \quad \text{times}.$$

1.4. Free Products with Amalgamation. Let H and K be groups and suppose we have two homomorphisms $i : A \to H$ and $j : A \to K$. We want to define something like the free product but where elements of the form $i(a)$ and $j(a)$ are identified. To this end, consider the normal subgroup N of $H * K$ generated by all elements of the form

$$i(a)j(a)^{-1} \qquad \text{for } a \in A.$$

We denote the quotient group $H * K/N$ by $H *_A K$, and call it the free product with amalgamation. (The notation has some defects since the group depends not only on A but also on the homomorphisms i and j. This concept was first studied in the case A is a subgroup of both H and of K.)

A general element of $H *_A K$ can be written

$$g_1 g_2 \cdots g_n$$

where either $g_i \in H$ or $g_i \in K$. In addition to equalities which result from equivalences of words in $H * K$, we have for each $a \in A$, the rule

$$i(a) = j(a)$$

in $H *_A K$. Note that we have decided to engage in a certain amount of 'abuse of notation' in order to keep the notaion relatively simple. An element of $H * K$ is actually an equivalence class of words, and a element of $H *_A K$ is a coset of the normal subroup N determined by an element of $H * K$. Hence, when we assert that an element of $H *_A K$ is of the form

$$g_1 g_2 \dots g_n$$

where either $g_i \in H$ and $g_i \in K$, we are ignoring several levels of abstraction.

The free product with amalgamation may also be characterized by a universal mapping property. Define $p : H \to H *_A K$ first mapping h to the word h in $H * K$ and then to the coset of h in $H * K/N$. Similarly, define $q : K \to H *_A K$. Then we have a commutative diagram

Suppose generally, we have $p' : H \to G$ and $q' : K \to G$ such that $p' \circ i = q' \circ j$, i.e.,

By the universal mapping property of $H * K$, there exists a unique map $f' : H * K \to G$ such that

$$f'(\underbrace{h_1 k_1 \dots h_l k_l}_{\text{in } H * K}) = p'(h_1) q'(k_1) \dots p'(h_l) q'(k_l).$$

In particular, for the element $i(a) j(a)^{-1} \in H * K$, we have

$$f'(\underbrace{i(a) j(a)^{-1}}_{\text{in } H * K}) = p'(i(a)) q'(j(a))^{-1} = 1$$

so the normal subroup of $H * K$ generated by all such elements is contained in $Ker f'$. Hence, f' induces a homomorphism $f : H * K/N = H *_A K \to G$ making the following diagram commute.

On the other hand, there can be at most one such homomorphism, since $H *_A K$ is generated by the cosets of words of the form $h_1 k_1 \dots h_l k_l$, so the commutativity of the diagram assures us that

$$f(\underbrace{h_1 k_1 \dots h_l k_l}_{\text{in } H *_A K}) = p'(h_1)q'(k_1)\dots p'(h_l)q'(k_l).$$

The argument above was made a bit more confusing than it needs to be by the ambiguity of the notation, but if you pay careful attention, you should have no trouble following it.

The universal mapping property described above is often said to specify $H *_A K$ as the *pushout* in the diagram

2. The Seifert–Van Kampen Theorem

Theorem 4.6. *Let X be a path connectedspace and suppose $X = U \cup V$ where $U, V, U \cap V$ are open sets of X which are also path connected. Denote the inclusion maps as indicated below*

Let $x_0 \in U \cap V$. Then

presents $\pi(X, x_0)$ as a pushout, i.e.,

$$\pi(X, x_0) \cong \pi(U, x_0) *_{\pi(U \cap V, x_0)} \pi(V, x_0).$$

We defer the proof until later.

Example 4.7 (The Figure eight). Let X be the figure eight space described earlier. Let U be the subspace obtained by deleting the closed half of one loop and let V be the subspace obtained by doing the same with the other loop.

Then U and V are each deformation retracts of spaces homeomorphic to S^1 and $U \cap V$ is contractible. Hence,

$$\pi(X) \cong \pi(U) *_1 \pi(V) \cong \mathbf{Z} * \mathbf{Z}$$

so it is free on two generators. Indeed, checking the details of the contruction, we may take as generators the generators of the fundamental groups of the two loops.

EXAMPLE 4.8 (The 2-torus). We shall show yet again that $\pi(T^2) \cong \mathbf{Z} \times \mathbf{Z}$. Realize T^2 as a square with opposite edges identified.

Let U be the open square without boundary, and let V be the open set obtained by deleting the center point. It is not hard to see that the boundary is homeomorphic to the figure eight space and that it is a deformation retract of V. Hence, $\pi(V) \cong \mathbf{Z} * \mathbf{Z}$ is free on two generators $[a]$ and $[b]$, each representing a circle in T^2. U is contractible so $\pi(U) = \{1\}$. Finally, $U \cap V$ is a punctured disk, so it has the homotopy type of S^1 and $\pi(U \cap V) \cong \mathbf{Z}$. Better yet, $\pi(U \cap V)$ is generated by any simple loop enclosing the deleted center point, so we may choose in particular a square h as indicated in the diagram.

It is clear that h is homotopic in V to the path $a * b * \overline{a} * \overline{b}$. Hence, the pushout diagram describing $\pi(T^2)$ becomes

It follows without two much diffculty that $\pi(T^2)$ is the free group on two generators $\alpha = [a], \beta = [b]$ modulo the normal subgroup generated by the relation $\alpha\beta\alpha^{-1}\beta^{-1}$. It follows that $\pi(T^2)$ is free abelian of rank two.

Proof. Since we use the common base point x_0 for all fundamental groups, we shall omit it from the notation.

By the universal mapping property of the pushout, we know there is a unique homomorphism $\pi(U) *_{\pi(U\cap V)} \pi(V) \to \pi(X)$ making the appropriate diagram commute, and we can even describe it explicitly on a general element in the free product with amalgamation. (See the previous section.) We already proved (when showing that S^n is simply connected for $n > 1$) that that any loop in X at x_0 is homotopic reltative to $\dot{I}$ to a product of loops each of which is either in U or is in V. Translation of this asserts that $\pi(X)$ is generated by products of the form

$$\gamma_1\gamma_2 \dots \gamma_k$$

where for each i, either $\gamma_i = p_*(\alpha_i), \alpha_i \in \pi(U)$ or $\gamma_i = q_*(\alpha_i), \alpha_i \in \pi(V)$. It follows that the homomorphism $G = \pi(U) *_{\pi(U\cap V)} \pi(V) \to \pi(X)$ is an epimorphism.

The hard work of the proof is to show that the homomorphism is a monomphism.

Let

$$h_1 * h_2 * \dots * h_k \sim_{\dot{I}} e_{x_0}$$

in X where $h_1, \dots, h_k$ are loops at x_0, each of which is either in U or in V. Let U_i be either U or V accordingly, and let $\alpha_i = [h_i]_{U_i}$ denote the class of h_i in $\pi(U_i)$. We want to show that the correponding element

$$\underbrace{\alpha_1\alpha_2 \dots \alpha_k = 1}_{\text{in } \pi(U) *_{\pi(U\cap V} \pi(V)}.$$

Choose $h : I \to X$ such that $h \sim_{\dot{I}} h_1 * h_2 * \dots * h_{k-1} * h_k$ in X by subdividing I into k subintervals such that the restriction of h to the ith subinterval is h_i after a suitable parameter change. Let $H : I \times I \to X$ be a homotopy realizing $h \sim_{\dot{I}} e_{x_0}$, i.e.,

$$\begin{aligned} H(t, 0) &= h(t) \\ H(t, 1) &= x_0 \\ H(0, s) &= H(1, s) = x_0. \end{aligned}$$

By the Lebesgue Covering Lemma, we may choose partitions

$$\begin{aligned} 0 = t_0 < t_1 < \dots < t_n = 1 \\ 0 = s_0 < s_1 < \dots < s_m = 1 \end{aligned}$$

which subdivide $I \times I$ into closed subrectangles $R_{ij} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$ such that for each i, j, either $H(R_{ij}) \subseteq U$ or $H(R_{ij}) \subseteq V$.

As above, let U_{ij} be U or V accordingly. We can also arrange the partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

to include the endpoints of the domains of the functions h_r. Just throw those points in as needed; the rectangles in the subdivision may become smaller. Now *redefine*

$$h_i = h|[t_{i-1}, t_i] \qquad i = 1, 2, \ldots, n.$$

This does not affect the equivalence

$$h_1 * h_2 * \cdots * h_n \sim_{\dot{I}} e_{x_0}$$

in X, but it does create one technical problem. Namely, the new paths h_i need not be loops.

Avoid this difficulty as follows. For each point $x \in X$, choose a path g_x from x_0 to x such that the image of g_x lies in U, V, or $U \cap V$ as x itself does. (This is possible since U, V, and $U \cap V$ are path connected.) Also, let g_{x_0} be constant. For each $i = 0, \ldots, n$, abbreviate $g_i = g_{h(t_i)}$. Then, $g_{i-1} * h_i * \overline{g}_i$ is a loop, and we shall abuse notation and abbreviate

$$[h_i] = [g_{i-1} * h_i * \overline{g}_i].$$

Similar conventions will be needed for the other vertices in the subdivision of $I \times I$ into rectangles. We shall be interested in paths in $X, U, V, U \cap V$ obtained by restricting H to the edges of these rectangles. These won't generally be loops at x_0. However, we may choose paths as above from x_0 to the endpoints of these edges and reinterpet the notation '$[-]$' accordingly. These base point shifting paths should be chosen so they lie in U, V, or $U \cap V$ if their endpoints do, and they should be constant if the endpoint happens to be x_0.

Consider paths in X defined by restricting H to a rectilinear path in $I \times I$ which starts at the left edge, follows a sequence of horizontal edges of the rectangles R_ij, perhaps drops down through one vertical edge, and then continues horizontally to the right hand edge.

This will be a loop in X based at x_0 since H takes on the value x_0 on both left and right edges. We may proceed by a sequence of such paths from h on the bottom edge of $I \times I$ to e_{x_0} on the top edge. The elementary step in this sequence is to 'add' a rectangle as indicated in the diagram. Consider the effect of adding the rectangle R_{ij}. Let h_{ij} be the path associated with the horizontal edge from (t_{i-1}, s_j) to (t_i, s_j) and let v_{ij} be the path associated with the vertical edge from (t_i, s_{j-1}) to (t_i, s_j). Then, in U_{ij} and in X we have

$$h_{i,j-1} * v_{ij} \sim_{\dot{I}} v_{i-1,j} * h_{i,j}.$$

This has two consequences. First, the effect of adding the rectangle is to produce an equivalent path in X. Second,

$$[h_{i,j-1}][v_{ij}] = [v_{i-1,j}][h_{i,j}]$$

is a true in equation in $\pi(U_{ij})$. We should like to conclude that it is also a true equation in $\pi(U) *_{\pi(U\cap V)} \pi(V)$. If that is true, then we can conclude that adding the rectangle changes the word associated with one path to the word associated to the other in $\pi(U) *_{\pi(U\cap V)} \pi(V)$. However, there is one problem associated with this. A path h_{ij} or v_{ij} may appear in the free product with amalgamation as coming from either of the two rectangles which it abuts. Hence, the path could lie in both U and in V and be considered to yield both an element in $\pi(U)$ and an element of $\pi(V)$. However, in the free product with amalgamation, these elements are the same because they come from a common element of $\pi(U \cap V)$.

It now follows that in $\pi(U) *_{\pi(U\cap v)} \pi(V)$, we have a sequence of equalities

$$[h_1][h_2]\dots[h_n] = \dots = [e_{x_0}]^n = 1.$$

□

We do a couple more examples.

EXAMPLE 4.9 (The Klein Bottle). As in the example of the 2-torus, the boundary is homeomorphic to a figure eight. The decomposition $X = U \cup V$ is the same as in that case, but the path h is homotopic instead to the path representing $\alpha\beta\alpha^{-1}\beta$. Thus $\pi(X)$ is isomorphic the the free group on two generators α, β modulo the relation $\alpha\beta\alpha^{-1} = \beta^{-1}$. Let B be the subgroup generated by β. It is not hard to see it is normal and the quotient is generated by the coset of α. However, for any n, m we can contruct a finite group of order nm generated by elements a, b and such that $a^n = b^m = 1$ and $aba^{-1} = b^{-1}$. (You can actually write out a multiplication table for such a group and check laboriously that it is a group.) This group is an epimorphic image of the above group. Hence, it follows that the subgroup B must be infinite cyclic as must the quotient group. In fact, the group under consideration is an example of what is called a semi-direct product.

EXAMPLE 4.10 (The Real Projective Plane). Use the square with the identification below.

Apply exactly the same reasoning as for the Klein bottle and T^2. However, in this case the bounding edges form a space homomorphic to S^1 (divided into two parts). The generator of $\pi(V)$ is $[ab]$ and the path h is homotopic to $[ab]^2$. Hence, $\pi(X)$ is isomorphic to the groups generated by $\gamma = [ab]$ and subject to the relation $\gamma^2 = 1$. Thus it is cyclic of order two.