CHAPTER 9

Products and the Künneth Theorem

1. Introduction to the Künneth Theorem

Our aim is to understand the homology of the cartesian product $X \times Y$ of two spaces. The Künneth Theorem gives a complete answer relating $H_*(X \times Y)$ to $H_*(X)$ and $H_*(Y)$, but the answer is a bit complicated. One important special case says that *if* $H_*(X)$ *and* $H_*(Y)$ *are free*, then

$$H_*(X \times Y) = H_*(X) \otimes H_*(Y).$$

Example 9.1 (Products of Spheres). We saw in the previous chapter that $H_i(S^m \times S^n)$ is $\mathbf{Z}$ in dimensions $0, n, m, m+n$ if $m \neq n$ and $\mathbf{Z} \oplus \mathbf{Z}$ in dimension $n = m$ when the two are equal. This is accounted for by the Künneth Theorem as follows. Write

$$H_*(S^m) = \mathbf{Z}e_0^m \oplus \mathbf{Z}e_m^m$$
$$H_*(S^n) = \mathbf{Z}e_0^n \oplus \mathbf{Z}e_n^n$$

where the subscript of each generator indicates its degree (dimension). If we take the tensor product of both sides and use the additivity of the tensor product and the fact that $\mathbf{Z} \otimes \mathbf{Z} = \mathbf{Z}$, we obtain

$$\begin{aligned} H_*(S^m) \otimes H_*(S^n) &= (\mathbf{Z}e_0^m \oplus \mathbf{Z}e_m^m) \otimes (\mathbf{Z}e_0^n \oplus \mathbf{Z}e_n^n) \\ &= \mathbf{Z}e_0^m \otimes e_0^n \oplus \mathbf{Z}e_0^m \otimes e_n^n \oplus \mathbf{Z}e_m^m \otimes e_0^n \oplus \mathbf{Z}e_m^m \otimes e_n^n. \end{aligned}$$

We see then how the answer is constructed. The rule is that if we tensor something of degree r with something of degree s, we should consider the result to have *total degree* $r + s$. With this convention, the terms in the above sum have degrees $0, n, m, m+n$ as required. Note also that we don't have to distinguish the $m = n$ case separately since in that case, there are two summands of the same total degree $n = 0 + n = n + 0$.

The above example illustrates the more explicit form of the Künneth Theorem in the free case

$$H_n(X \times Y) = \bigoplus_{r+s=n} H_r(X) \otimes H_s(Y).$$

If X and Y are CW complexes, it is not hard to see how such tensor products might arise. Namely, $C_*(X)$ and $C_*(Y)$ are each free abelian groups on the open cells of the respective CW complexes. Also, $X \times Y$ has a CW complex structure with open cells of the form $e \times f$ where e is an open cell of X and f is an open cell of Y. Hence,

$$C_*(X \times Y) = \bigoplus_{e,f} \mathbf{Z} e \times f \cong \bigoplus_{e,f} \mathbf{Z} e \otimes \mathbf{Z} f \cong \bigoplus_{e} \mathbf{Z} e \otimes \bigoplus_{f} \mathbf{Z} f \cong C_*(X) \otimes C_*(Y).$$

Note also that if $\dim(e \times f) = \dim e + \dim f$ which is consistent with the rule enunciated for degrees.

Unfortunately, the above decomposition is not the whole answer because we still have to relate $H_*(C_*(X) \otimes C_*(Y))$ to $H_*(C_*(X)) \otimes H_*(C_*(Y)) = H_*(X) \otimes H_*(Y)$. This comparison itself requires considerable work, i.e., we need a Künneth Theorem for chain complexes *before* we can derive such a theorem for spaces.

The analysis for CW complexes is not complete because we have not discussed the boundary homomorphism in $C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$. It turns out that this is fairly straightforward provided we know the boundary homomorphism for each term, but the latter morphisms are not easy to get at in general. (Of course, in specific cases, they are easy to compute, which is one thing that makes the CW theory attractive.) For theoretical purposes, we know that the singular chain complex is the 'right' thing to use because of its functorial nature. Unfortunately, it is not true in general that the 'product' of two simplices is again a simplex.

In fact, we spent considerable time studying how to simplicially decompose the product of an n-simplex and a 1-simplex, i.e, a prism, when proving the homotopy axiom. In general, in order to make use of singular chains, we need to relate $S_*(X \times Y)$ to $S_*(X) \otimes S_*(Y)$. It turns out that these are not isomorphic but there are chain maps between them with compositions which are chain homotopic to the identity. Thus, the two chain complexes are chain homotopy equivalent, and they have the same homology. This relationship is analyzed in the *Eilenberg–Zilber Theorem.*

Our program then is the following: first study the homology of the tensor product of chain complexes, then prove the Eilenberg–Zilber

Theorem, and then apply these results to obtain the Künneth Theorem for the product of two spaces.

2. Tensor Products of Chain Complexes

Let C' and C'' denote chain complexes. We make the tensor product $C' \otimes C''$ into a chain complex as follows. Define

$$(C' \otimes C'')_n = \bigoplus_{r+s=n} C'_r \otimes C''_s.$$

Also define boundary homomorphisms $\partial_n : (C' \otimes C'')_n \to (C' \otimes C'')_{n-1}$ by

$$\partial_n(x' \otimes x'') = \partial'_r x' \otimes x'' + (-1)^r x' \otimes \partial''_s x''$$

where $x' \in C'_r$ and $x'' \in C''_s$ and $r + s = n$. Note that the expression on the right is biadditive in x' and x'', so the formula does defined a homomorphism of $X'_r \otimes X''_s$. Since $(C' \otimes C'')_n$ is the direct sum of all such terms with $r + s = n$, this defines a homomorphism.

Proposition 9.2. $\partial_n \circ \partial_{n+1} = 0$.

Proof. Exercise. In doing this notice how the sign comes into play. □

When you do the calculation, you will see that the sign is absolutely essentially to prove that $\partial_n \circ d_{n+1} = 0$. We can also see on geometric grounds why such a sign is called for by considering the following example. Let $C' = C_*(I)$ where I is given a CW structure with two 0-cells and one 1-cell as indicated below. Similarly, let $C'' = C_*(I^2)$ with four 0-cells, four 1-cells and one 2-cell. We may view $C' \otimes C''$ as the chain complex of the product CW complex $I^3 = I \times I^2$. The diagram indicates how we expect the orientations and boundaries to behave. Note how the boudaries of various product cells behave. Note in particular how the boundary of $e'_1 \times e''_2$ ends up with a sign.

Our immediate problem then is to determine the homology of $C' \otimes C''$ in terms of the homology of the factors. First note that there is a natural homorphism

$$\times : H_*(C') \otimes H_*(C'') \to H_*(C' \otimes C'')$$

defined as follows. Given $z' \in \mathbf{Z}(C')$ of degree r represents $\overline{z'} \in H_r(C')$ and $z'' \in \mathbf{Z}(C'')$ of degree s represents $\overline{z''} \in H_s(C'')$ define

$$\overline{z'} \times \overline{z''} = \overline{z' \otimes z''} \in H_{r+s}(C' \otimes C'').$$

Note that the right hand side is well defined because

$$\begin{aligned}(z' + \partial' c') \otimes (z'' + \partial'' c'') &= z' \otimes z'' + \partial' c' \otimes z'' + z' \otimes \partial'' c'' + \partial' c' \otimes \partial'' c'' \\ &= z' \otimes z'' + \partial(c' \otimes z'' \pm z' \otimes c'' + c' \otimes \partial'' c'').\end{aligned}$$

It is also easy to check that it is biadditive, so

$$\overline{z'} \otimes \overline{z''} \to \overline{z'} \times \overline{z''}$$

defines a homomorphism $H_r(C') \otimes H_s(C'') \to H_{r+s}(C' \otimes C'')$ as required. We shall call this homomorphism the *cross product*, and as above we shall denote it by infix notation rather than the usual functional prefix notation.

We shall show that under reasonable circumstances, the homomophism is a monomorphism, and if $H_*(C')$ or $H_*(C'')$ is free then it is an isomorphism.

Suppose in all that follows that C' and C'' are free abelian groups, i.e., free $\mathbf{Z}$-modules. Then tensoring with any component C'_r or C''_s of either is an exact functor, i.e., it preserves short exact sequences. The importance of this fact is that any exact functor preserves homology. In particular, if A is a free abelian group and C is a chain complex, then $C \otimes A$ is also a chain complex (with boundary $\partial \otimes \mathrm{Id}$) and

$$H_*(C) \otimes A \cong H_*(C \otimes A).$$

The isomorphism is provided by $\overline{z} \otimes a \mapsto \overline{z \otimes a}$. It is clear that the isomorphism in natural with respect to morphisms of chain complexes and homomorphisms of groups.

Consider the exact sequence

$$0 \to Z(C'') \to C'' \to C''/Z(C'') \to 0. \tag{39}$$

This of course yields a collection of short exact sequences of groups, one in each degree s, but we may also view it as a short exact seqeunce of chain complexes, where $Z(C'')$ is viewed as a subcomplex of C'' with *zero* boundary homomorphism. Note also that since $\partial'' C'' \subseteq Z(C'')$, the quotient complex also has zero boundary. In fact, $C''/Z(C'')$ may be identified with $B(C'')$ (also with zero boundary) but with degrees shifted by one, i.e., $(C''/Z(C''))_r = B(C'')_{r-1}$. We shall denote the shifted complex by $B_+(C'')$.

Tensor the sequence 39 with C' to get the exact sequence of chain complexes

$$0 \to C' \otimes Z(C'') \to C' \otimes C'' \to C' \otimes B_+(C'') \to 0. \tag{40}$$

Note that each of these complexes is a tensor product complex, but for the two complexes on the ends, the contribution to the boundary from the second part of tensor product is trivial, e.g., $\partial_{p+q} c' \otimes z'' = \partial_p c' \otimes z''$. Since $Z(C'')$ and $B_+(C'')$ are also free, we have by the above remark

$$H_*(C') \otimes Z(C'') \cong H_*(C' \otimes Z(C''))$$
$$H_*(C') \otimes B_+(C'') \cong H_*(C' \otimes B_+(C'')).$$

Note however that because these are actually tensor products of complexes, we must still keep track of degrees, i.e., we really have

$$\bigoplus_{r+s=n} H_r(C') \otimes Z_s(C'') \cong H_n(C' \otimes Z(C''))$$
$$\bigoplus_{r+s=n} H_r(C') \otimes B_{+s}(C'') \cong H_n(C' \otimes B_+(C'')).$$

Now consider the long exact sequence in homology of the SES 40

$$\dots \xrightarrow{\delta_{n+1}} H_n(C' \otimes Z(C'')) \to H_n(C' \otimes C'')$$
$$\to H_n(C' \otimes B_+(C'')) \xrightarrow{\delta_n} H_{n-1}(C' \otimes Z(C'')) \to \dots$$

From this sequence, we get a SES

$$0 \to \mathrm{Coker}(\delta_{n+1}) \to H_n(C' \otimes C'') \to \mathrm{Ker}(\delta_n) \to 0. \tag{41}$$

We need to describe δ_{n+1} and δ_n in order to determine these groups. By the above discussion, $H_n(C' \otimes Z(C''))$ may be identified with the direct sum of components

$$H_r(C') \otimes Z_s(C'') \cong H_r(C' \otimes Z_s(C''))$$

where $r + s = n$. Similarly, $H_{n+1}(C' \otimes B_+(C''))$ may be identified with the direct sum of components of the form

$$H_r(C') \otimes B_{+,s+1}(C'') \cong H_r(C') \otimes B_s(C'')$$

where $r + s + 1 = n + 1$. Fix a pair, r, s with $r + s = n$.

LEMMA 9.3. *δ_{n+1} on $H_r(C') \otimes B_s(C'')$ is just $(-1)^r \operatorname{Id} \otimes i_s$ where i_s is the inclusion of $B_s(C'')$ in $Z_s(C'')$.*

PROOF. We just have to trace through the various identifications and definitions. Let $\overline{z'} \otimes \partial'' c''$ be a typical image we want to apply δ_{n+1} to. This should first be identified with $\overline{z' \otimes \partial'' c''}$. This is of course represented by $z' \otimes \partial'' c'' \in Zr + s + 1(C' \otimes B_+(C''))$. However, this comes from $z' \otimes c'' \in (C' \otimes C'')_{r+s+1}$, so taking its boundary, we obtain

$$\partial(z' \otimes c'') = (-1)^r z' \otimes \partial'' c''.$$

Not so surprisingly, this comes from a cycle in $(C' \otimes Z(C''))_{r+s}$, namely $(-1)^r z' \otimes \partial'' c''$. However, this is just what we want. □

Suppose now that $H_*(C'')$ is free. Then, for each s, the ses

$$0 \to B_s(C'') \to Z_s(C'') \to H_s(C'') \to 0$$

splits. Hence, since tensor products preserve direct sums, the sequence

$$0 \to H_r(C') \otimes B_s(C'') \xrightarrow{\delta_{n+1}} H_r(C') \otimes Z_s(C'') \to H_r(C') \otimes H_s(C'') \to 0$$

also splits; hence it is certainly exact. This yields two conclusions: $\operatorname{Coker}(\delta_{n+1}) \cong H_r(C') \otimes H_s(C'')$ and $\operatorname{Ker} \delta_n = 0$. Hence, it follows that

$$\bigoplus_{r+s=n} H_r(C') \otimes H_s(C'') \cong H_n(C' \otimes C'').$$

If you trace through the argument, you will find that the isomorphism is in fact given by the homomorphism '$\times$' defined earlier.

Clearly, we could have worked the argument with the roles of C' and C'' reversed if it were true that $H_*(C'')$ were free.

THEOREM 9.4 (Kúnneth Theorem, restricted form). *Let C' and C'' be free chain complexes such that either $H_*(C')$ is free or $H_*(C'')$ is free. Then*

$$\times : H_*(C') \otimes H_*(C'') \to H_*(C' \otimes C'')$$

is an isomorphism.

In the next section, we shall determine $\operatorname{Coker} \delta_{n+1}$ and $\operatorname{Ker} \delta_n$ in the case neither $H_*(C')$ nor $H_*(C'')$ is free.

2.1. A Digression. There is another useful way to thing of the chain complex $C' \otimes C''$. It has two boundary homomorphisms $d' = \partial' \otimes \mathrm{Id}, d'' = \pm \mathrm{Id} \otimes \partial''$ where the sign $\pm = (-1)^r$ is given as above. These homomorphisms satisfy the rules

$$d'^2 = d''^2 = 0, \quad d'd'' + d''d' = 0.$$

What we have is the first important example of what is called a *double complex*.

We may temporarily set the first boundary in this chain complex to zero, so taking its homology amounts to taking the homology with respect the the second boundary. Since C' is free, tensoring with it is exact, and we have

$$H_*(C' \otimes C'', d'') \cong C' \otimes H_*(C'').$$

d' induces a boundary on this chain complex, and it makes sense to take the boundary on $H_*(C'')$ to be zero. Denote the reulst

$$H_*(C' \otimes H_*(C'')).$$

Note that it would be natural to conclude that this is $H_*(C') \otimes H_*(C'')$, but this is wrong except in the case $H_*(C')$ or $H_*(C'')$ is free as above. To study this quantity in general, consider as above the exact sequence of chain complexes (with trivial boundaries)

$$0 \to B(C'') \to Z(C'') \to H(C'') \to 0.$$

Tensoring as above with the free complex C' yields the exact sequence of chain complexes.

$$0 \to C' \otimes B(C'') \to C' \otimes Z(C') \to C' \otimes H(C'') \to 0.$$

This yields a long exact sequence in homology

$$\begin{aligned} H_n(C' \otimes B(C'')) \to H_n(C' \otimes Z(C')) \to H_n(C' \otimes H(C'')) \\ \to H_{n-1}(C' \otimes B(C'') \to H_{n-1}(C' \otimes Z(C'')) \to \dots \end{aligned}$$

However, as above

$$\begin{aligned} H_n(C' \otimes B(C'')) &\cong H_n(C') \otimes B(C'') \\ H_n(C' \otimes Z(C'')) &\cong H_n(C'') \otimes Z(C'') \end{aligned}$$

so we get an exact sequence

$$0 \to \mathrm{Coker}(\delta_{n+1}) \to H_n(C' \otimes C'') \to \mathrm{Ker}(\delta_n) \to 0. \tag{42}$$

You might conclude from this that this is the same sequence as 40 above, i.e., that the middle terms are isomorphic. In fact they are but only because, as we shall see, both sequences split. Thus

$$H_n(C' \otimes C'') \cong H_n(C' \otimes H(C''))$$

but *there is no natural isomorphism between them.*

3. Tor and the Künneth Theorem for Chain Complexes

To determine the homology of $C' \otimes C''$, we came down to having to determine the kernel and cokernel of the homorphism

$$\mathrm{Id} \otimes i_s : H_r(C') \otimes B_s(C'') \to H_r(C') \otimes Z_s(C'')$$

which (except for a sign) is the r, s component of δ_{r+s+1}. Thus, in effect we need to know what happens to the exact sequence

$$0 \to B_s(C'') \to Z_s(C'') \to H_s(C'') \to 0$$

when we tensor with $H_r(C')$.

Recall in general that if $0 \to B' \to B \to B'' \to 0$ is a ses of abelian groups, and A is any abelian group, then

$$A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$$

is exact. What we need to know for the Künneth Theorem is the cokernel and the kernel of the homomorphism on the left. It is clear from the right exactness that the cokernel is always isomorphic to $A \otimes B''$, but we don't yet have a way to identify the kernel. (We know the kernel is trivial if A is torsion free or if the 'B' sequence splits.) We shall define a new functor $\operatorname{Tor}(A, B)$ which will allow us to continue the seqeunce to the left:

(43)

$$\operatorname{Tor}(A, B') \to \operatorname{Tor}(A, B) \to \operatorname{Tor}(A, B'') \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0.$$

It is pretty clear that this functor $\operatorname{Tor}(A, B)$ should have certain properties:

(i) It should be a functor of both variables.

(ii) If A is torsion free, if would make sense to require that $\operatorname{Tor}(A, -) = 0$, because that would insure that tensoring with A is an exact functor.

(iii) It should preserve direct sums.

(iv) Since $A \otimes B \cong B \otimes A$, we should have $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(B, A)$. More generally, its properties should be symmetric in A and B.

(v) A sequence like 43 and its analogue with the roles of the arguments reversed should hold.

Property (ii) suggests the following approach. Let B be an abelian groups and choose a free presentation of it

$$0 \to R \xrightarrow{i} F \xrightarrow{j} B \to 0,$$

i.e., pick an epimorphism of a free abelians groups $j : F \to B$, and let R be the kernel. R is also free because *any subgroup of a free abelian group is free.* Define

$$\operatorname{Tor}(A, B) = \operatorname{Ker}(\operatorname{Id} \otimes i : A \otimes R \to A \otimes F).$$

Note that whatever the definition, if (ii) and (v) hold, we have an exact seqeunce

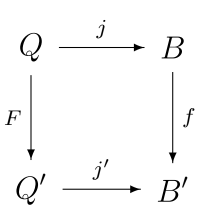
$$0 \to 0 \to \operatorname{Tor}(A, B) \to A \otimes R \to A \otimes F \to A \otimes B \to 0$$

so in any event $\mathrm{Tor}(A, B) \cong \mathrm{Ker}(\mathrm{Id} \otimes i)$. However, there is clearly a problem with this definition: *it appears to depend on the choice of presentation.*

THEOREM 9.5. *There is a functor* $\mathrm{Tor}(A, B)$ *such that for each presentation* $0 \to R \xrightarrow{i} F \to B \to 0$*, there is an isomorphism* $\mathrm{Tor}(A, B) \cong \mathrm{Ker}\,\mathrm{Id} \otimes i$*. Furthermore this isomorphism is natural with respect to homomorphisms of both arguments* A, B *and also with respect to morphisms of presentations.*

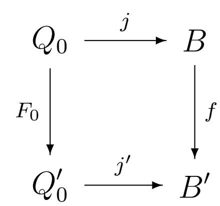
PROOF. It will be helpful to describe presentations from a slightly different point of view. Given a presentation, of B construct a chain complex with two non-zero groups by putting $Q_0 = F, Q_1 = R$ and $d_1 = i$. Then j may be viewed as a morphism of chain complexes $f \to B$ where the latter is the trivial chain complex with only one non-zero group, B, in degree 0. Then j induces an isomorphism of homology $H_*(Q) \cong B$, i.e., $H_0(Q) = B$, and $H_k(Q) = 0$ otherwise. Now consider the chain complex $A \otimes Q$. We have a natural isomorphism $H_0(A \otimes Q) \cong A \otimes B$, and $H_1(A \otimes Q)$ is supposed to be isomorphic to $\mathrm{Tor}(A, B)$.

LEMMA 9.6. *Let* Q *be a chain complex for a presentation of* B *and* Q' *a chain complex for a presentation of* B'*. Let* $f : B \to B'$ *be a homomorphism. Then there is a morphism of chain complexes* $F : Q \to Q'$ *such that*



commutes. Moreover, any two such morphisms are chain homotopic.

PROOF. Since Q_0 is free, it is easy to see there is a homomorphism $F_0 : Q_0 \to Q_0'$ such that



commutes. Since $Q_1 \to Q_0$ is a monomorphism, it is easy to see that the restriction F_1 of F_0 to Q_1 has image in Q_1'. (Since Q_1 is free, you could define F_1 even if $Q_1 \to Q_0$ were not a monomorphism.)

Suppose F, F' are two such chain morphisms. Since $j' \circ (F_0 - F_0') = 0$, it follows from the freeness of Q_0, that there is a homomorphism $L_0 : Q_0 \to Q_1'$ such that $i' \circ L_0 = F_0 - F_0'$. Let $L_i = 0$ otherwise. It is easy to see $L : Q \to Q'$ is a chain homotopy of F to F'. □

Suppose then $f : B \to B'$ and $F : Q \to Q'$ are morphisms as above. Then we obtain morphisms $\mathrm{Id} \otimes f : A \otimes B \to A \otimes B'$ and $\mathrm{Id} \otimes F : A \otimes Q \to A \otimes Q'$ consistent with it. The latter morphism induces a homomorphism $H_*(A \otimes Q) \to H_*(A \otimes Q')$. Replacing F by F' will also yield a chain homotopic morphism $\mathrm{Id} \otimes F'$, so the morphism in homology will be the same. Note that the morphism $H_0(A \otimes Q) \to H_0(A \otimes Q')$ corresponds to $\mathrm{Id} \otimes f : A \otimes B \to A \otimes B'$. However it is $H_1(A \otimes Q)$, and the morphism $H_1(A \otimes Q')$, we want to think about now.

First note that one consequence of the above lemma is that up to isomorphism $H_1(A \otimes Q)$ depends only on B (and of course also on A). For if we choose two different presentations $j : Q \to B$ and $j' : Q' \to B$, then there are chain morphisms between Q and Q'

It is clear that both compositions of these morphisms are chain homotopic to the identity, so tensoring with A and taking homology gives the desired conclusion. Now for each B, choose one specific presentation. (It could be for example formed by letting Q_0 be the free abelian group on the elements of B as basis with the obvious epimorphism $Q_0 \to B$, and Q_1 the kernel of that epimorphism. However, if B is finitely generated, you might want to restrict attention to finitely generated Q.) Define $\mathrm{Tor}(A, B) = H_1(A \otimes Q)$ for this specific presentation. For any other presentation $Q' \to B$, we have an isomorphism $\mathrm{Tor}(A, B) \to H_1(A \otimes Q')$ as above.

Another consequence of the above lemma is that $\mathrm{Tor}(A, B)$ is a functor in B. We leave it to the student to check that. $\mathrm{Tor}(A, B)$ is also a functor in A in the obvious way. □

Warning. The arguments we employed above give short shrift to some tricky issues. Namely, we rely on certain naturality properities of the objects being studied without going into detail as to what those properties are and how they relate to particular points. Hence, the student should treat this development as a preliminary treatment to be done more thoroughly in a later course devoted specifically to homological algebra.

With the above definition, we may derive the remaining properties of the Tor functor.

Proposition 9.7. *$\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.*

Proof. Let $P \to A$ be a free presentation of A and $Q \to B$ a free presentation of B as above. Then $P \otimes Q$ is a chain complex also. We shall show that $H_1(P \otimes Q) \cong H_1(A \otimes Q) \cong \mathrm{Tor}(A, B)$ and similarly $H_1(P \otimes Q) \cong H_1(P \otimes B) \cong H_1(B \otimes P) \cong \mathrm{Tor}(B, A)$.

The relations between the complexes $P \otimes Q$, $A \otimes Q$, and $P \otimes B$ and $A \otimes B$ are given by the following diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & P_1 \otimes B & \longrightarrow & P_0 \otimes B & \longrightarrow & A \otimes B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_1 \otimes Q_0 & \longrightarrow & P_0 \otimes Q_0 & \longrightarrow & A \otimes Q_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_1 \otimes Q_1 & \longrightarrow & P_0 \otimes Q_1 & \longrightarrow & A \otimes Q_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The two vertical homomorphisms on the left yield a morphism $P \otimes Q \to P \otimes B$ of chain complexes, and some diagram chasing shows that it induces an isomorphism of H_1. □

PROPOSITION 9.8. *Let A be an abelian group. If A or B is torsion free, then* $\mathrm{Tor}(A, B) = 0$*.*

PROOF. We need only prove it for A by the commutativity of Tor.
If $Q \to B$ is a presentation, then

$$A \otimes Q_1 \to A \otimes Q_0$$

is an injection and $H_1(A \otimes Q) = 0$. □

PROPOSITION 9.9. Tor *preserves direct sums.*

PROOF. We need only prove it for A as above.
Let $A = A' \oplus A''$. We have as chain complexes $A \otimes Q \cong A' \otimes Q \oplus A'' \otimes Q$. □

PROPOSITION 9.10. *Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence. Then there is a natural connecting homomoprhism* $\mathrm{Tor}(A'', B) \to A' \otimes B$ *such that*

$$0 \to \mathrm{Tor}(A', B) \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A'', B) \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$$

is exact. Similarly for the roles of the arguments reversed.

PROOF. Again the commutativity of $\otimes$ and of Tor allows us to just prove the first assertion.

Let $Q \to B$ be a free presentation. Since Q is free,

$$0 \to A' \otimes Q \to A \otimes Q \to A'' \otimes Q \to 0$$

is an exact sequence of chain complexes. Hence, since each of these complexes has trivial homology for $k \neq 0, 1$, we get a long exact sequence

$$0 \to H_1(A' \otimes Q) \to H_1(A \otimes Q) \to H_1(A'' \otimes Q) \\ \to H_0(A' \otimes Q) \to H_0(A \otimes Q) \to H_0(A'' \otimes Q) \to 0.$$

Now replace H_0 by the tensor product and H_1 by Tor using suitable isomorphisms. $\square$

Note that this is the only place in the development where we use the fact that Q_1 is free.

Note. Because any finitely generated abelian group is a direct sum of cyclic groups, the above results allow us to calculate $\mathrm{Tor}(A, B)$ for any finitely generated abelian groups if we do so for cyclic groups. Use of the short exact sequence $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ allows us to conclude

$$\mathrm{Tor}(\mathbf{Z}/n\mathbf{Z}, B) \cong {}_nB = \{b \in B \mid nb = 0\}.$$

From this, it is not hard to see that

$$\mathrm{Tor}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}) \cong \mathbf{Z}/\gcd(n, m)\mathbf{Z}.$$

THEOREM 9.11. *Let C' and C'' be free chain complexes. Then there are natural short exact sequences*

$$0 \to \bigoplus_{r+s=n} H_r(C') \otimes H_s(C'') \xrightarrow{\times} H_n(C' \otimes C'') \to \bigoplus_{r+s=n} \mathrm{Tor}(H_r(C'), H_{s-1}(C'')) \to 0.$$

Moreover, each of these sequences splits, but not naturally.

PROOF. We saw previously that $H_{r+s}(C' \otimes C'')$ fits in a short exact sequence between sums of terms made up from the cokernels of the homomorphisms

$$\mathrm{Id} \otimes i_s : H_r(C') \otimes B_s(C'') \to H_r(C') \otimes Z_s(C'')$$

and the kernels of the homomorphisms

$$\mathrm{Id} \otimes i_{s-1} : H_r(C') \otimes B_{s-1}(C'') \to H_r(C') \otimes Z_{s-1}(C'').$$

For fixed r, s, the cokernel is the tensor product, and since

$$0 \to B_{s-1} \to Z_{s-1} \to H_{s-1} \to 0$$

is a free presentation of H_{s-1}, it follows that the kernel is the required Tor term.

To establish the splitting, argue as follows. Since $B(C')$ is free, the sequence

$$0 \to Z_r(C') \to C'_r \to B_{r-1}(C') \to 0$$

splits. Thus, there is a retraction $p'_r : C'_r \to Z_r(C')$. Similarly, there is a retraction $p''_s : C''_s \to Z_s(C'')$. Define $\overline{p} : C' \otimes C'' \to H_*(C') \otimes H_*(C'')$ by

$$\overline{p}(c' \otimes c'') = \overline{p'(c')} \otimes \overline{p''(c'')}.$$

Since $p'(\partial' c') = \partial' c'$ and $p''(\partial'' c'') = \partial'' c''$—because both are cycles—it follows that $\overline{p}$ takes boundaries in $C' \otimes C''$ to zero. Restrict $\overline{p}$ to $Z(C' \otimes C'')$. This yields a homomophism $H_*(C' \otimes C'') \to H_*(C') \otimes H_*(C'')$ which by direct calculation is seen to be a retraction of $\times$. □

Note. Except of the splitting, the Künneth Theorem is in fact true if either C' or C'' consists of torsion free components. Refer to any book on homological algebra, e.g., Hilton and Stammbach, for a proof.

4. Tensor and Tor for Other Rings

In algebra courses, you will study the tensor product over an arbitrary ring. For the case of a commutative ring K, we have the following additional relation in the tensor product $M \otimes_K N$ of two K-modules M, N.

$$rx \otimes y = x \otimes ry \qquad r \in K, x \in M, y \in N$$

For $K = \mathbf{Z}$, this condition follows from the biadditivity conditions. In the general case, these conditions, together with biadditivity are called *bilinearity*. The universal mapping property of the tensor product then holds for *bilinear* functions into an arbitrary K-module.

The theory of Tor_K is developed analagously for modules over a ring K, but it is much more involved. In the special case that K is a (commutative) principal ideal domain, then the theory proceeds exactly as in the case of $\mathbf{Z}$. (That is because every submodule of a free K-module is free for such rings.)

Similarly, we may define the concept of a chain complex C over a ring K by requiring all the components to be K-modules and the boundary homomorphisms to be K-homomorphisms. Also, we may define the tensor product $C' \otimes_K C''$ of two such complexes. If K is a principal ideal domain, the Künneth Theorem remains true, except that we need to put K as a subscript on $\otimes$ and Tor.

The most important case is that in which the ring K is a field. In this case, every module is free so $\mathrm{Tor}_K(M, N) = 0$ for all K-modules, i.e., vector spaces, M, N. Thus the Künneth Theorem takes the specially simple form

THEOREM 9.12. *Let C' and C'' be chain complexes over a field K. Then $\times$ provides an isomorphism*

$$H_*(C' \otimes_K C'') \cong H_*(C') \otimes_K H_*(C'').$$

The fields used most commonly in algebraic topology are $\mathbf{Q}$ (used already in the Lefshetz Fixed Point Theorem), $\mathbf{R}$, and the finite prime fields $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. Note that if A, B are vector spaces over $\mathbf{F}_p$ (which is the same as saying they are abelians groups with every non-zero element of order p), then $A \otimes_{\mathbf{F}_p} B \cong A \otimes_{\mathbf{Z}} B$. (Exercise.) However, $\mathrm{Tor}_{\mathbf{F}_p}(A, B) = 0$ while $\mathrm{Tor}_{\mathbf{Z}}(A, B) \neq 0$.

5. Homology with Coefficients

Before continuing with our study of the homology of products of spaces, we discuss a related matter for which the homological algebra is a special case of the development in the previous sections.

Recall that when discussing the Lefshetz Fixed Point Theorem, we considered the chain complex $C_*(K) \otimes \mathbf{Q}$. This is part of a more general concept. We shall discuss it in the context of singular theory, but it could be done also for simplicial or cellular theory. Let X be a space and A any abelian group. Define

$$H_*(X; A) = H_*(S_*(X) \otimes A).$$

This is called the homology of X *with coefficients in* A. Homology with coefficients has may advantages. Thus, if A has some additional structure, then that structure can usually be carried through to the homology with coefficeints in A. For example, suppose K is a field, then $S_*(X) \otimes K$ becomes a vector space under the action $a(\sigma \otimes b) = \sigma \otimes ab$. It is easy to see that the boundary homomorphism is a linear homomorphism, and it follows that $H_*(X; K)$ is also a vector space over K. (If K is a ring other than a field, the same analysis works, but we use the term 'module' instead of 'vector space'.) We made use of this idea implicitly in the case of simplicial homology when discussing the Lefshetz Theorem, because we could take traces and use other tools available for vector spaces.

5.1. The Universal Coefficient Theorem for Chain Complexes. We can do the same thing for any chain complex. Define $H_*(C; A) = H_*(C \otimes A)$. We have the following 'special case' of the Künneth Theorem.

THEOREM 9.13. *Let C be a free chain complex and A an abelian group. There are natural short exact sequences*

$$0 \to H_n(C) \otimes A \xrightarrow{i_n} H_n(C; A) \to \mathrm{Tor}(H_{n-1}(C), A) \to 0.$$

i_n is defined by $\overline{z} \otimes a \mapsto \overline{z \otimes a}$. These sequences split but not naturally.

Note. As in the case of the Künneth Theorem, this result is true in somewhat broader circumstances. See a book or course on homological algebra for details.

PROOF. It would seem that we could deduce this directly from the Künneth Theorem since we can treat A as a chain complex which is zero except in degree 0. Unfortunately, our statement of the Künneth Theorem did not have suffiently general hypotheses for this to hold. However, we can just use the same proof as follows. Let $0 \to P_1 \to P_0 \xrightarrow{j} A \to 0$ be a free presentation of A. Then

$$0 \to C \otimes P_0 \to C \otimes P_1 \to C \otimes A \to 0$$

is exact since C is free, and we get a long exact sequence

$$H_n(C \otimes P_0) \to H_n(C \otimes P_1) \to H_n(C \otimes A) \to H_{n-1}(C \otimes P_0) \to H_{n-1}(C \otimes P_1).$$

This puts $H_n(C \otimes A)$ in a short exact sequence with the cokernel of the left morphisms on the left and the kernel of the right morphism on the right. This is exactly the same situation as before, so we get exactly the same result.

The proof of splitting is done as before by using a retraction $C \to Z(C)$. ☐

COROLLARY 9.14. *Let X be a topological space, and A an abelian group. Then there is a natural short exact sequence*

$$0 \to H_*(X) \otimes A \to H_*(X; A) \to \mathrm{Tor}(H_*(X), A) \to 0$$

which splits but not naturally.

EXAMPLE 9.15. Let $X = \mathbf{R}P^n$. Then for each $k \leq n$, we have

$$H_k(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) \cong H_k(\mathbf{R}P^n) \otimes \mathbf{Z}/2\mathbf{Z} \oplus \mathrm{Tor}(H_{k-1}(\mathbf{R}P^n), \mathbf{Z}/2\mathbf{Z}).$$

Since $H_k(\mathbf{R}P^n)$ is zero for $k > 0$ even and either $\mathbf{Z}/2\mathbf{Z}$ for k odd, and since $\mathrm{Tor}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$, it follows that

$$H_k(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z} \qquad k = 1, \dots, n.$$

We may also study the universal coefficient theorem for chain complexes over an arbitrary commutative ring K. As above the most interesting case is that of a PID, and the most interesting subcase is that of a field. The univeral coefficient theorem reads as before. The results are as above except that we must use $\otimes_K$ and Tor_K. Also if we work over a field k, then $\mathrm{Tor}_K = 0$.

6. The Eilenberg-Zilber Theorem

It might be worth your while at this point to go back and review some of the basic definitions and notation for singular homology. Remember in particular that $[p_0, \dots, p_n]$ denoted the affine map of Δ^n in some Euclidean space sending $\mathbf{e}_i$ to p_i.

Let X, Y be spaces. As noted previously, we shall show that $S_*(X \times Y)$ and $S_*(X) \otimes S_*(Y)$ are chain homotopy equivalent. To this end, we need to define chain morphisms in both directions between them. In fact we can show the existence of such morphisms with the right properties by an abstract approach called the *method of acyclic models*. In so doing we don't actually have to write down any explicit formulas. However, we can define an explicit morphism

$$A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$$

fairly readily, and this will help us do some explicit calculations later.

First, note that any singular n-simplex $\sigma : \Delta^n \to X \times Y$ is completely specified by giving its component maps $\alpha : \Delta^n \to X$ and $\beta : \Delta^n \to Y$. We abbreviate this $\sigma = \alpha \times \beta$. Define

$$A(\sigma) = \sum_{r+s=n} \alpha \circ [\mathbf{e}_0, \dots, \mathbf{e}_r] \otimes \beta \circ [\mathbf{e}_r, \dots, \mathbf{e}_n].$$

$[\mathbf{e}_0, \dots, \mathbf{e}_r]$ is an affine r-simplex called the front r-face of Δ^n, and $[\mathbf{e}_r, \dots, \mathbf{e}_n]$ is an affine s-simplex called the back s-face of Δ^n. The morphism A is called the *Alexander–Whitney map*.

Proposition 9.16. *A is a chain morphism. Moreover, it is natural with respect to maps $X \to X'$ and $Y \to Y'$ of both arguments.*

Proof. The second assertion is fairly clear from the fomula and the diagram

The first assertion is left as an exercise for the student. $\square$

To get a morphism $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ we shall use a more abstract approach.

6.1. Acyclic Models. In some of the arguments we used earlier in this course, we defined chain maps inductively. A general algebraic result which often allows us to make such constructions is the following.

PROPOSITION 9.17. *Let C, C' be two (non-negative) chain complexes with C $\mathbf{Z}$-free and C' acyclic, i.e., $H_n(C') = 0$ for $n > 0$. Let $\phi_0 : H_0(C) \to H_0(C')$ be a homomorphism.*

(i) There is a chain morphism $\Phi : C \to C'$ such that $H_0(\Phi) = \phi_0$.

(ii) Any two such chain morphisms are chain homotopic.

COROLLARY 9.18. *If C, C' are free acyclic chain complexes with $H_0(C) \cong H_0(C')$, then C, C' are chain homotopy equivalent.*

PROOF. The proof is encapsulated in the following diagrams.

□

Note on the Proof. Note that by the inductive nature of the proof, we may assume that $\Phi_i, 0 \leq i < n$, with the desired property for Φ_0, have already been specified, and then we may continue defining Φ_i for $i \geq n$. Similarly, if Φ, Φ' are two such chain morphisms, and a partial chain homotopy has been specified between them, we may extend it.

Note that the above results would suffice to show that $S_*(X \times Y)$ is chain homotopy equivalent to $S_*(X) \otimes S_*(Y)$ for acyclic spaces. How to extend this to arbitrary spaces is the purpose of the theory of 'acyclic models'. The idea is to define the desired morphisms for acylic spaces inductively and the extend them to arbitrary spaces by naturality. However, the exact order in which the definitions are made is crucial. Also, the theory is stated in very great generality to be sure it applies in enough interesting cases. Hence, we need some general categorical 'nonsense' in order to proceed.

Let $\mathcal{T}$ and $\mathcal{A}$ be categories, and let $F, G : \mathcal{T} \to \mathcal{A}$ be functors. A *natural transformation* $\phi : F \to G$ is a collection of morphisms $\phi_X : F(X) \to G(X)$ in $\mathcal{A}$, one for each object X in $\mathcal{T}$ such that for each morphism $f : X \to Y$ in $\mathcal{T}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi_X} & G(X) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\phi_Y} & G(Y) \end{array}$$

commutes.

This is a formalization of the concept of 'natural homomorphism' or 'natural map', so it should be familiar to you. You should go back and examine the use of this term before. Sometimes the categories and functors are a bit tricky to identify.

It is clear how to define the composition of two natural transformations. The identity morphisms $F(X) \to F(X)$ provide a natural transformation of any functor to itself. A natural transformation is called a natural equivalence if it has an inverse in the obvious sense.

We now want to set up the context for the acyclic models theorem. To understand this context concentrate on the example of the category of topological spaces and the functor $S_*(X)$ to the category $\mathcal{C}h$ of chain complexes. (Recall that by convention a chain complex for us is zero in negative degrees.)

We assume there is given a set $\mathcal{M}$ of objects of $\mathcal{T}$ which we shall call *models*. A functor $F : \mathcal{T} \to \mathcal{C}h$ is said to be *acyclic with respect to* $\mathcal{M}$ if the chain complex $F(M)$ is acyclic for each M in $\mathcal{M}$. (That means $H_i(F(M)) = 0$ for $i \neq 0$.) Such a functor is called *free in degree n with respect to* $\mathcal{M}$ if there is given an indexed subset $\mathcal{M}^n = \{M_j\}_{j \in J_n}$ of models and elements $i_j^n \in F_n(M_j)$ for each $j \in J_n$, such that for every X, an object of $\mathcal{T}$, $F_n(X)$ is free on the basis consisting of the elements

$$\{F_n(f)(i_j^n) | \quad f \in \mathrm{Hom}_{\mathcal{T}}(M_j, X),\, j \in J_n\}$$

(which are assumed to be distinct). The functor F is called *free with respect to* $\mathcal{M}$ if it is free in each degree $n \geq 0$.

Example 9.19. Let $\mathcal{T}$ be the category of $\mathcal{T}op$ of topological spaces and continuous maps. Let $\mathcal{M}$ be the collection of standard simplices Δ^n. Then $S_*(-)$ is acyclic because $S_*(\Delta^n)$ is acylic for every standard simplex. It is also free. For let $\mathcal{M}^n = \{\Delta^n\}$ and let $i^n = \mathrm{Id} : \Delta^n \to \Delta^n$. Then $S_*(\sigma)(i^n) = \sigma$ is a singular n-simplex, and $S_n(X)$ is free on the set of singular simplices.

Example 9.20. Let $\mathcal{T}$ be the category of pairs (X, Y) of topological spaces and pairs of maps (f, g). (We don't assume $Y \subseteq X$. The components may be totally unrelated.) Let the model set be the set of pairs (Δ^r, Δ^s) of standard simplices. Consider first the the functor $S_*(X \times Y)$. This is acyclic because each of the spaces $\Delta^r \times \Delta^s$ is acylic. It is also free. For let $\mathcal{M}^n = \{(\Delta^n, \Delta^n)\}$ and let $i^{n,n}$ be the *diagonal* map $\Delta^n \to \Delta^n \times \Delta^n$ defined by $x \mapsto (x, x)$. Let $\sigma = \alpha \times \beta$ be a singular n-simplex in $X \times Y$. Then $\sigma = S_n(\alpha, \beta)(i^{n,n})$ so $S_n(X \times Y)$ is free as required.

Consider next the functor $S_*(-) \otimes S_*(-)$. This is acyclic because for standard simplices $H_i(\Delta^r) = 0$ for $r \neq 0$ and similarly for $H_j(\Delta^s) = 0$, and these are both $\mathbf{Z}$ in degree zero. We may now apply the Künneth Theorem for chain complexes to conclude that $S_*(\Delta^r) \otimes S_*(\Delta^s)$ is acyclic. This functor is also free. To see this, fix an $n \geq 0$. Let $\mathcal{M}^n = \{(\Delta^r, \Delta^s) \,|\, r + s = n\}$, and let $i^{r,s} = i^r \otimes i^s$. Then $F_n(X, Y) = \bigoplus_{r+s=n} S_r(X) \otimes S_s(Y)$ is free on the basis consisting of all $\alpha \otimes \beta$ where α is a singular r-simplex in X, β is an singular s- simplex in Y, and $r + s = n$. However, $\alpha \otimes \beta = S_r(\alpha) \otimes S_s(\beta)(i^{r,s})$.

THEOREM 9.21. *Let $\mathcal{T}$ be a category with a set of models $\mathcal{M}$. Let F and G be functors to the category of (non-negative) chain complexes. Suppose F is free with respect to $\mathcal{M}$ and G is acyclic with respect to $\mathcal{M}$. Suppose there is given a natural transformation of functors $\phi_0 : H_0 \circ F \to H_0 \circ G$. Then there is a natural transformation of functors $\Phi : F \to G$ which induces ϕ_0 is homology in degree zero. Moreover, any two such natural transformations Φ, Φ' are chain homotopic by a natural chain homotopy. That is, there exist natural transformations $D_n : F_n \to G_{n+1}$ such that for each X in $\mathcal{T}$, we have*

$$\Phi_{n,X} - \Phi'_{n,X} = \partial_{n+1}^{G(X)} \circ D_{n,X} + D_{n-1,X} \circ \partial_n^{F(X)}.$$

COROLLARY 9.22. *Let $\mathcal{T}$ be a category with models $\mathcal{M}$. Let F and G be functors to the category of (non-negative) chain complexes which are both acyclic an free with respect to $\mathcal{M}$. If $H_0 \circ F$ is naturally equivalent to $H_0 \circ G$, then F is naturally chain homotopically equivalent to G.*

We leave it to the student to work out exactly what the Corollary means and to prove it.

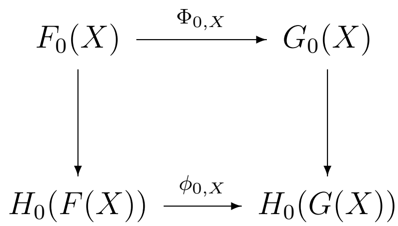
PROOF. By hypothesis, for each model M in $\mathcal{M}$, $F(M)$ is a free chain complex and $G(M)$ is acyclic. Also, $\phi_{0,M} : H_0(F(M)) \to H_0(G(M))$ is given. By the previous proposition, there is a chain morphism $\psi_M : F(M) \to G(M)$ which induces ϕ_0 in degree zero homology. For X in $\mathcal{T}$, $M_j, j \in J_0$, a model of degree 0, $f \in \mathrm{Hom}_{\mathcal{T}}(M_j, X)$, define

$$\Phi_{0,X}(F(f)(i_j^0)) = G_0(f)(\psi_{0,M_j}(i_j^0)) \in G_0(X).$$

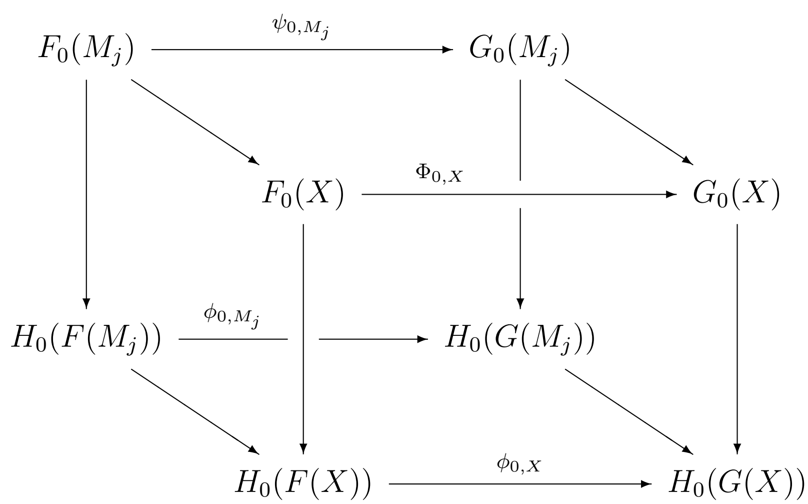
This specifies $\Phi_{0,X}$ on a basis for $F_0(X)$, so it yields a homomophism

$$\Phi_{0,X} : F_0(X) \to G_0(X).$$

It is not hard to see that the collection of these homomorphisms is a natural transformation $\Phi_0 : F_0 \to G_0$. It is also true that each of the diagrams



commutes. To see this note that for each M_j of degree 0, and each $f : M_j \to X$ in $\mathcal{T}$, there is a cubical diagram:



The back face commutes by the defining property of the top arrow. The two side faces of this cube commute because the morphism from cycles to homology is a natural transformation of functors on chain complexes. The bottom face commutes because ϕ_0 is a natural transformation of functors. The top face is not necessarily commutative but for the element i_j^0, it commutes by the definition of $\Phi_{0,X}$. This establishes that the front face commutes on a basis for the upper left corner $F_0(X)$, so it commutes.

Now suppose inductively that natural transformations $\Phi_i : F_i \to G_i$ have been defined which commute with the boundary homomorphisms for $0 \leq i < n$. Assume also that the morphisms $\psi_M : F(M) \to G(M)$ have been modified so that $\psi_{i,M_j} = \Phi_{i,M_j}, 0 \leq i < n$ for each model M_j

of degree n. For such an M_j and $f : M_j \to X$ in $\mathcal{T}$, define

$$\Phi_{n,X}(F_n(f)(i_j^n)) = G_n(f)(\psi_{n,M_j}(i_j^n))$$

and extend by linearity to get a homomorphism

$$\Phi_{n,X} : F_n(X) \to G_n(X).$$

As above, this defines a natural transformation of functors. Also,

$$\begin{array}{ccc} F_n(X) & \xrightarrow{\Phi_{n,X}} & G_n(X) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ F_{n-1}(X) & \xrightarrow{\Phi_{n-1,X}} & G_{n-1}(X) \end{array}$$

commutes by a three dimensional diagram chase as above. We leave the details to the student.

The arguments for chain homotopies are done essentially the same way. If $\Phi, \Phi' : F \to G$ both induce ϕ_0 in degree 0, then by the proposition there is a chain homotopy from Φ_M to Φ'_M for each model M. Using the value of this chain homotopy on i_j^0 for M_j, a model of degree 0, we may define a natural transformation $D_0 : F_0 \to G_1$ with the right property. This may then be lifted inductively to $D_n : F_n \to G_{n+1}$ as above. We leave the details to the student.

□

THEOREM 9.23. *The Alexander–Whitney chain morphism*

$$A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$$

is a chain homotopy equivalence. In particular, there is a natural chain morphism $B : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ such that $A \circ B$ and $B \circ A$ are both naturally chain homotopic to the respective identities.

PROOF. This follows directly from the acyclic models theorem with the categories and models specified in Example 9.20. In degree 0, the Alexander–Whitney morphism sends $\alpha \times \beta$ to $\alpha \otimes \beta$, and it is an isomorphism

$$S_0(X \times Y) \cong S_0(X) \otimes S_0(Y).$$

This in turn yields the natural isomorphism $H_0(S_*(X \times Y)) \to H_0(S_*(X) \otimes S_*(Y)) \cong H_0(S_*(X)) \otimes H_0(S_*(Y))$. □

THEOREM 9.24 (Künneth Theorem for Singular Homology). *Let X, Y be spaces. There are natural short exact sequences*

$$0 \to \bigoplus_{r+s=n} H_r(X) \otimes H_s(Y) \xrightarrow{\times} H_n(X \times Y) \to \bigoplus_{r+s=n-1} \mathrm{Tor}(H_r(X), H_s(Y)) \to 0.$$

These sequences split but not naturally.

EXAMPLE 9.25. Let $X = Y = \mathbf{R}P^2$. Then

$H_0(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathbf{Z}$

$H_1(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathbf{Z} \otimes \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \otimes \mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$

$H_2(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathbf{Z}/2\mathbf{Z} \otimes \mathbf{Z}/2\mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z}$

$H_3(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathrm{Tor}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}.$

Note that each of these isomorphisms is actually natural because in each case one term in the non-natural direct sum is trivial.

The acyclic models theorem works just as well for functors into chain complexes over a PID. In particular, if K is a field, we get the following stronger result.

THEOREM 9.26. *Let X, Y be spaces, and let K be a field. Then $\times$ provides a natural isomorphism*

$$H_*(X;K) \otimes_K H_*(Y;K) \cong H_*(X \times Y;K)$$

Note that although the chain complex Künneth Theorem is relatively simple in this case, working over a field does not materially simplify the Eilenberg–Zilber Theorem.