CHAPTER 1

Introduction

1. Introduction

Topology is the study of properties of topological spaces invariant under homeomorphisms. See Section 2 for a precise definition of topological space.

In algebraic topology, one tries to attach algebraic invariants to spaces and to maps of spaces which allow us to use algebra, which is usually simpler, rather than geometry. (But, the underlying motivation is to solve geometric problems.)

A simple example is the use of the Euler characteristic to distinguish closed surfaces. The Euler characteristic is defined as follows. Imagine the surface (say a sphere in $\mathbf{R}^3$) triangulated and let n_0 be the number of vertices, ... Then $\chi = n_0 - n_1 + n_2$. As the picture indicates, this is 2 for a sphere in $\mathbf{R}^3$ but it is 0 for a torus.

This analysis raises some questions. First, how do we know that the number so obtained does not depend on the way the surface is triangulated? Secondly, how do we know the number is a topological invariant?

Our approach will be to show that for reasonable spaces X, we can attach certain groups $H_n(X)$ (called homology groups), and that invariants b_n (called Betti numbers) associated with these groups can be used in the definition of the Euler characteristic. These groups don't depend on particular triangulations. Also, homeomorphic spaces have isomorphic homology groups, so the Betti numbers and the Euler characteristic are topological invariants. For example, for a 2-sphere, $b_0 = 1, b_1 = 0$, and $b_2 = 1$ and $b_0 - b_1 + b_2 = 2$.

Another more profound application of this concept is the Brouwer Fixed Point Theorem.

Let $D^n = \{x \in \mathbf{R}^n \mid |x| \leq 1\}$, and $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$.

THEOREM 1.1 (Brouwer). *Let $f : D^n \to D^n$ $(n \geq 1)$ be a continuous map. Then f has a fixed point, i.e., $\exists x \in D^n$ such that $f(x) = x$.*

PROOF. (modulo this course) Suppose $f : D^n \to D^n$ does not have a fixed point. Define $r : D^n \to S^{n-1}$ as follows. Extend the ray which starts at $f(x)$ and goes to x until it hits the boundary S^{n-1} of the closed ball D^n. Let that be $r(x)$. Note that this is well defined if there are no fixed points. It is also not hard to prove it is continuous. (Do it!) Finally, it has the property $r(x) = x$ for every point in S^{n-1}. (Such a map of a space into a subspace is called a *retraction*. We shall show using homology groups that such a map can't exist.

Some properties of homology theory that will be proved.

(i) $H_{n-1}(D^n) = 0$.

(ii) $H_{n-1}(S^{n-1}) = \mathbf{Z}$, the infinite cyclic group.

(iii) If $r : X \to Y$ is a continuous map of spaces, then for each k, there are induced group homomorphisms $r_k : H_k(X) \to H_k(Y)$.

(iv) Moreover, these group homomorphisms are consistent with composition of functions, i.e., $(r \circ s)_k = r_k \circ s_k$.

(v) The identity map $X \to X$ induces identity homomorphisms of homology groups.

Let $i : S^{n-1} \to D^n$ be the inclusion map. Then $r \circ i = \mathrm{Id}$. Thus

$$H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$$

is the identity homomorphism of $\mathbf{Z}$ which is inconsistent with the middle group being trivial. □

Note that since there are so many conceivable continuous maps, it is not at all clear (even for $n = 2$) on purely geometric grounds that there can't be a retraction r, although it seems intuitively reasonable that no such map can exist. However, by bringing in the homology groups, we reduce the issue to a question of whether a certain type of homomorphism can exist, and the answer to that question is much simpler, basically because there are many fewer homomorphisms between groups than maps between spaces, so it easier to tell the former apart than the latter.

2. Point Set Topology, Brief Review

A metric space is a set X with a real valued function $d(x, y)$ satisfying

(i) $d(x, y) = 0 \Leftrightarrow x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

Given such a space, one can define the concept of continuous function and a variety of other concepts such as compactness, connectedness, etc.

However, metric spaces are not sufficiently general since even in cases where there may be a metric function $d(x, y)$, it may not be apparent what it is. (Also, there are cases of interest where there is no such function.)

Example. The Klein bottle is often defined by a picture of the following type.

Here the two horizontal edges are *identified* in same the direction and the vertical edges are *identified* in opposite directions. Something homeomorphic to this space may be embedded in an appropriate $\mathbf{R}^n$ so using the metric inherited from that, it can be viewed as a metric space. However, that particular representation is hard to visualize.

We need then some other way to describe spaces without using a metric. We do that by means of open sets. In a metric space X, denote $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$ (open ball centered at x or radius ϵ.) A set is open if every point in the set is the center of some open ball contained in the set. Open sets have the following properties.

(i) $\emptyset$ and X are both open sets.
(ii) Any union whatsoever of open sets is open.
(iii) Any finite intersection of open sets is open.

Then we define a *topology* on a set X to be a collection of subsets (which will be called open sets) satisfying these three axioms. A *topological space* is a set, together with some topology on it. (Note that the same set can have many different topologies placed on it.)

Here is an example of a topological space which is not a metric space. Let $X = \{x, y\}$ be a set with two elements. Let the topology be the collection consisting of the following subsets of X: $\emptyset, \{x\}$, and $\{x, y\} = X$.

This can't be a metric space because it doesn't satisfy the following *Hausdorff separation property*: For any two points $x, y \in X$, there exist non-intersecting open sets U and V containing x and y respectively. (These are usually called open *neighborhoods* of the points.) It is easy to see that any metric space has this property, but clearly X in the above example doesn't have it.

A function $f : X \to Y$ of topological spaces is called continuous if the inverse image $f^{-1}(U)$ of every open set U is open. (Similarly for closed sets.) A continuous function $f : X \to Y$ is called a homeomorphism if there is a continuous function $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are respectively the identity maps of X and Y. This implies f is one-to-one and onto as a map of sets. Conversely, if f is one-to-one and onto, it has a set theoretic inverse $g = f^{-1}$, and f is homeomorphism when this inverse is continuous.

Any subset A of a topological space X becomes a topological space by taking as open sets all intersections of A with open sets of X. Then the inclusion map $i : A \to X$ is continuous.

The collection of all topological spaces and continuous maps of topological spaces forms what is called a category. This means among other things that the composition of two continuous functions is continuous and the identity map of any space is continuous. (Later we shall study the concept of category in more detail.) Another important category is the category of groups and homomorphisms of groups. The homology groups $H_n(X)$ together with induced maps $H_n(f) : H_n(X) \to H_n(Y)$ describe what is called a functor from one category to another. This also is a concept we shall investigate in great detail later.

There are various concepts defined for metric spaces which extend easily to topological spaces since they depend only on the concept open set.

A subset A of a topological space X is called compact if every covering of A by a union of open sets can be reduced to a finite subcovering which also covers A. (Some authors, e.g., Bourbaki, insist that a compact space also be Hausdorff.) Compact subsets of $\mathbf{R}^n$ are exactly the closed bounded subsets. (This is not necessarily true for any metric space.) Here are some other facts about compactness.

A closed subset of a compact space is always compact.

In a Hausdorff space, a compact subset is always closed.

If $f : X \to Y$ is continuous and $A \subset X$ is compact, then $f(A)$ is compact.

A topological space X is called connected if it cannot be decomposed $X = U \cup V$ into a disjoint union ($U \cap V = \emptyset$) of two non-empty open sets.

In $\mathbf{R}$, the connected subspaces are precisely the intervals. (See the exercises.)

If $f : X \to Y$ is continuous, and A is a connected subspace of X, then $f(A)$ is a connected subspace of Y.

A set X may have more than one topology, so it can be the underlying set of more than one topological space. In particular, a set X may always be given the discrete topology in which every set is open. (When is a space with the discrete topology Hausdorff? compact? connected?)

One of the important functors we shall describe is the fundamental group. For this purpose, we need a stronger notion than connectedness called path connectedness. To define this we need a preliminary notion. A path in X is a continuous function $\alpha : [0, 1] \to X$.

A space X is called path connected if any two points x, y may be connected by a path, i.e., there is a path α with $\alpha(0) = x$ and $\alpha(1) = y$.

Proposition 1.2. *A path connected space X is connected.*

Proof. Let $X = U \cup V$ be a decomposition into disjoint open sets. If neither is empty, pick $x \in U$ and $y \in V$, and pick a path α joining x to y.

Since $[0, 1]$ is connected, so is $A = \mathrm{Im}(\alpha)$. On the other hand, $A = (U \cup A) \cap (V \cup A)$ is a decomposition of A into disjoint open sets of A, and neither is empty, so that is a contradiction. □

A connected space is not necessarily path connected, but a locally path connected space which is connected is path connected.

If X, Y are topological spaces, then the Cartesian product $X \times Y$ (consisting of all pairs (x, y)) is made into a topological space as follows.

If U is open in X and V is open in Y, then $U \times V$ is open in $X \times Y$. Moreover, any union of such 'rectangular sets' is also taken to be open in $X \times Y$, and this gives the collection of all open sets. This is in fact the smallest topology which can be put on $X \times Y$ so that the projections $X \times Y \to X$ and $X \times Y \to Y$ are both continuous. A finite product of topological spaces is made into a topological space is an analogous manner. However, an infinite Cartesian product of topological spaces requires more care. (You should look that up if you don't know it.)

A product of two Hausdorff (respectively compact, connected, or path connected) spaces is Hausdorff (respectively, ...).

An n dimensional manifold is a Hausdorff space X with the property that each point $x \in X$ has an open neighborhood which is homeomorphic to an open ball in $\mathbf{R}^n$. Most of the spaces we are interested in algebraic topology are either manifolds or closely related to manifolds. For example, what we usually think of as a surface in $\mathbf{R}^3$ is a 2-manifold. However, there are 2-manifolds (e.g., the Klein bottle) which can't be embedded in $\mathbf{R}^3$ as subspaces.