

CHAPTER 7

Simplicial Complexes

1. Simplicial Complexes

Singular homology is defined for arbitrary spaces, but as we have seen it may be quite hard to calculate. Also, if the spaces are bizarre enough, the singular homology groups may not behave quite as one expects. For example, there are subspaces of \mathbf{R}^n which have non-zero singular homology groups in every dimension. We now want to devote attention to nicely behaved spaces and to derive alternate approaches to homology which give reasonable approaches. We start with the notion of a simplicial complex.

We need to clarify some issues which were previously ignored. The notation $[p_0, p_1, \dots, p_n]$ for an affine n -simplex in \mathbf{R}^N , assumes an implicit order for the vertices. This order played an important role in the development of the theory. Certainly, the order is essential if we interpret the notation as standing for the affine map $\Delta^n \rightarrow \mathbf{R}^N$ defined by $\mathbf{e}_i \mapsto p_i$. However, even without that, the order is used implicitly when numbering the $(n - 1)$ -dimensional faces of the simplex. Earlier, the notation took care of itself, so we didn't always have to distinguish between an affine simplex as a convex subset of a Euclidean space and an ordered affine simplex in which an order is specified for the vertices. In what follows, we shall have to be more careful. We shall use the term 'affine simplex' to mean the point set and 'ordered affine simplex' to mean the set together with an order for the vertices, or what is the same thing, the affine map $\Delta^n \rightarrow \mathbf{R}^N$.

In what follows, by the term 'face of an affine simplex', we shall mean any affine simplex spanned by a non-empty subset of its vertices, rather than specifically one of the faces in its boundary.

A simplicial complex K consists of a collection of affine simplices α in some \mathbf{R}^n which satisfy the following conditions.

- (1) Any face of a simplex in K is in K .
- (2) Two simplices in K either are disjoint or intersect in a common face.

Clearly, any affine simplex together with all its faces provide a collection of simplices which are a simplicial complex. Other examples are indicated in the diagram below.

We shall consider only finite simplicial complexes, i.e., ones made up of a finite number of simplices. Although this assumption is implicit, we shall usually state it explicitly in important theorems and propositions, but on occasion we will forget. A simplicial complex K will be called n -dimensional if n is the largest dimension of any simplex in K .

If K is a simplicial complex, we denote by $|K|$ the space which is the union of all the simplices in K . (Note that if K were not finite, there might be some ambiguity about what topology to use on $|K|$. For example, in \mathbf{R}^∞ one can consider the convex set spanned by the standard basis vectors $e_i, i = 0, 1, \dots$. In the finite case, $|K|$ will be a compact subset of some finite dimensional \mathbf{R}^N .)

A space X which is homeomorphic to $|K|$ for some finite simplicial complex K will be called a *polyhedron* and K will be called a *triangulation* of X . Note that a space can have more than one triangulation. For example, the 2-faces of a tetrahedron form a triangulation of S^2 , but by dividing each face of a cube into two triangles, we obtain a different triangulation.

More generally, if K is a simplicial complex, then the process of barycentric subdivision defined previously, when applied to the simplices in K produces another simplicial complex we shall denote $Sd(K)$. (You should review the definitions used in barycentric subdivision to see that the subdivision of a simplicial complex is again a simplicial complex.) Clearly, $|Sd(K)| = |K|$, but they are different simplicial complexes.

If K is a simplicial complex, let $vert(K)$ denote its set of vertices or 0-simplices.

If K and L are simplicial complexes, we say that L is a subcomplex of K if $L \subseteq K$ and $\text{vert}(L) \subseteq \text{vert}(K)$. Note that a subcomplex can have exactly the same vertices as the complex without being the same. For example, consider the boundary of a simplex.

If K and L are simplicial complexes, a morphism $\phi : K \rightarrow L$ is a function $\phi : \text{vert}(K) \rightarrow \text{vert}(L)$ such that if σ is a simplex of K spanned by the affinely independent set $\{p_0, \dots, p_m\}$, then the distinct elements of the set elements in the list $\phi(p_0), \dots, \phi(p_m)$ form an affinely independent set spanning a simplex $\phi(\sigma)$ in L . Note that ϕ need not be one-to-one, and $\phi(\sigma)$ could have lower dimension than σ .

The collection of simplicial complexes and morphisms of such forms a category we denote \mathcal{K} .

Given a morphism $\phi : K \rightarrow L$ of simplicial complexes, for each simplex, $p_i \mapsto \phi(p_i)$ defines a unique affine map of the affine simplex σ to $\phi(\sigma)$. This induces a piecewise-affine map of spaces $|\phi| : |K| \rightarrow |L|$. It is not hard to see that $|\cdot|$ is a functor from the category \mathcal{K} of simplicial complexes to the category Top of topological spaces.

Generally, given a space X , it may have no triangulation. (There are even n -manifolds for large n with that property.) Even for simplicial complexes, one can have a map $f : |K| \rightarrow |K|$, which is not the realization of a simplicial morphism. For example, S^1 may be triangulated, say as the edges of a triangle, but we can define a degree two map $S^1 \rightarrow S^1$ which won't take all vertices to vertices.

However, it turns out that, given a map f of polyhedra, *after sufficiently many subdivisions*, we can find a simplicial morphism ϕ such that $|\phi|$ is homotopic to f .

To prove this we must first introduce some notation. If σ is an affine p -simplex, let $\dot{\sigma}$ denote the boundary of σ and let $\overset{\circ}{\sigma} = \sigma - \dot{\sigma}$ denote the interior of σ *except* if $p = 0$ in which case let $\overset{\circ}{\sigma} = \sigma$.

Let K be a simplicial complex, and let p be a vertex of K .

$$St(p) = \bigcup_{p \in \text{vert}(\sigma), \sigma \in K} \overset{\circ}{\sigma}$$

is an open subset of $|K|$ called the *star* of p .

Let K and L be simplicial complexes, and let $f : |K| \rightarrow |L|$ be a map. A simplicial morphism $\phi : K \rightarrow L$ is called a *simplicial approximation* if for every vertex p of K we have $f(St(p)) \subseteq St(\phi(p))$.

Note first that if ϕ is a simplicial approximation for f , then for each $x \in |K|$, $|\phi|(x)$ and $f(x)$ are in a common simplex of L . For, choose a simplex σ of minimal dimension in K , such that $x \in \overset{\circ}{\sigma}$. Because simplices intersect only in faces, σ is unique. We claim that $\phi(\sigma)$ is a *face* of some simplex in L containing $f(x)$, (Of course, $|\phi(x)| \in \phi(\sigma)$, so that suffices.) The claim follows from the Lemma below since for each of the vertices p_i of σ we have $f(x) \in St(\phi(p_i))$.

LEMMA 7.1. *Let L be a simplicial complex, and let $\{q_0, \dots, q_r\}$ be a set of distinct vertices of L . If a point $y \in |L|$ lies in $\bigcap_{i=0}^r St(q_i)$, then there is a simplex τ in L such that $y \in \tau$, and $[[q_0, \dots, q_r]]$ is a face of τ .*

PROOF. Since $y \in St(q_i)$, there is a simplex τ_i in L with q_i a vertex and such that $y \in \overset{\circ}{\tau}_i$. However, as above, y belongs to $\overset{\circ}{\tau}$ for precisely one simplex τ in L . Hence, all the τ_i are the same simplex τ , and each q_i is a vertex of τ . Hence, the q_i span a face of τ as claimed. \square

It follows from the above discussion that if ϕ is a simplicial approximation to f , then $|\phi|$ is homotopic to f . For, since for every x , $f(x)$ and $|\phi|(x)$ always belong to a common convex subset of a Euclidean space, the formula

$$F(x, t) = tf(x) + (1 - t)|\phi|(x) \quad x \in |K|, 0 \leq t \leq 1$$

makes sense and defines a homotopy from one to the other.

THEOREM 7.2. *K and L be finite simplicial complexes, and let $f : |K| \rightarrow |L|$ be a map. Then there exists a simplicial approximation $\phi : Sd^k K \rightarrow L$ of f for some $k \geq 0$.*

PROOF. The sets $f^{-1}(St(q))$ for q a vertex of L form an open covering of $|K|$. Since K is finite, $|K|$ is compact. Hence, by the Lebesgue Covering Lemma, there is a $\delta > 0$ such that every subset of $|K|$ of diameter less than δ is carried by f into a star of some vertex of L . Choose k such that the mesh of $Sd^k \sigma$ is less than $\delta/2$ for each $\sigma \in K$. It follows easily that each $St(p)$ for p a vertex of $Sd^k K$ has diameter

less than δ , so $f(St(p)) \subseteq St(q)$ for some vertex q in L . Choose such a vertex and set $\phi(p) = q$.

This defines a map $\phi : \text{vert}(Sd^k K) \rightarrow \text{vert}(L)$. To complete the proof, we need to show that $\phi(\sigma)$ is a simplex in L whenever $\sigma = [[p_0, \dots, p_m]]$ is a simplex in $Sd^k K$. Let $x \in \sigma$. Then $x \in \cap_i St(p_i)$. Hence,

$$f(x) \in f(\cap_i St(p_i)) \subseteq \cap_i f(St(p_i)).$$

By the above Lemma, the distinct elements in the list $\phi(p_0), \dots, \phi(p_m)$ span a simplex in L which is a face of a simplex in L containing $f(x)$. It follows that $\phi(\sigma)$ is a simplex in L . \square

For any simplicial complex K , the collection of all simplices σ in K of dimension less than or equal to r is again a simplicial complex called the r -skeleton of K . (Check this for yourself.) We shall denote it $K^{(r)}$. Note $K^{(n)} = K$ if K is n -dimensional.

Suppose $\phi : K \rightarrow L$ is a simplicial morphism. If σ has dimension $\leq r$, then certainly $\phi(\sigma)$ has dimension $\leq r$. It follows easily that ϕ induces a simplicial morphism $\phi_r : K^{(r)} \rightarrow L^{(r)}$. Clearly,

$$|\phi|(|K^{(r)}|) \subseteq |L^{(r)}|.$$

THEOREM 7.3. *Any map $f : S^r \rightarrow S^n$ with $0 < r < n$ is null homotopic, i.e., homotopic to a trivial map.*

In the language of homotopy groups, this asserts $\pi_r(S^n) = 0$ for $0 < r < n$. This generalizes the fact that S^n is simply connected for $n \geq 2$.

PROOF. The difficulty is that f might be onto. Otherwise, the image of f is contained in a subspace of S^n homeomorphic to \mathbf{R}^n which is contractible. S^n is homeomorphic to $|L|$ where L is the simplicial complex obtained from the boundary of Δ^{n+1} . Similarly, S^r is homeomorphic to $|K|$ where K is the boundary of Δ^{r+1} . Choose a k for which there is a simplicial approximation $\phi : Sd^k K \rightarrow L$ to f . $|\phi|(|K|) \subseteq |L^{(r)}|$, so $|\phi|$ is not onto. Hence, $|\phi|$ is null homotopic. Since $f \sim |\phi|$, it follows that f is also null homotopic. \square

2. Abstract Simplicial Complexes

Let K be a finite simplicial complex. We think of K as a subset of a Euclidean space decomposed into simplices. However, all the relevant information about K is contained in knowledge of its set of vertices and which subsets of that set span simplices. This suggests the following more abstract approach.

An *abstract simplicial complex* K consists of a set V (called vertices) and a collection of distinguished non-empty finite subsets of V (called simplices) such that

- (a) every singleton subset $\{p\}$ of V is a simplex;
- (b) every non-empty subset of a simplex is also a simplex (called a face of the simplex).

Note that it follows that the intersection of two simplices is either empty or a face of both simplices.

A morphism of abstract simplicial complexes is a function from the set of vertices of the first to the set of vertices of the second which takes simplices to simplices. It is not hard to check that the collection of abstract simplicial complexes and morphisms of such forms a category.

As noted above, a simplicial complex K defines an abstract simplicial complex K^a in the obvious way. Similarly, a simplicial morphism $\phi : K \rightarrow L$ of simplicial complexes defines a morphism $\phi^a : K^a \rightarrow L^a$ in the obvious way. It is not hard to check that $(-)^a$ is a functor from the category of simplicial complexes to the category of abstract simplicial complexes.

There is a (non-functorial) way to go in the opposite direction. Let K be a finite abstract simplicial complex. If K' is a simplicial complex with $K'^a \cong K$, i.e., there are a pair of abstract simplicial morphisms relating $(K')^a$ and K which are inverses of one another. Then $|K'|$ is called a *geometric realization* of K . There is a standard way we may always construct such a geometric realization. Let $n+1$ be the number of vertices in K . Consider the simplicial complex L^n obtained from the standard simplex Δ^n and all its faces. Choose some one-to-one correspondence between the vertices of K and the vertices of Δ^n . This amounts to an ordering

$$v_0, v_1, \dots, v_n$$

of the vertices of K . Let K^g be the subcomplex of L^n consisting of those simplices

$$\sigma = |[e_{i_0}, \dots, e_{i_r}]|$$

such that $\{v_{i_0}, \dots, v_{i_r}\}$ is a simplex of K . It is clear that $|K^g|$ is a geometric realization of K .

It is not hard to see that any two geometric realizations of an abstract simplicial complex are homeomorphic. For suppose that $K_1^a \cong K_2^a$. Then we can define simplicial morphisms $\phi : K_1 \rightarrow K_2$ and $\psi : K_2 \rightarrow K_1$ in the obvious manner such that the compositions in both directions are the identity simplicial morphisms of K_1 and K_2 respectively. It follows that $|\phi| : |K_1| \rightarrow |K_2|$ and $|\psi| : |K_2| \rightarrow |K_1|$ are inverse maps of spaces, so the spaces are homeomorphic.

3. Homology of Simplicial Complexes

There are several possible ways to define homology for a simplicial complex. The difficulty is deciding which signs to attach to the faces of a simplex in the formula for the boundary. This requires some way to specify an orientation for each simplex. Fortunately, any reasonable method will result in homology groups isomorphic to the singular homology groups of the support of the simplicial complex, so it doesn't matter which we choose.

Let K be a simplicial complex (or an abstract simplicial complex). The naive thing to do would be to define $C_*(K)$ as the free abelian group with the set of simplices as basis. In fact this is pretty much what we will actually do. Unfortunately, if we just specify a simplex as by its set of vertices, we have no way to determine the signs for the faces in the formula for its boundary. To deal with this issue we must introduce some notion of *orientation*. Here is one approach, which is a little indirect, but has some conceptual advantages.

Let $O_n(K)$ be the free abelian group with basis the set of all symbols of the form $[p_0, p_1, \dots, p_n]$ where $\{p_0, p_1, \dots, p_n\}$ is the set of vertices of some simplex σ in K , but *we allow possible repetitions in the list*. Call such symbols *abstract ordered n -simplices*. Note that there are $(n+1)!$ abstract ordered n -simplices elements associated with each n simplex σ in K , but there are also many additional *degenerate* degenerate abstract ordered m -simplices with $m > n$. We can define a boundary homomorphism $\partial_n : O_n(K) \rightarrow O_{n-1}(K)$ in the usual way with

$$\partial_n[p_0, \dots, p_n] = \sum_{i=0}^n (-1)^i [p_0, \dots, \hat{p}_i, \dots, p_n]$$

for $n > 0$. As before, $\partial_{n-1} \circ \partial_n = 0$.

In $O_n(K)$, let $T_n(K)$ be the subgroup generated by all elements of the form

$$[p_0, \dots, p_n] \quad \text{if there is a repetition of vertices,}$$

and

$$[p_0, p_1, \dots, p_n] - s(\pi)[q_0, q_1, \dots, q_n] \quad \text{if there is no repetition of vertices,}$$

where $q_i = p_{\pi(i)}$, $i = 0, \dots, n$ for some permutation π of degree $n+1$, and where $s(\pi) = \pm 1$ is the sign of that permutation. It is not hard to check that $\partial_n(T_n) \subseteq T_{n-1}$. (See the Exercises)

Now let $C_n(K) = O_n(K)/T_n(K)$ and let $\partial_n : C_n(K) \rightarrow C_{n-1}(K)$ be the induced homomorphism, and again note that $\partial_{n-1} \circ \partial_n = 0$. Hence, we can define the group $Z_n(K) = \text{Ker } \partial_n$ of cycles, the group

$B_n(K) = \text{Im } \partial_{n+1}$ of boundaries, and the homology group $H_n(K) = Z_n(K)/B_n(K)$.

Let $\langle p_0, \dots, p_n \rangle$ denote the element of $C_n(K)$ represented by $[p_0, \dots, p_n]$. Then, if there is no repetition in the list, and we permute the vertices in the list, we either obtain the same element or its negative depending on whether the permutation is even or odd. ($\langle p_0, \dots, p_n \rangle$ is zero if there is a repetition in the list, so it seems a bit superfluous to bother with the degenerate abstract ordered simplices at all, but they will yield some technical advantages later.) It follows that we may form a basis for $C_n(K)$ by choosing for each n -simplex in K , one of these two possible representatives which are the same except for sign. This may be done explicitly as follows. Choose an ordering for the vertices of K with the property that the vertices of any simplex σ in K are linearly ordered

$$p_0 < p_1 < \dots < p_n.$$

Then, the element $\langle p_0, \dots, p_n \rangle$ of $C_n(K)$ represented by the abstract ordered simplex $[p_0, \dots, p_n]$ is the basis element corresponding to σ . Thus, the ordering of K yields a preferred orientation for each simplex, but different orderings may produce the same set of preferred orientations. Generally, if we were to choose some other acceptable ordering of the vertices of K , we would get another basis differing from the first in that some of the basis elements might be negatives of those determined by the first ordering. To make calculations, one would pick some ordering of the vertices. Then each basis element of $C_n(K)$ may be identified with the appropriate ordered simplex and we just use the usual formula for its boundary.

The simplicial homology groups defined this way have some advantages over the singular homology groups of $|K|$ in the case of a finite simplicial complex K . For, it is clear that the groups $C_n(K)$, $Z_n(K)$, $B_n(K)$, and $H_n(K)$ are all finitely generated abelian groups. In addition, if K is n -dimensional, it is clear that $H_r(K) = 0$ for $r > n$.

The most important disadvantage of simplicial homology is that it appears to depend on the complex rather than just the space $|K|$. However, we shall see below that simplicial homology gives the same result as singular homology for polyhedra.

Let $\phi : K \rightarrow L$ be a simplicial map. It is not hard to check that ϕ induces a chain morphism $O_*(K) \rightarrow O_*(L)$ taking $T_*(K)$ into $T_*(L)$. Hence, it induces a chain map $\phi_{\#} : C_*(K) \rightarrow C_*(L)$ which may be described on basis elements by

$$\phi_{\#} \langle p_0, \dots, p_n \rangle = \langle \phi(p_0), \dots, \phi(p_n) \rangle.$$

Here, according to our conventions the vertices on the left are in the proper order for K , but on the right we might have to introduce a sign in order to get the vertices in the proper order for L . If there is a repetition of vertices on the right, our conventions tell us the symbol stands for zero. This in turn induces homomorphisms $\phi_n : H_n(K) \rightarrow H_n(L)$. It is not hard to check that simplicial homology defined in this way is a functor on the category of simplicial complexes and simplicial morphisms.

One may also define *reduced simplicial homology* much as in the singular theory. Define $C_{-1}(K) = \mathbf{Z}$ and define a homomorphism $\tilde{\partial}_0 : C_0(K) \rightarrow C_{-1}(K)$ by $\sum_i n_i p_i \mapsto \sum_i n_i$. Then let $\tilde{H}_0(K) = \text{Ker } \tilde{\partial}_0 / \text{Im } \partial_1$. This is also a functor as above.

Call a simplicial complex simplicially connected if given any two vertices, there is a sequence of intermediate vertices such that successive pairs of vertices are the 0-faces of 1-simplices in the complex. Using this notion, one could define simplicial components of a simplicial complex. It is not hard to see that for a finite complex, these notions coincide with the notions of path connectedness and path components for the support of the complex. It is also easy to see that $H_0(K)$ is the free group with the set of components as basis, and analogously for $\tilde{H}_0(K)$.

Relative homology is also defined as in the singular case. Let K be a simplicial complex, appropriately ordered in some way, and let L be a subcomplex with the inherited order. Then $C_*(L)$ may be identified with a subcomplex of the chain complex $C_*(K)$ and we may define the relative chain complex $C_*(K, L) = C_*(K)/C_*(L)$, and $H_*(K, L)$ is defined to be its homology. It follows from basic homological algebra that we have connecting homomorphisms and a long exact sequence

$$\cdots \rightarrow H_n(L) \rightarrow H_n(K) \rightarrow H_n(K, L) \rightarrow H_{n-1}(L) \rightarrow \cdots$$

and similarly for reduced homology. Moreover, the connecting homomorphisms are natural with respect to simplicial morphisms of pairs (K, L) of simplicial complexes. The proofs of these facts parallel what we did before in singular theory and we shall omit them here.

Note that everything we did above could just as well have been done for finite abstract simplicial complexes.

3.1. An Example. Use the diagram below to specify an abstract simplicial complex with geometric realization homeomorphic to a 2-torus. Note that an actual geometric simplicial complex would in fact have to be imbedded in a higher dimensional space than \mathbf{R}^2 , say in \mathbf{R}^3 . Call such a simplicial complex K .

Note that K has 9 vertices (0-simplices), 27 1-simplices, and 18 2-simplices. The numbering of the vertices indicates an acceptable ordering, and the arrows show induced orientations for the 1-simplices. For the 2-simplices, the arrows indicate the simplex should be counted with \pm according to whether they are consistent with the ordering or not.

$H_0(K) = \mathbf{Z}$ since the complex is connected. We next use a mixture of theory and explicit calculation to see that $H_1(K) = \mathbf{Z} \oplus \mathbf{Z}$ and $H_2(K) = \mathbf{Z}$ as we would expect from our calculations in the singular case. Let L be the subcomplex of K consisting of the 6 1-simplices on the edges and their vertices. We first determine the homology groups of L . (Of course, $|L|$ is homomorphic to a wedge $S^1 \vee S^1$.) $H_n(L) = 0$ for $n > 1$ and $\tilde{H}_0(L) = 0$. To compute $H_1(L)$ consider the two cycles which are the sums of the 1-simplices on either horizontal or vertical edges. Since $B_1(L) = 0$, these form an independent pair. However, it is easy to see that any 1-chain is a one cycle if and only if it is a linear combination of these two 1-cycles. For, if two adjacent 1-simplices abut in any vertex but that numbered 1, they must have the same coefficient or else some multiple of the common vertex would be non-zero in the boundary. It follows that $H_1(L) = \mathbf{Z} \oplus \mathbf{Z}$.

We next calculate $H_1(K, L)$ and $H_2(K, L)$.

The second is easier. Clearly, the sum of all the 2-simplices (with the appropriate signs as discussed above) represents a 2-cycle in $Z_2(K, L)$. By considering adjacent 2-simplices *inside the square* it is clear that they must have the same coefficient in any 2-cycle in $Z_2(K, L)$ or else the boundary would contain their common edge with a non-zero coefficient. Hence, $H_2(K, L) = \mathbf{Z}$. However, the boundary of the basic 2-cycle in $Z_2(K, L)$ is zero, so the homomorphism $H_2(K, L) \rightarrow H_1(L)$ is trivial.

Now consider $H_1(K, L)$. The diagram below suggests a sequence of reductions to subcomplexes of K containing L . If we start with a 1-cycle modulo $C_1(L)$, and reduce modulo boundaries, we can reduce appropriate 1-simplices to chains in a smaller complex. Any time we have a 1-simplex ‘hanging’ with a vertex not in L exposed, we can drop it since the boundary in that case would contain some multiple of that vertex. Hence, the original chain could not have been a 1-cycle. At the end of this sequence, we end up in $Z_1(L, L) = 0$. Hence, $H_1(K, L) = 0$.

We may now compute the homology of K from the long exact sequence

$$0 \rightarrow H_2(K) \rightarrow H_2(K, L) = \mathbf{Z} \xrightarrow{0} H_1(L) = \mathbf{Z} \oplus \mathbf{Z} \rightarrow H_1(K) \rightarrow H_1(K, L) = 0$$

to obtain

$$(1) \quad H_1(K) = \mathbf{Z} \oplus \mathbf{Z}$$

$$(2) \quad H_2(K) = \mathbf{Z}$$

The argument also shows that the two basic 1-cycles in L map to a basis of $H_1(K)$ and the sum of all the 2-simplices generates $H_2(K)$.

The student should check that these results are consistent with the ranks of $C_n(K)$, $n = 0, 1, 2$. Note that a direct brute force calculation of the homology groups using the formulas for the boundaries would have been quite horrendous. Fortunately, the geometry helps us organize the calculation. Note also that the calculation above could have been done directly in $C_*(K)$ with appropriate modifications. (Try it yourself!) We used the machinery of the long exact sequence because it illuminates the calculation somewhat, but it isn't really necessary.

4. The Relation of Simplicial to Singular Homology

Let K be a finite simplicial complex, and suppose that we choose an acceptable ordering of its vertices. We define a chain morphism $h_{\#} : C_*(K) \rightarrow S_*(|K|)$ as follows. For σ an n -simplex in K , let $[p_0, \dots, p_1]$ be the ordered affine simplex corresponding to it for the specified order. Let $h_{\#}(\sigma)$ be the same ordered affine simplex viewed now as a singular simplex (i.e., an affine map $\Delta^n \rightarrow |K|$) in $|K|$. Because the boundaries are defined the same way, it is easy to see that $h_{\#}$ is in fact a chain morphism. Hence, it induces homomorphisms $h_n : H_n(K) \rightarrow H_n(|K|)$. Using the corresponding reduced complexes, we also get $\tilde{h}_0 : \tilde{H}_0(K) \rightarrow \tilde{H}_0(|K|)$ in dimension 0. It is the purpose of this section to show that these homomorphisms are isomorphisms. These isomorphisms appear to depend on the ordering of the vertices of K , but in fact they do not so depend, as we shall see later.

To establish that h_* is an isomorphism, we need to verify some of the 'axioms' we described in singular theory for simplicial theory.

PROPOSITION 7.4. *Let T^n denote the simplicial complex consisting of an affine n -simplex and all its faces. Then $H_q(T^n) = 0$ for $n > 0$, and $H_0(T^n) = \mathbf{Z}$.*

PROOF. We shall prove this by constructing a contracting homotopy for the chain complex $C_*(K)$. Assume a specified ordering $p_0 <$

$p_1 < \cdots < p_n$ for the vertices of the underlying affine n -simplex. Define a homomorphism $t_k : C_k(T^n) \rightarrow C_{k+1}(T^n)$ by

$$t_k \langle q_0, \dots, q_k \rangle = \langle p_0, q_0, \dots, q_k \rangle \quad k \geq 0,$$

where $q_0 < \cdots < q_k$ is the set of vertices (in proper order) of some k -face of the n -simplex if $k \geq 0$ and set $t_k = 0$ otherwise. Note that on the right hand side, $p_0 \leq q_0$ is necessarily true and the result is zero if they are equal. We have for $k > 0$,

$$(3) \quad \partial_{k+1}(\langle p_0, q_0, \dots, q_k \rangle) = \langle q_0, \dots, q_k \rangle - \sum_{i=0}^k \langle p_0, q_0, \dots, \hat{q}_i, \dots, q_k \rangle$$

$$(4) \quad = \langle q_0, \dots, q_k \rangle - t_{k-1}(\partial_k \langle q_0, \dots, q_k \rangle).$$

For $k = 0$, the calculation yields

$$(5) \quad \partial_1 \langle p_0, q_0 \rangle = q_0 - p_0$$

$$(6) \quad = q_0 - p_0 - t_{-1}(\partial_0 q_0).$$

Let ϵ denote the simplicial morphism of T^n which sends all vertices to p_0 . Clearly, $\epsilon_k = 0$ in homology for $k > 0$, and the above formulas show that the homomorphisms t_k form a chain homotopy of $C_*(K)$ from $\epsilon_{\#}$ to $\text{Id}_{\#}$. \square

Let K be a simplicial complex, K_1, K_2 subcomplexes. The set of simplices common to both is denoted $K_1 \cap K_2$ and it is easy to see that it is a subcomplex. Similarly, the set of simplices in one or the other is denoted $K_1 \cup K_2$. As in the case of singular homology, we have homomorphisms

$$(7) \quad i_{1*} \oplus -i_{2*} : H_*(K_1 \cap K_2) \rightarrow H_*(K_1) \oplus H_*(K_2)$$

$$(8) \quad j_{1*} + j_{2*} : H_*(K_1) \oplus H_*(K_2) \rightarrow H_*(K).$$

THEOREM 7.5. *be a finite simplicial complexes, K_1 and K_2 subcomplexes such that $K = K_1 \cap K_2$. Then there are homomorphisms $\partial_n : H_n(K) \rightarrow H_{n-1}(K_1 \cap K_2)$ which are natural with respect to triples K, K_1, K_2 ($K = K_1 \cup K_2$) and such that*

$$\cdots \rightarrow H_n(K_1 \cap K_2) \rightarrow H_n(K_1) \oplus H_n(K_2) \rightarrow H_n(K) \rightarrow H_{n-1}(K_1 \cap K_2) \rightarrow \cdots$$

is exact. If $K_1 \cap K_2 \neq \emptyset$, then the corresponding sequence in reduced homology is also exact.

PROOF. We have a short exact sequence of chain complexes

$$0 \rightarrow C_*(K_1 \cap K_2) \rightarrow C_*(K_1) \oplus C_*(K_2) \rightarrow C_*(K_1) + C_*(K_2) = C_*(K) \rightarrow 0.$$

We leave the chain morphisms for you to invent, after considering the corresponding morphisms in the singular case. This induces the desired long exact sequence. \square

There is also a simplicial version of excision. Since we don't have the notion of interior and closure just with the category of simplicial complexes, this 'axiom' must be stated a bit differently, but it is very easy to prove. (See the Exercises.)

Our strategy in relating simplicial homology to singular homology is to proceed by induction on the number of simplices in K . For zero dimensional simplicial complexes, it is clear that h_* is an isomorphism, so we suppose K has dimension greater than zero. We choose a simplex σ of maximal dimension, and we let K_1 be the subcomplex of K all faces of σ let K_2 be the subcomplex of all simplices in K with the exception of σ . If $K_1 = K$, then $H_n(K) = 0$ for $n > 0$ and $H_0(K) = 0$, and similarly for the singular homology of $|K|$. It is easy to check in this case that h_* must be an isomorphism. Suppose instead that K does not consist of σ and its faces. Then $K = K_1 \cup K_2$, where K_1, K_2 , and $K_1 \cap K_2$ all have fewer simplices than K , and we have a Mayer–Vietoris sequence for K, K_1, K_2 . If we can show that we also have a Mayer–Vietoris sequence

$$\rightarrow H_n(|K_1| \cap |K_2|) \rightarrow H_n(|K_1|) \oplus H_n(|K_2|) \rightarrow H_n(|K|) \rightarrow H_{n-1}(|K_1| \cap |K_2|) \rightarrow \dots,$$

then by use of induction and the five-lemma, we may conclude that $H_n : H_n(K) \rightarrow H_n(|K|)$ is an isomorphism.

Unfortunately, $|K| = |K_1| \cup |K_2|$ is not a covering with the property that the interiors of the subspaces cover $|K|$. Hence, we must use a 'fattening' argument to get the desired sequence. Let $U_2 = |K_2| \cup (\sigma - \{b\})$ where b is the barycenter of σ . Since $|K_1| \cap |K_2| = \dot{\sigma}$ is a deformation retract of $\sigma - \{b\}$, it follows that $|K_2|$ is a deformation retract of U_2 . Also, $|K|$ is the union of the open set U_2 and the interior of $|K_1| \xrightarrow{\circ} \sigma$. Finally, $|K_1| \cap U_2 = \sigma - \{b\}$.

It follows that we have a Mayer–Vietoris sequence

$$\rightarrow H_n(\sigma - \{b\}) \rightarrow H_n(|K_1|) \oplus H_n(U_2) \rightarrow H_n(|K|) \rightarrow H_{n-1}(\sigma - \{b\}) \rightarrow .$$

Now consider the diagram

$$\begin{array}{ccccccc}
 H_n(|K_1| \cap |K_2|) & \longrightarrow & H_n(|K_1|) \oplus H_n(|K_2|) & \longrightarrow & H_n(|K|) & \xrightarrow{?} & H_{n-1}(|K_1| \cap |K_2|) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_n(\sigma - \{b\}) & \longrightarrow & H_n(|K_1|) \oplus H_n(U_2) & \longrightarrow & H_n(|K|) & \longrightarrow & H_{n-1}(\sigma - \{b\})
 \end{array}$$

where the morphism ‘?’ has not been defined. Each vertical homomorphism is an isomorphism induced by inclusion. It follows that we may use the diagram to define the homomorphism ‘?’ and the resulting sequence is exact.

To complete the proof, we need to consider the diagram

$$\begin{array}{ccccccc}
 H_n(K_1 \cap K_2) & \longrightarrow & H_n(K_1) \oplus H_n(K_2) & \longrightarrow & H_n(K) & \longrightarrow & H_{n-1}(K_1 \cap K_2) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_n(|K_1| \cap |K_2|) & \longrightarrow & H_n(|K_1|) \oplus H_n(|K_2|) & \longrightarrow & H_n(|K|) & \longrightarrow & H_{n-1}(|K_1| \cap |K_2|)
 \end{array}$$

As discussed above, we may assume inductively that all the vertical homomorphisms—except the desired one—are isomorphisms, so the five lemma will give the desired result, provided we know the diagram commutes. Consider the following diagram of chain complexes.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_*(K_1 \cap K_2) & \longrightarrow & C_*(K_1) \oplus C_*(K_2) & \longrightarrow & C_*(K) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S_*(|K_1| \cap U_2) & \longrightarrow & S_*(|K_1|) \oplus S_*(U_2) & \longrightarrow & S^u(|K|) \longrightarrow \cdots
 \end{array}$$

where each sequence may be used to derive the appropriate Mayer-Vietoris sequence. Note that under h_* , it is in fact true that $C_*(K)$ maps to $S^u(|K|)$. This shows that the diagram with ‘fattened’ objects on the bottom row commutes. We leave it to the student to verify that replacing the ‘fattened’ row by the original row still yields a commutative diagram.

We have now proved

THEOREM 7.6. *The homomorphism $h_* : H_*(K) \rightarrow H_*(|K|)$ is an isomorphism for every finite simplicial complex K .*

There is one point which was left unresolved. Namely, the homomorphism h_* appeared to depend on the order chosen for K . In fact it does not so depend. This may be proved by considering the homology of the chain complex $O_*(K)$. Namely, there is an obvious chain map $h'_\# : O_*(K) \rightarrow S_*(|K|)$; just map an ordered affine simplex (even with repetitions) to itself viewed as a singular simplex in $|K|$. We clearly have a factorization

Hence, to show that h_* does not depend on the ordering, it would suffice to show that the chain morphism $O_*(K) \rightarrow C_*(K)$ induces an isomorphism in homology. We won't do this in this course. If you want to see a proof, look in one of the texts on algebraic topology such as Munkres, Rotman, or Spanier. In any event, an important consequence of this result is the following

PROPOSITION 7.7. *The homomorphism h_* is natural.*

The above arguments would have worked just as well for reduced homology. For relative homology, we may use the Five Lemma as follows. Let L be a subcomplex of K . $h'_\#$ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(L) & \longrightarrow & C_*(K) & \longrightarrow & C_*(K, L) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_*(|L|) & \longrightarrow & S_*(|K|) & \longrightarrow & S_*(|K|, |L|) & \longrightarrow & 0 \end{array}$$

which in turn yields the commutative diagram

$$\begin{array}{ccccccccc} H_n(L) & \longrightarrow & H_n(K) & \longrightarrow & H_n(K, L) & \longrightarrow & H_{n-1}(L) & \longrightarrow & H_{n-1}(K) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(|L|) & \longrightarrow & H_n(|K|) & \longrightarrow & H_n(|K|, |L|) & \longrightarrow & H_{n-1}(|L|) & \longrightarrow & H_{n-1}(|K|) \end{array}$$

Now use the Five Lemma to conclude

COROLLARY 7.8. *Let K be a finite simplicial complex and L a subcomplex. Then we have a natural isomorphism $H_*(K, L) \rightarrow H_*(|K|, |L|)$.*

5. Some Algebra. The Tensor Product

Let A and B be abelian groups. We shall define a group $A \otimes B$, called the *tensor product*. The tensor product plays an important role in what is called multilinear algebra. It is useful in subjects ranging from algebraic topology to differential geometry.

Here is the definition. Let $F(A, B)$ be the free abelian group with basis the set $A \times B$, i.e., the set of pairs (a, b) with $a \in A, b \in B$. In $F(A, B)$ consider the subgroup $T(A, B)$ generated by all elements of the form

$$(9) \quad (a_1 + a_2, b) - (a_1, b) - (a_2, b) \quad a_1, a_2 \in A, b \in B$$

$$(10) \quad (a, b_1 + b_2) - (a, b_1) - (a, b_2) \quad a \in A, b_1, b_2 \in B$$

Then, define $A \otimes B = F(A, B)/T(A, B)$.

Denote by $a \otimes b$ the element of $A \otimes B$ represented by $(a, b) \in A \times B$. Note that the elements $a \otimes b$ generate $A \otimes B$, but not every element is of that form. Generally, an element will be expressible $\sum_{i=1}^k a_i \otimes b_i$, but not necessarily in a unique way.

What we have done is to force the relations

$$(11) \quad (a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

$$(12) \quad a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

in $A \otimes B$. In fact, $A \otimes B$ is the largest possible group in which such relations hold. To explain this, first define $p : A \times B \rightarrow A \otimes B$ by $p(a, b) = a \otimes b$. Then the above formulas may be rewritten

$$(13) \quad p(a_1 + a_2, b) = p(a_1, b) + p(a_2, b)$$

$$(14) \quad p(a, b_1 + b_2) = p(a, b_1) + p(a, b_2).$$

A function $g : A \times B \rightarrow C$ is called *biadditive* if it satisfies these conditions, so we may say instead that p is biadditive.

PROPOSITION 7.9. *Let A and B be abelian groups. If $g : A \times B \rightarrow C$ is a biadditive function, then there exists a unique group homomorphism $G : A \otimes B \rightarrow C$ such that the diagram*

$$\begin{array}{ccc} A \times B & \xrightarrow{p} & A \otimes B \\ & \searrow g & \swarrow G \\ & & C \end{array}$$

commutes.

PROOF. There exists such a homomorphism. For, g induces a homomorphism $g_1 : F(A, B) \rightarrow C$, namely, $g_1(a, b) = g(a, b)$ specifies g_1 on basis elements. Since

$$(15) \quad g(a_1 + a_2, b) - g(a_1, b) - g(a_2, b) = 0$$

$$(16) \quad g(a, b_1 + b_2) - g(a, b_1) - g(a, b_2) = 0$$

it follows that $g_1(T(A, B)) = 0$. Thus, g_1 induces $G : A \otimes B = F(A, B)/T(A, B) \rightarrow C$ which clearly has the right properties.

Any such G is unique. For,

$$G(a \otimes b) = G(p(a, b)) = g(a, b)$$

determines G on a generating set for $A \otimes B$. □

The tensor product is a bit difficult to get at directly. The explicit definition starts with a very large free group, and the set of relations—i.e., the subgroup $T(A, B)$ —is not particularly easy to compute with. Hence, one must usually determine tensor products from their properties. The most fundamental property is the universal mapping property asserted in the above Proposition.

5.1. Example. We claim that $\mathbf{Z}/n\mathbf{Z} \otimes \mathbf{Z}/m\mathbf{Z} = 0$ if $(n, m) = 1$. To see this, note that

$$(17) \quad n(a \otimes b) = a \otimes b + \cdots + a \otimes b \quad n \text{ times}$$

$$(18) \quad = (a + \cdots + a) \otimes b = (na) \otimes b = 0 \otimes b = 0.$$

(Can you prove $0 \otimes b = 0$?) Similarly, $m(a \otimes b) = 0$. Hence, both n and m kill every generator of $A \otimes B$, so they both kill $A \otimes B$. Since $1 = nr + ms$, it follows that every element of $A \otimes B$ is zero.

PROPOSITION 7.10. *For any group A , we have $A \otimes \mathbf{Z} \cong A$.*

PROOF. Define $A \times \mathbf{Z} \rightarrow A$ by $(a, n) \mapsto na$. It is clear that this is biadditive. Hence, it induces $j : A \otimes \mathbf{Z} \rightarrow A$ such that $j(a \otimes n) = j(p(a, b)) = na$. Define a homomorphism $i : A \rightarrow A \otimes \mathbf{Z}$ by $i(a) = a \otimes 1$. (It is not hard to check this is a homomorphism.) Some simple calculation shows that i and j are inverses. □

Note that the argument used in the above proof is often abbreviated as follows. ‘Define $j : A \otimes \mathbf{Z} \rightarrow A$ by $j(a \otimes n) = na$ ’. For this to make sense, there has to be a ‘overlying’ biadditive map, but in writing ‘ $j(a \otimes n) = na$ ’, it is presumed that the reader has checked the biadditivity of what is on the right, so the formula makes sense. Of course, we can’t in general specify $j(a \otimes b)$ in an arbitrary way and get a well defined homomorphism. The elements $a \otimes b$ form a set

of generators for $A \otimes B$, but there are many relations among these generators which may not be preserved by what we want to set $j(a \otimes b)$ to.

PROPOSITION 7.11. *If A and B are abelian groups, then $A \otimes B \cong B \otimes A$.*

PROOF. Use $a \otimes b \mapsto b \otimes a$. Why does this work? \square

The tensor product is our first example of a *bifunctor*. Namely, suppose $f : A \rightarrow A'$ and $g : B \rightarrow B'$ are homomorphisms of abelian groups. Then, it is easy to check that the map $A \times B \rightarrow A' \otimes B$ defined by $(a, b) \rightarrow f(a) \otimes g(b)$ is biadditive, so it induces a homomorphism $A \otimes B \rightarrow A' \otimes B'$ which is denoted $f \otimes g$. This homomorphism takes $a \otimes b = p(a, b)$ to the same $f(a) \otimes g(b)$, so $f \otimes g$ is characterized by the property

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b) \quad a \in A, b \in B.$$

We leave it to the student to check that this is all functorial, i.e.,

$$(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2)$$

whenever it makes sense.

The tensor product is also an example of an additive functor, i.e., it is consistent with direct sums.

PROPOSITION 7.12. *Let $A_i, i \in I$ be a collection of abelian groups indexed by some set I and let B be an abelian group. Then*

$$\left(\bigoplus_i A_i\right) \otimes B \cong \bigoplus_i (A_i \otimes B).$$

PROOF. We have inverse homomorphisms

$$(19) \quad (a_i)_{i \in I} \otimes b \mapsto (a_i \otimes b)_{i \in I}$$

$$(20) \quad (a_i \otimes b)_{i \in I} \mapsto (a_i)_{i \in I} \otimes b$$

(Why are these well defined? You should think it out carefully. You might find it easier to understand if you consider the case where there are only two summands: $A = A' \oplus A''$, and each element of the direct sum is a pair (a', a'') .) \square

Tensor products *in certain cases* preserve exact sequences. First, we always have the following partial result.

THEOREM 7.13. *Suppose*

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$$

is an exact sequence of abelian groups and B is an abelian group. Then

$$A' \otimes B \xrightarrow{i \otimes \text{Id}} A \otimes B \xrightarrow{j \otimes \text{Id}} A'' \otimes B \longrightarrow 0$$

is exact.

This property is called *right exactness*.

PROOF. (i) $j \otimes \text{Id}$ is an epimorphism. For, given a generator $a'' \otimes b$, it is the image of $a \otimes b$, where $j(a) = a''$.

(ii) $\text{Ker}(j \otimes \text{Id}) \supseteq \text{Im}(i \otimes \text{Id})$. For, $(j \otimes \text{Id}) \circ (i \otimes \text{Id}) = j \circ i \otimes \text{Id} = 0 \otimes \text{Id} = 0$.

(iii) $\text{Ker}(j \otimes \text{Id}) \subseteq \text{Im}(i \otimes \text{Id}) = I$. To see this, first define a homomorphism $j_1 : A'' \otimes B \rightarrow (A \otimes B)/I$ as follows. Define a function $A'' \times B \rightarrow (A \otimes B)/I$ by

$$(a'', b) \mapsto \overline{a \otimes b} \in (A \otimes B)/I$$

where $j(a) = a''$. This is a well defined map, since if $j(a_1) = j(a_2) = a''$, we have $j(a_1 - a_2) = 0$, so by the exactness of the original sequence, $a_1 - a_2 = i(a')$. Hence,

$$a_1 \otimes b - a_2 \otimes b = (a_1 - a_2) \otimes b = i(a') \otimes b \in \text{Im } i \otimes \text{Id} = I.$$

Thus,

$$\overline{a_1 \otimes b} = \overline{a_2 \otimes b} \in (A \otimes B)/I.$$

It is easy to check that this function is biadditive, so it induces the desired homomorphism j_1 satisfying

$$j_1(a'' \otimes b) = \overline{a \otimes b} \quad \text{where } j(a) = a''.$$

This may be rewritten

$$j_1(j \otimes \text{Id})(a \otimes b) = \overline{a \otimes b},$$

so we get the following commutative diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{j \otimes \text{Id}} & A'' \otimes B \\ & \searrow & \swarrow j_1 \\ & (A \otimes B)/I & \end{array}$$

Thus,

$$\text{Ker}(j \otimes \text{Id}) \subseteq \text{Ker}(j_1 \circ (j \otimes \text{Id})) = I$$

as claimed. \square

5.2. Example. The sequence in the Theorem is not always exact at the left hand end. To see this, consider

$$0 \rightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow 0$$

and tensor this with $\mathbf{Z}/2\mathbf{Z}$. We get

$$\mathbf{Z}/2\mathbf{Z} \xrightarrow{0} \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow 0$$

so the homomorphism on the left is not a monomorphism.

However, the sequence always is exact on the left if the group B is torsion free.

THEOREM 7.14. *Suppose $i : A' \rightarrow A$ is a monomorphism of abelian groups and B is a torsion free abelian group. Then $i \otimes \text{Id} : A' \otimes B \rightarrow A \otimes B$ is a monomorphism.*

PROOF. First, assume that B is free and finitely generated, i.e., $B = \bigoplus_j \mathbf{Z}$. Then

$$A' \otimes B \cong \bigoplus_j A' \otimes \mathbf{Z} \cong \bigoplus_j A'$$

and similarly

$$A \otimes B \cong \bigoplus_j A.$$

These isomorphisms are natural in an appropriate manner, so the conclusion follows from the fact that $\bigoplus_j A' \rightarrow \bigoplus_j A$ is a monomorphism.

Next, assume only that B is torsion free. Suppose $\sum_j a'_j \otimes b_j \in A' \otimes B$ is in the kernel of $i \otimes \text{Id}$, i.e.,

$$\sum_j i(a'_j) \otimes b_j = 0 \in A \otimes B.$$

Note first that there are only a finite number of b_j in this sum. So in $A \otimes B = F(A, B)/T(A, B)$ only a finite number of basis elements $(i(a'_j), b_j)$ of $F(A, B)$ are needed to represent the left hand side of the above equation. Moreover, the assertion that the element is zero in $F(A, B)/T(A, B)$ means that

$$\sum_j ((a'_j), b_j) \in T(A, B)$$

which means that it is a linear combination of *finitely many* of the generators of the subgroup $T(A, B)$. Hence, only a finite number of elements of B are needed to represent the desired element *and* the fact that it is zero. Let B' be the subgroup of B generated by these elements. $\sum_j i(a'_j) \otimes b_j = 0 \in A \otimes B'$ by the choice of B' . However,

B' is finitely generated and torsion free. Hence, by the first part of the argument, $i \otimes \text{Id} : A' \otimes B' \rightarrow A \otimes B'$ is a monomorphism. Hence, $\sum_j a'_j \otimes b_j = 0 \in A' \otimes B'$. Hence, $\sum_j a'_j \otimes b_j = 0 \in A' \otimes B$ as claimed. \square

The notion of tensor product is considerably more general than what we did here. If A and B are modules over a commutative ring R , then one may define the tensor product $A \otimes_R B$ in a manner similar to what we did above. The result is again an R -module. (There is an even more general definition for modules over non-commutative rings.) In this context, a module is called *flat* if tensoring with it preserves monomorphisms. We leave most of this discussion for your algebra course.

5.3. Applications to Rank. The tensor product may be used to reduce questions about finitely generated abelian groups to questions about vector spaces. We do this by tensoring with the abelian group \mathbf{Q} , the additive group of rational numbers. If A is any abelian group, the abelian group $A \otimes \mathbf{Q}$ may be given the structure of a vector space over \mathbf{Q} . Namely, for $c \in \mathbf{Q}$ define

$$c(a \otimes b) = a \otimes (cb) \quad a \in A, b \in \mathbf{Q}.$$

There is more than meets the eye in this definition. The formula only tells us how to multiply generators $a \otimes b$ of $A \otimes \mathbf{Q}$ by rational numbers. To see that this extends to arbitrary elements of $A \otimes \mathbf{Q}$, it is necessary to make an argument as before about a bi-additive function. We shall omit that argument. It is also necessary to check that the distributive law and all the other axioms for a vector space over \mathbf{Q} hold. We shall also omit those verifications. The student is encouraged to investigate these issues on his/her own.

Let $A = F \oplus T$ where F is free of rank r and T is a torsion group. Then

$$A \otimes \mathbf{Q} \cong F \otimes \mathbf{Q} \oplus T \otimes \mathbf{Q}.$$

PROPOSITION 7.15. *If A is a torsion group, then $A \otimes \mathbf{Q} = 0$.*

PROOF. Exercise. \square

It follows that $A \otimes \mathbf{Q} \cong F \otimes \mathbf{Q}$. Suppose $F = \bigoplus_{i=1}^r \mathbf{Z}x_i$ is free with basis $\{x_1, x_2, \dots, x_r\}$. Then, by additivity, we see that

$$A \otimes \mathbf{Q} \cong \bigoplus_{i=1}^r \mathbf{Z}x_i \otimes \mathbf{Q} \cong \bigoplus_i \mathbf{Q}(x_i \otimes 1).$$

Thus, the rank of A as an abelian group is just the dimension of $A \otimes \mathbf{Q}$ as a vector space over \mathbf{Q} .

6. The Lefschetz Fixed Point Theorem

As mentioned earlier, one advantage of using simplicial homology is that we can reduce calculations to a chain complex of finitely generated abelian groups. One example of this is the definition of the so-called *Euler characteristic*. Suppose X is a polyhedron. Then the Euler characteristic of X is

$$\chi(X) = \sum_i (-1)^i \text{rank}(H_i(X)).$$

Since $H_i(X) = 0$ outside a finite range, this sum makes sense. For example,

$$(21) \quad \chi(S^n) = 1 + (-1)^n \quad n > 0$$

$$(22) \quad \chi(T^2) = 1 - 2 + 1 = 0.$$

The integers $\text{rank}H_i(X)$ are called the *Betti numbers* of X , so $\chi(X)$ is the alternating sum of the Betti numbers.

The Euler characteristic derives some of its importance from the fact that it may be calculated directly from a simplicial complex triangulating X .

PROPOSITION 7.16. *Let K be a finite simplicial complex.*

$$\sum_i (-1)^i \text{rank}(C_i(K)) = \sum_i (-1)^i H_i(K).$$

Note $H_i(|K|) = H_i(K)$. Also, $\text{rank}(C_i(K))$ is just the number of simplices in K of dimension i . Thus, triangulating T^2 as we did previously, we see that

$$\chi(T^2) = 9 - 27 + 18 = 0,$$

and we may make this calculation without computing the homology groups of T^2 .

PROOF. We have short exact sequences

$$(23) \quad 0 \rightarrow Z_i(K) \rightarrow C_i(K) \rightarrow B_{i-1}(K) \rightarrow 0$$

$$(24) \quad 0 \rightarrow B_i(K) \rightarrow Z_i(K) \rightarrow H_i(K) \rightarrow 0.$$

Hence,

$$(25) \quad \text{rank}(C_i) = \text{rank}(Z_i) + \text{rank}(B_{i-1})$$

$$(26) \quad \text{rank}(Z_i) = \text{rank}(B_i) + \text{rank}(H_i)$$

Thus,

$$\begin{aligned}
 (27) \quad \sum_i (-1)^i \text{rank}(C_i) &= \sum_i (-1)^i B_i + \sum_i (-1)^i \text{rank}(H_i) + \sum_i (-1)^i \text{rank}(B_{i-1}) \\
 (28) \quad &= \sum_i (-1)^i \text{rank}(H_i).
 \end{aligned}$$

□

We shall see later that it is not even necessary to decompose the space into simplices. Cells or more elaborate ‘polyhedra’ will do. In particular, suppose S^2 is decomposed into ‘polygons’ F_1, F_2, \dots, F_r . Inside each polygon, we may choose a point p_i and joining each p_i to each of the vertices of F_i will yield a triangulation of S^2 . Suppose F_i has v_i vertices. Then the effect of doing this will be to replace one ‘face’ F_i by v_i triangles, to add v_i edges, and to add one vertex. The net change in the Euler characteristic from the i -face is

$$1 - v_i + (v_i - 1) = 0.$$

We illustrate the use of the Euler characteristic by determining all regular solids in \mathbf{R}^3 —the Platonic solids. Each of these may be viewed as a 3-ball with its boundary S^2 decomposed into f r -gons such that, at each vertex, k faces meet. Let e be the number of edges and v the number of vertices. Then since each face has r edges and each edge belongs to precisely two faces, we have $fr = 2e$. Also, since each vertex belongs to k edges, and each edge has 2 vertices, we have $kv = 2e$. Since the Euler characteristic of S^2 is 2, we have

$$(29) \quad v - e + f = 2$$

$$(30) \quad 2e/k - e + 2e/r = (1/k - 1/2 + 1/r)2e = 2$$

$$(31) \quad \left(\frac{1}{k} + \frac{1}{r} - \frac{1}{2}\right) = 1$$

Since $k, r \geq 3$, and since $1/k + 1/r - 1/2$ must be positive, the only possibilities are summarized in the following table

k	r	$1/k + 1/r - 1/2$	e	v	f	Solid
3	3	1/6	6	4	4	Tetrahedron
3	4	1/12	12	8	6	Cube
4	3	1/12	12	6	8	Octahedron
3	5	1/30	30	20	12	Dodecahedron
5	3	1/30	30	12	20	Icosohedron

There is an important generalization which applies to a self map $f : X \rightarrow X$ where X is a polyhedron. To discuss this we need some

preliminary concepts. If $g : A \rightarrow A$ is an endomorphism of a finitely generated abelian group, we may consider the induced \mathbf{Q} -linear transformation $f \otimes \text{Id} : A \otimes \mathbf{Q} \rightarrow A \otimes \mathbf{Q}$. We denote the *trace* of this linear transformation by $tr(g)$. Note that $tr(f) \in \mathbf{Z}$. For, if T is the torsion subgroup of A , g induces $\bar{g} : A/T \rightarrow A/T$ where A/T is free of finite rank. Since $T \otimes \mathbf{Q} = 0$, the epimorphism $A \otimes \mathbf{Q} \rightarrow (A/T) \otimes \mathbf{Q}$ is an isomorphism, so we may identify $g \otimes \text{Id}$ with $\bar{g} \otimes \text{Id}$. However, if we choose a basis for A/T over \mathbf{Z} , it will also be a basis for $(A/T) \otimes \mathbf{Q}$ over \mathbf{Q} . Hence, the matrix of $\bar{g} \otimes \text{Id}$ will be the same as the (integer) matrix of \bar{g} . Hence, the trace will be an integer.

The trace is additive on short exact sequences, i.e.,

LEMMA 7.17. *Let A be a finitely generated abelian group, A' a subgroup. Suppose g is an endomorphism of A such that $g(A') \subseteq A'$. Let g' be the restriction of g to A' and g'' the induced homomorphism on $A'' = A/A'$. Then $tr(g) = tr(g') + tr(g'')$.*

PROOF. Consider the short exact sequence

$$0 \rightarrow A' \otimes \mathbf{Q} \rightarrow A \otimes \mathbf{Q} \rightarrow A'' \otimes \mathbf{Q} \rightarrow 0$$

of vector spaces. It suffices to prove that

$$tr(g \otimes \text{Id}) = tr(g' \otimes \text{Id}) + tr(g'' \otimes \text{Id}).$$

Modulo some identifications, we may consider $A' \otimes \mathbf{Q}$ a \mathbf{Q} -subspace of $A \otimes \mathbf{Q}$ and $A'' \otimes \mathbf{Q}$ the resulting quotient space. By standard vector space theory, we can choose a basis for $A \otimes \mathbf{Q}$ such that the matrix of $g \otimes \text{Id}$ with respect to this basis has the form

$$\begin{bmatrix} C' & 0 \\ * & C'' \end{bmatrix}$$

where C' is a matrix representation of $g' \otimes \text{Id}$ and C'' is a matrix representation of $g'' \otimes \text{Id}$. Taking traces yields the desired formula. \square

Let $f : X \rightarrow X$. Define the Lefschetz number of f to be

$$L(f) = \sum_i (-1)^i tr(H_i(f)).$$

By the above remarks, $L(f)$ is an integer.

6.1. Examples. Let $f : S^n \rightarrow S^n$ be a self map. Then

$$L(f) = 1 + (-1)^n deg(f).$$

This uses the fairly obvious fact that if X is path connected, and f is a self map, then $H_0(f)$ is the identity isomorphism of $H_0(X) = \mathbf{Z}$, so $tr(H_0(f)) = 1$.

For any polyhedron, the Euler characteristic is the Lefschetz number of the identity map since $tr(H_i(\text{Id})) = rank(H_i(X))$.

Like the Euler characteristic, the Lefschetz number may be calculated using simplicial chains. For, suppose $g : C_* \rightarrow C_*$ is a chain morphism of a chain complex inducing homomorphisms $H_i(g) : H_i(C_*) \rightarrow H_i(C_i)$ in homology. As before we have the exact sequences

$$(32) \quad 0 \rightarrow Z_i \rightarrow C_i \rightarrow B_{i-1} \rightarrow 0$$

$$(33) \quad 0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0$$

and g will induce endomorphism $Z_i(g)$, $C_i(g)$, and $B_i(g)$ of each of these groups. Then

$$(34) \quad tr(C_i(g)) = tr(Z_i(g)) + tr(B_{i-1}(g))$$

$$(35) \quad tr(Z_i(g)) = tr(B_i(g)) + tr(H_i(g)).$$

Taking the alternating sum yields as in the case of the Euler characteristic

$$(36) \quad \sum_i (-1)^i tr(g_i) = \sum_i (-1)^i tr(H_i(g)) = L(g)$$

THEOREM 7.18. (*Lefschetz Fixed Point Theorem*) *Let $f : X \rightarrow X$ be a self map of a polyhedron. If $L(f) \neq 0$ then f has a fixed point.*

The following corollary is a generalization of the Brouwer Fixed Point Theorem

COROLLARY 7.19. *Let $f : X \rightarrow X$ be a self map of an acyclic polyhedron (i.e., $H_i(X) = 0, i > 0$). Then f has a fixed point*

PROOF. $L(f) = 1 \neq 0$. □

PROOF. Suppose f does not have a fixed point. We shall show that there is a triangulation K of X and a chain morphism $F_{\sharp} : C_*(K) \rightarrow C_*(K)$ which induces $H_*(f) : H_*(X) \rightarrow H_*(X)$ in homology, and such that for each simplex $\sigma \in K$, the chain $F_{\sharp}(\sigma)$ does not involve σ . That says that for each i , the matrix of the homomorphism $F_i : C_i(K) \rightarrow C_i(K)$ has zero diagonal entries. Hence, $tr(F_i) = 0$ so by formula (36), $L(f) = 0$.

The idea behind this argument is fairly clear. Since X is compact, if f does not have any fixed points, it must be possible to subdivide X finely enough into simplices so that no simplex is carried by f into itself. However, there are many technical complications in making this argument precise, and we shall now deal with them.

By picking a homeomorphic space, we may assume $X = |L|$ for some finite simplicial complex (contained in an appropriate \mathbf{R}^N). Under the assumption that f has no fixed points, $|f(x) - x|$ has a lower bound. By taking sufficiently many subdivisions, so the mesh of L is small enough, we can assume

$$(37) \quad \overline{St_L(v)} \cap f(\overline{St_L(v)}) = \emptyset$$

for every vertex in L .

Our first problem is that f won't generally come from a simplicial morphism of L , so we choose an iterated barycentric subdivision L' of L and a simplicial approximation $\phi : L' \rightarrow L$ to f . Thus,

$$(38) \quad f(\overline{St_{L'}(v')}) \subseteq \overline{St_L(\phi(v'))}$$

for every vertex v' in L' . Note that $|\phi|$ is homotopic to f , but it is not quite the simplicial map we want since it takes $C_*(L')$ to $C_*(L)$ instead of to $C_*(L')$. We shall deal with that problem below, but note for the moment that ϕ meets our needs in the following partial sense. Let σ be a simplex of L and suppose σ' is a simplex of L' which is contained in σ . Let v be any vertex of σ . Then

$$\sigma' \subseteq \sigma \subseteq \overline{St_L(v)},$$

it follows from (37) that $f(\sigma')$ is disjoint from $\overline{St_L(v)}$. Hence, by (38), we can't have $v = \phi(v')$ for any vertex v' of σ' . In other words, if σ' is a simplex of L' contained in a simplex σ of L , then

$$\phi(\sigma') \neq \sigma.$$

We now deal with the problem that ϕ_{\sharp} does not end up in $C_*(L')$. For this, we define a chain morphism $j_{\sharp} : C_*(L) \rightarrow C_*(L')$ which will have appropriate properties. This is done by barycentric subdivision. Namely, for each simplex σ in L define a chain $Sd(\sigma)$ in $C_*(SdL)$ exactly as we did for singular homology. This presumes that we have chosen a fixed order for the vertices of L and a consistent order for the vertices of SdL , but otherwise the definition (by induction) uses the same formulas as in the singular case. j_{\sharp} will be consistent with the chain morphism to singular theory, i.e.,

$$\begin{array}{ccc} C_*(L) & \xrightarrow{Sd_{\sharp}} & C_*(SdL) \\ h_{\sharp} \downarrow & & \downarrow h_{\sharp} \\ S_*(X) & \xrightarrow{Sd_{\sharp}} & S_*(X) \end{array}$$

commutes. Hence, in homology, we get a commutative diagram

$$\begin{array}{ccc} H_*(L) & \xrightarrow{Sd_*} & H_*(SdL) \\ h_* \downarrow & & \downarrow h_* \\ H_*(X) & \xrightarrow{=} & H_*(X) \end{array}$$

Here we use the fact that for singular theory, the subdivision operator is chain homotopic to the identity. If we iterate this process, we get the desired chain map $j_{\sharp} : C_*(L) \rightarrow C_*(L')$ and

$$\begin{array}{ccc} H_*(L) & \xrightarrow{j_*} & H_*(L') \\ h_* \downarrow & & \downarrow h_* \\ H_*(X) & \xrightarrow{=} & H_*(X) \end{array}$$

commutes.

To complete the proof, consider the diagram

$$\begin{array}{ccc} H_*(L') & \xrightarrow{\phi_*} & H_*(L) \\ h_* \downarrow & & \downarrow h_* \\ H_*(X) & \xrightarrow{|\phi|_*} & H_*(X) \end{array}$$

This commutes by the naturality of the h_* homomorphism. (We did not prove naturality in general, but in the present case we can assume that the orderings needed to define the h_{\sharp} morphisms are chosen so that ϕ is order preserving, in which case the commutativity of the diagram is clear.) Putting this diagram alongside the previous one and using formula (36) shows that the chain morphism $j_{\sharp} \circ \phi_{\sharp} : C_*(L') \rightarrow C_*(L)$ may be used to calculate $L(|\phi|)$. However, by our assumption $\phi(\sigma')$ does not contain σ' , so under barycentric subdivision, σ' does not appear in $j_{\sharp}(\phi(\sigma'))$, and as above each trace is zero. Since $f \sim |\phi|$, it follows that $L(f) = L(|\phi|) = 0$ as claimed.

□