CHAPTER 6

Singular Homology

1. Homology, Introduction

In the beginning, we suggested the idea of attaching algebraic objects to topological spaces in order to discern their properties. In language introduced later, we want functors from the category of topological spaces (or perhaps some related category) and continuous maps (or perhaps homotopy classes of continuous maps) to the category of groups. One such functor is the fundamental group of a path connected space. We saw how to use such functors in proving things like the Brouwer Fixed Point Theorem and related theorems.

The fundamental group is the first of a sequence of functors called homotopy groups. These are defined roughly as follows. Let X be a space and fix a base point x_0. For each $n \geq 1$ consider the set $\pi_n(X, x_0)$ of base point preserving homotopy classes of maps of S^n into X. Note that $\pi_1(X, x_0)$ is just the fundamental group. It is possible with some care to defined a group structure on $\pi_n(X, x_0)$. (Think about the case $n = 1$ and how you might generalize this to $n > 1$.) The resulting group is abelian for $n > 1$ and is called the nth homotopy group. It is also not too hard to see that a (base point preserving) map of spaces $X \to Y$ induces a homomorphism of corresponding homotopy groups.

The homotopy groups capture quite a lot about the geometry of spaces and are still the subject of intense study. Unfortunately, they are very difficult to compute. However, there is an alternate approach, the so-called homology groups which historically came first. The intuition behind homology groups is a bit less clear, but they are much easier to calculate than homotopy groups, and their use allows us to solve many important geometric problems. Also, homology theory is a basic tool in further study of the subject. We shall spend the rest of this year studying homology theory and related concepts. You will get to homotopy theory later if you continue with your study of algebraic topology.

One way to explain to roots of homology theory is to consider the basic integral theorems of vector analysis. (You probably studied this as an undergraduate, and in any case you will probably have to teach

it as a teaching assistant.) Green's Theorem asserts that for an appropriate region D in $\mathbf{R}^2$ and a differential form $Pdx + Qdy$, we have

$$\int_{\partial D} Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

Note that the orientation of the boundary plays an important role.

We want to know the theorem for fairly arbitrary regions, including those with 'holes' where the boundary may be disconnected. The strategy for proving it is first to do it for regions with a relatively simple shape, e.g., 'curvilinear' triangles, and then approach general regions by 'triangulating' them as discussed previously in the section on classifying surfaces.

Here two important concepts play a role. First, we have the concept of a 'chain' as a formal sum of 'triangles' (or perhaps other elementary regions). Such an object helps us think about dissecting the region for integration on the right of the formula. Secondly, we see that the *oriented* boundary of such a chain will break up into many separate segments, some of which will be the same segment repeated twice but with opposite orientation. In the integral on the right, such segments will cancel and we will be left only with the integral on the external (oriented) boundary. A useful way to think of this is that we have an algebraic boundary which attaches to any chain the formal sum of its boundary segments but where segments in opposite directions are give opposite signs so they cancel in the formal sum.

Let's focus on the external boundary ∂D of the region. This can be viewed as a formal sum of segments, and each segment has a boundary consisting of its two end points with opposite 'orientation' or sign. Since the boundary is closed, if we take its algebraic boundary in this sense, each division point is counted twice with opposite signs and so the zero dimensional 'boundary' of ∂D is zero.

Note however, that other closed loops may have this property. e.g., the unit circle in $\mathbf{R}^2 - \{(0,0)\}$. A closed curve (or collection of such) decomposed this way into segments is called a *cycle*. The extent to which cycles in a region X differ from boundaries of subregions D is a question which is clearly related to the fundamental group of the region. In fact, we shall measure this property by a functor called the first homology group $H_1(X)$. It will turn out that $H_1(X)$ is $\pi_1(X)$ made abelian!

The above considerations can be generalized to regions in $\mathbf{R}^n$ for $n > 2$. In $\mathbf{R}^3$, there is a direct generalization of Green's Theorem called Stokes's Theorem. The relates a line integral over the boundary of a surface to a surface integral over the surface. (If you don't remember Stokes's Theorem, you should go now and look it up.) The same considerations apply except now the curvilinear triangles dissect the surface. Another generalization is $Gauss's$ theorem which asserts for a solid region D in $\mathbf{R}^3$

$$\iint_{\partial D} P dy\, dz + Q dx\, dz + R dx\, dy = \iiint_D \left(\frac{\partial P}{\partial X} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV.$$

The first integral is a surface integral of a 'two-form' and the second is an ordinary triple integral. The orientation of the surface relative the the region D plays an crucial role. We have stated the theorem the way it was commonly stated in the 19th century. You may be more familiar with the more common form using vector fields. The proof of this theorem parallels that of Green's theorem. We first prove it for basic regions such as 'curvilinear tetrahedra'. We then view D as dissected as an appropriate union of such tetrahedra. This leads to the concept of a 3-chain as a formal sum of tetrahedra. As above, the boundary of a tetrahedron can be thought of as a formal sum of 'triangles' with appropriate orientations. The same triangle may appear as a face on adjacent tetrahedral cells with opposite orientations. In the sum of all the surface integrals for these cells, these two surface integrals cancel, so we are left with the integral over the external boundary. This cancellation can be treated formally in terms of algebraic cancellation of the terms in the formal sums.

Let X be a solid region in $\mathbf{R}^3$. As above, the boundary of a subregion D dissected into triangles may be though of as a 2-cycle (because its boundary is trivial), but there may be other closed surfaces in X which don't bound a subregion. Thus again we have the question of the extent to which 2-cycles differ from boundaries. This is measured by a group called the second homology group and denoted $H_2(X)$.

The issue of when cycles are boundaries is clearly something of some significance for understanding the geometry of a region. It is also important for other reasons. A differential form $Pdx + Qdy$ defined on a path connected region D is called exact if it is of the form df for some function f defined on D. It is important to know if a differential form is exact when we want to solve the differential equation $Pdx + Qdy = 0$. (Why?) It is easy to see that every exact form satisfies the relation $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$. Such forms are called closed. We would like to know if every closed form on D is exact. One way to approach this issue is as follows. Choose an arbitrary point x_0 in D and given any other point x, choose a path $\mathcal{C}$ from x_0 to x and define $f(x) = \int_{\mathcal{C}} Pdx + Qdy$. If this function f is well defined, it is fairly obvious that $df = Pdx + Qdy$. Unfortunately, f is not always well defined because the integral on the right might depend on the path. Consider two different paths $\mathcal{C}$ and $\mathcal{C}'$ going from x_1 to x.

For simplicitly assume the two paths together bound a subregion of D. It is easy to see by an application of Green's theorem that the two line integrals are the same. Hence, if every cycle is a boundary, it follows that the function is well defined. Hence, in that case, every closed form is exact.

This sort of analysis is quite fascinating and lies at the basis of many beautiful theorems in analysis and geometry. We hope you will pursue such matters in other courses. Now, we shall drop our discussion of the motivation for homology theory and begin its formal development

2. Singular Homology

The material introduced earlier on abelian groups will be specially important, so you should review it now. In particular, make sure you

are comfortable with free abelian groups, bases for such groups, and defining homomorphisms by specifying them on bases.

The *standard n-simplex* is defined to be the set Δ^n consisting of all $(t_0, t_1, \dots, t_n) \in \mathbf{R}^{n+1}$ such that $\sum_i t_i = 1$ and $t_i \geq 0, i = 0, 1, \dots n$. (Note that we start the numbering of the coordinates with 0.) The cases $n = 0, 1, 2$ are sketched below. Δ^3 is a solid tetrathedron but imbedded in $\mathbf{R}^4$.

We could of course choose the standard simplex in $\mathbf{R}^n$, but imbedding it in $\mathbf{R}^{n+1}$ gives a more symmetric description and has some technical advantages.

Let X be a space. A *singular n-simplex* is any (continuous) map $\sigma : \Delta^n \to X$. You can think of this as the generalization of a 'curvilinear triangle', but notice that it need not be one-to-one. For example, any constant map is a singular n-simplex. Note that a 0-singular simplex can be identified with a point in X. A singular n-chain is any formal linear combination $\sum_i n_i\sigma_i$ of singular n-simplices with integer coefficients n_i. (Note, by implication, 'linear combination' always means 'finite linear combination'. Unless explicitly stated, in any sum which potentially involves infinitely many terms, we shall always assume that *all but a finite number of terms are zero*.)

More explicitly, let $S_n(X)$ be the free abelian group with basis the set of all singular n-simplices in X. Note that this is an enormous group since the basis certainly won't be countable for any interesting space. Thus, $S_0(X)$ can be viewed as the free abelian group with X as basis. A *singular n-chain* then is any element of this group. By the usual conventions, $S_n(X)$ is the trivial group $\{0\}$ for $n < 0$ since it is the free abelian group on the empty basis.

A bit of explanation for this definition is called for. Suppose X is a compact surface with a triangulation. Each 'triangle' may be viewed as a singular 2-simplex where the map is in fact one-to-one. Also, there are restrictions about how these triangles intersect. In fact, up to homeomorphism, we may view X as a two dimensional 'polyhedron' (but we may not be able to imbed it in $\mathbf{R}^3$ if it is not orientable). The generalization of this to higher dimensions is called a *simplicial complex*. Simplicial complexes and their homeomorphs were the original subject matter of algebraic topology. The reason is fairly clear. As in

the previous section, there are many reasons to consider formal sums of the triangles (in general n-simplices) in a triangulation. Unfortunately, we are then left with the problem of showing that the results we obtain are not dependent on the triangulation. This was the original approach. Our approach will allow us to avoid this problem. We consider arbitrary singular n-simplices and linear combinations of such. This theory will introduce many *degenerate* n-simplices, but there is no question that it depends only on the space X. This large advantage of singular theory is counterbalanced by the fact that it is difficult to make computations because the groups invovled are so large. Hence, later we shall introduce simplicial complexes and related objects where the groups have much smaller, even finite, bases in most cases, so the computations are much easier.

The considerations in the previous section suggest that we want to be able to define a homomorphism $\partial_n : S_n(X) \to S_{n-1}(X)$ which reflects algebraically the properties of an oriented boundary. Since $S_n(X)$ is free on the set of singular n-simplices $\sigma : \Delta^n \to X$, it suffices to define ∂_n on all singular n-simplices σ and then extend by linearity. A moment's thought suggests that the boundary should be determined somehow by restricting σ to the 'faces' of the standard simplex and then summing with appropriate signs to handle the issue of orientation. In order to get this right, we need a rather long digression on so called affine simplices.

Let $T = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\}$ be any finite set of points in $\mathbf{R}^{n+1}$. We shall call this set *affinely independent* if the set of differences $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_p - \mathbf{x}_0\}$ is linearly independent. (You should prove this does not depend on the use of $\mathbf{x}_0$ rather than some other element in the set.) We shall assume below that any such set is affinely independent, but some of what we do will in fact work for any finite set. The set $\{\sum_i t_i\mathbf{x}_i \,|, t_i \in \mathbf{R}, \sum_i t_i = 1, t_i \geq 0\}$ is called the *affine simplex* spanned by $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\}$. It is not too hard to see that it is a closed convex set, and in fact it is the smallest convex set containing the set T. We denote it $[T] = [\mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_p]$.

The set A of all points of the form $\sum_{i=0}^p t_i\mathbf{x}_i$ where $t_i \in \mathbf{R}$ and $\sum_i t_i = 1$ is called the affine subspace spanned by those points. It is a translate (by $\mathbf{x}_0$ for example) of the linear subspace of $\mathbf{R}^{n+1}$ spanned by $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_p - \mathbf{x}_0\}$. Namely,

$$\sum_{i=0}^p t_i\mathbf{x}_i = \sum_{i=0}^p t_i(\mathbf{x}_i - \mathbf{x}_0) + (\sum_{i=0}^p t_i)\mathbf{x}_0 = \sum_{i=1}^p t_i(\mathbf{x}_i - \mathbf{x}_0) + \mathbf{x}_0.$$

It follows from this that if two points are in A, then the line they determine is in A. Also, (in the default case where the points are independent), this set is homeomorphic to $\mathbf{R}^p$, and we call it a p-dimensional affine subspace. (Since $[\mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_p]$ is the closure of an open subset of this space, it is also reasonable to consider it a p-dimensional object.) (Abstractly, an affine subspace A may be defined as any subspace having the property $\mathbf{x}, \mathbf{y} \in A, s, t \in \mathbf{R}, s + t = 1 \to s\mathbf{x} + t\mathbf{y} \in A$. The p dimensional affine subspaces of $\mathbf{R}^{n+1}$ are exactly the same as the translates of p-dimensional linear subspace.)

The coefficients $(t_0, t_1, \dots, t_p)$ of a point $\sum_i t_i\mathbf{x}_i$ in the affine subspace are determined uniquely by the point. We leave this as an exercise for the student. These coefficients are called the *barycentric coordinates* of the point. The reason for the terminology is as follows. If you imagine weights $m_0, m_1, \dots, m_p$ at the points $\mathbf{x}_0, \dots, \mathbf{x}_p$, then the center of mass of these points is $\dfrac{1}{\sum_i m_i} \sum_i m_i\mathbf{x}_i$. Setting $t_i = \dfrac{m_i}{\sum_i m_i}$, we get the barycentric coordinates of the center of mass. This makes a lot of sense if the masses are all non-negative (as it the case for points in the simplex $[\mathbf{x}_0, \dots, \mathbf{x}_p]$) but we may also allow negative 'masses' (e.g., charges) and thereby encompass arbitrary points in the affine space as weighted sums.

For linear spaces, the nicest maps are linear maps, and there is a corresponding concept for affine spaces. Let $f : A \to B$ be a function which maps an affine subspace of $\mathbf{R}^{n+1}$ to an affine subspace of $\mathbf{R}^{m+1}$. n and m may be different as may be the dimensions of A and B. We shall say f is an affine map if for any points $\mathbf{x}, \mathbf{y} \in A$, we have $f(t\mathbf{x} + s\mathbf{y}) = tf(\mathbf{x}) + sf(\mathbf{y})$ for all $t, s \in \mathbf{R}$ such that $t + s = 1$. Note that this implies that f carries any line in A into a line in B or a point in B. It is not hard to show that an affine map is completely determined by its values on an affinely independent generating set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\}$ through the formula $f(\sum_i t_i\mathbf{x}_i) = \sum_i t_i f(\mathbf{x}_i)$ where $\sum_i t_i = 1$, and moreover an affine map may be defined by specifying it arbitrarily on such a set. Clearly, if the image points are also independent, then $f([\mathbf{x}_0, \dots, \mathbf{x}_p]) = [f(\mathbf{x}_0), \dots, f(\mathbf{x}_p)]$.

Note that any p-dimensional affine simplex $[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p]$ in $\mathbf{R}^{n+1}$ is a singular simplex under the affine map $\Delta^p \to \mathbf{R}^{n+1}$ defined by $\mathbf{e}_i \mapsto \mathbf{x}_i$. We shall abuse notation by using the symbol $[\mathbf{x}_0, \dots, \mathbf{x}_p]$ both for this singular simplex (which is a map) and for its image.

Consider now the standard simplex Δ^n. Let $\mathbf{e}_i = \mathbf{e}_i^{n+1}$ denote the ith standard basis vector in $\mathbf{R}^{n+1}$. Clearly, $\Delta^n = [\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n]$. (Under our abuse of notation, this means Δ^n also denotes the identity map of the standard simplex.) The barycentric coordinates of Δ^n

are just the ordinary coordinates in $\mathbf{R}^{n+1}$. The affine $(n-1)$-simplex $[\mathbf{e}_0, \dots, \hat{\mathbf{e}}_i, \dots, \mathbf{e}_n]$ is called the ith face of Δ^n. (Here we adopt the convention that putting a 'hat' over an element of a list means that it should be omitted from the list.) It is in effect obtained by intersecting Δ^n with the hyperlane with equation $t_i = 0$. With our abuse of notation, these faces are the images of the singular simplices

$$\epsilon_i^n = [\mathbf{e}_0, \dots, \hat{\mathbf{e}}_i, \dots, \mathbf{e}_n] : \Delta^{n-1} \to \Delta^n$$

(This is a shorthand way of saying that

$$\begin{aligned} \epsilon_i^n(\mathbf{e}_j^{n-1}) &= \mathbf{e}_j^n \qquad j < i \\ &= \mathbf{e}_{j+1}^n \qquad j \geq i.) \end{aligned}$$

These maps are called the face maps for Δ^n. Note that there are $n+1$ faces of an n-simplex.

We are now in a position to define the map $\partial_n : S_n(X) \to S_{n-1}(X)$. Let $\partial_n = 0$ for $n \leq 0$. For $n > 0$, for a singular n-simplex $\sigma : \Delta^n \to X$, define

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^n.$$

Note that each of the terms on the right is a singular $(n-1)$-simplex which is in some sense the restriction of σ to the ith face of σ. The diagram below shows why in the cases $n = 1, 2$ the signs make sense by reflecting a proper orientation for each simplex in the boundary.

Proposition 6.1. $\partial_n \circ \partial_{n+1} = 0$.

This proposition plays an absolutely essential role in all that follows. For, given it, we may now make the following definitions. The subgroup $Z_n(X) = \operatorname{Ker} \partial_n$ of $S_n(X)$ is called the subgroup of n-cycles. Its elements may be thought of as 'closed' n-dimensional objects in X. The subgroup $B_n(X) = \operatorname{Im} \partial_{n+1}$ is called the subgroup of n-boundaries. Because of the lemma, every n-boundary is an n-cycle, i.e., $B_n(X) \subseteq Z_n(X)$. Finally, we shall define the nth homology group of X as the factor group

$$H_n(X) = Z_n(X)/B_n(X).$$

This measures the extent to which it is not true that every n-cycle is an n-boundary.

Proof of Proposition 6.1. Note first that the proof is clear for $n \le 0$.

For $n > 0$, let $\sigma : \Delta^{n+1} \to X$ be a singular $(n+1)$-simplex. Then

$$\partial_{n+1}\sigma = \sum_{j=0}^{n+1} (-1)^j \sigma \circ \epsilon_j^{n+1}$$

so

$$\begin{aligned} \partial_n(\partial_{n+1}\sigma) &= \sum_{j=0}^{n+1} (-1)^j \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_j^{n+1} \circ \epsilon_i^n \\ &= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_i^n \end{aligned}$$

The strategy is to pair terms with opposite terms which cancel. The diagram below illustrates the argument for $n = 1$.

The general case follows from

Lemma 6.2. *For* $0 \le i < j \le n+1$, $\epsilon_j^{n+1} \circ \epsilon_i^n = \epsilon_i^{n+1} \circ \epsilon_{j-1}^n$.

This suffices to prove the proposition because the term

$$(-1)^{i+j} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_i^n \qquad 0 \le i < j \le n+1$$

cancels the term

$$(-1)^{j-1+i} \sigma \circ \epsilon_i^{n+1} \circ \epsilon_{j-1}^n.$$

The student should check that every term is accounted for in this way. □

Proof of Lemma 6.2. For $0 \le i < j \le n+1$, consider the $(n-1)$-dimensional affine simplex $[\mathbf{e}_0^{n+2}, \dots, \hat{\mathbf{e}}_i^{n+2}, \dots, \hat{\mathbf{e}}_j^{n+2}, \dots, \mathbf{e}_{n+1}^{n+2}]$ in Δ^{n+1}. This may be thought of as its intersection with the linear subspace defined by setting $t_i = t_j = 0$. Call this its (i, j)-face. (This is also the image of the affine map which maps the basis elements as below in the indicated order

$$\{\mathbf{e}_0^n, \dots, \mathbf{e}_{n-1}^n\} \to \{\mathbf{e}_0^{n+2}, \dots, \hat{\mathbf{e}}_i^{n+2}, \dots, \hat{\mathbf{e}}_j^{n+2} \dots \mathbf{e}_{n+1}^{n+2}\}.$$

This exhibits each of the $(n+2)(n+1)/2$ (i,j)-faces as a singular $(n-1)$-simplex.)

The right hand side of the equation can be viewed as folows. First take the ith face of Δ^{n+1} by setting $t_i = 0$. The barycentric coordinates of a point on this face relative to its vertices in the proper order (obtained from Δ^n) will be $(s_0, \dots, s_n)$, where $s_r = t_r$ for $r < i$ and $s_r = t_{r+1}$ for $r \geq i$. If we now take its $(j-1)$st face by setting $s_{j-1} = 0$, that is the same thing as setting $t_j = 0$. (Remember $i \leq j-1$.) This gives us the (i,j)-face. The left hand side of the equation also represents the (i,j) face by a similar analysis. First, take the jth face of Δ^{n+1} by setting $t_j = 0$. Then take the ith face of that by setting $t_i = 0$. (Since $i < j$, we do not have to worry about the shift for $r \geq j$.)

$\square$

3. Properties of Singular Homology

The singular homology groups $H_n(X)$ are generally so hard to compute that we won't be able to do much more than simple examples to start. We shall derive a series of results or 'axioms' and then show how we can compute singular homology just by use of these 'axioms'. The point of this is that there are several other versions of homology theory. If we can show that an alternate theory satisfies these same 'axioms', then it will follow that it is essentially the same as singular homology. (This axiomatic approach is due to Eilenberg and Steenrod and is worked out in detail in their book *Foundations of Algebraic Topology*.)

THEOREM 6.3 (The Dimension Axiom). *For a space $X = \{x\}$ consisting of a single point, we have*

$$\begin{aligned} H_n(\{x\}) &= \mathbf{Z} \qquad n = 0 \\ &= 0 \qquad n > 0. \end{aligned}$$

PROOF. For each $n \geq 0$ there is precisely one singular n-simplex σ_n which is the constant map with value x. For $n > 1$, we have

$$\begin{aligned} \partial_n \sigma_n &= 0 \qquad n \quad \text{odd} \\ \partial_n \sigma_n &= \sigma_{n-1} \qquad n \quad \text{even}. \end{aligned}$$

The reason is that when n is odd there are an even number of terms and they all cancel. In the even case there is one term left over with sign $(-1)^n = 1$. The above situation can be summarized by the diagram

$$\dots \xrightarrow{0} S_2(\{x\}) \xrightarrow{\cong} S_1(\{x\}) \xrightarrow{0} S_0(\{x\}).$$

Then for $n > 0$, $Z_n(\{x\})$ is alternately $\mathbf{Z}\sigma_n$ or zero and $B_n(\{x\})$ is alternately zero or $\mathbf{Z}\sigma_n$. Hence, the quotients are all zero. It is clear that $Z_0(\{x\}) = \mathbf{Z}\sigma_0$ and $B_0(\{x\}) = 0$, so $H_0(\{x\}) = \mathbf{Z}$. □

3.1. Homological Algebra. Algebraic topology generates algebra which is of interest in its own right and in fact has been used extensively outside algebraic topology. We shall periodically introduce such concepts as they are needed. In principle, you could separate these out and have a short course in homological algebra, but all the motivation of course comes from the geometric ideas.

A *chain complex* C consists of a collection of abelian groups C_n, for $n \in \mathbf{Z}$, and maps $d_n : C_n \to C_{n-1}$ with the property that $d_n \circ d_{n+1} = 0$ for each n. (Sometimes the collection of all d_n is denoted 'd' and we say simply $d \circ d = 0$.) Thus the collection of singular n-chains $S_n(X)$ for all n forms a chain complex $S_*(X)$.

Most of the chain complexes we shall consider will be non-negative, i.e., $C_n = 0$ for $n < 0$. *Unless otherwise stated, you should assume the term 'chain complex' is synonymous iwth 'non-negative chain complex'.*

Given a chain complex C, we may define its homology groups as follows. $Z_n(C) = \operatorname{Ker} d_n, B_n(C) = \operatorname{Im} d_{n+1}$ so $B_n(C) \subseteq Z_n(C)$. These are called the cylces and boundaries of the chain complex. Two chains are called *homologous* if they differ by a boundary. We define $H_n(C) = Z_n(C)/B_n(C)$, and its elements are called *homology classes*.

Note that $H_n(S_*(X)) = H_n(X)$.

Chain complexes are the objects of a category. Let C and C' be chain complexes. A morphism $f : C \to C'$ is a collection of group homomophisms $f_n : C_n \to C'_n$ such that $d'_n \circ f_n = f_{n-1} \circ d_n$ for each n.

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ C'_n & \longrightarrow & C'_{n-1} \end{array}$$

It is easy to see that the composition of two morphisms of chain complexes is again such a morphism.

A morphism of $f : C \to C'$ of chain complexes induces a homomorphism $H_n(f) : H_n(C) \to H_n(C')$ for each n. Namely, because f commutes with d, it follows that $f_n(Z_n(C)) \subseteq Z_n(C')$ and $f_n(B_n(C)) \subseteq B_n(C')$ whence it induces a homomorphism on the quotients $H_n(f) = H_n(C) = Z_n(C)/B_n(C) \to H_n(C') = Z_n(C')/B_n(C')$. $(H_n(f)(\bar{c}) = \overline{f_n(c)}.)$

PROPOSITION 6.4. *For each n, the homology group $H_n(\quad)$ is a functor from the category of chain complexes to the category of abelian groups.*

PROOF. It is easy to see that $H_n(\mathrm{Id}) = \mathrm{Id}$, so all we need to do is show is $H_n(g \circ f) = H_n(g) \circ H_n(f)$ for

$$C \xrightarrow{f} C' \xrightarrow{g} C''$$

morphisms of chain complexes. We leave this to the reader to verify. □

We may now apply the above homological algebra to singular homology. Let $f : X \to Y$ be a map of spaces. Define a morphism $f_\sharp : S(X) \to S(Y)$ as follows. For σ a singular n-simplex, let $f_n(\sigma) = f \circ \sigma : \Delta^n \to Y$. Since the set of singular n-simplices forms a basis for $S_n(X)$, this defines a homomorphism $f_n : S_n(X) \to S_n(Y)$. It commutes with the boundary operators since

$$\begin{aligned}\partial_n^Y(f_n(\sigma)) &= \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \epsilon_i^n \\ &= \sum_{i=0}^{n}(-1)^i f_{n-1}(\sigma \circ \epsilon_i^n) \\ &= f_{n-1}(\sum_{i=0}^{n}(-1)^i \sigma \circ \epsilon_i^n) = f_{n-1}(\partial_n^X \sigma).\end{aligned}$$

(Make sure you understand the reasons for each of the steps!)

It follows that $f_\sharp$ induces a homomoprhism

$$H_n(f) = H_n(f_\sharp) : H_n(X) = H_n(S_*(X)) \to H_n(Y) = H_n(S_*(Y)).$$

This homomorphism is often abbreviated f_n and the collection of all of them is often denoted f_*. (This yields a slight notational problem, since we are also using f_n to denote the induced maps of singular n-chains, but usually the context will make clear what is intended. Where there is any doubt, we may use the notation $H_n(f)$ or f_* for the collection of all the maps.)

PROPOSITION 6.5 (Functoriality Axiom). *$H_n(-)$ is a functor from the category of topological spaces to the category of abelian groups*

PROOF. First note that the associations $X \mapsto S_*(X)$ and $f \mapsto f_\sharp$ provide a functor from the category of topological spaces to the category of chain complexes. For, it is clear that the identity goes to the identity and it is easy to verify that $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. It is not

hard to see that the composition of two functors is a functor, and since $H_n(-)$ is clearly such a composition, it follows that it is a functor. $\square$

Proposition 6.6 (Direct Sum Axiom). *Let X be the disjoint union $\cup_{a\in A} X_a$ where each X_a is a path connected subspace. Then for each $n \geq 0$, we have $H_n(X) \cong \oplus_{a\in A} H_n(X_a)$.*

If X is itself path connected, then $H_0(X) \cong \mathbf{Z}$. Hence, in the general case $H_0(X)$ is a free abelian group with basis the path components of X.

Note that the indexing set A could be infinite. Also, you should reivew what you know about possibly infinite direct sums of abelians groups.

Proof. If σ is a singular n-simplex, then the image of σ must be contained in one of the X_a. Since, $S_n(X)$ is free with the set of singular n-simplices as basis, it follows that we can make an identification $S_n(X) = \oplus_{a\in A} S_n(X_a)$. Also, the faces of any singular n-simplex σ with image in X_a will be singular $(n-1)$-simplices with images in X_a. Thus, $\partial_n \sigma$ can be identified with a chain in $S_n(X_a)$. In other words, the above direct sum decomposition is consistent with the boundary operators. It follows that

$$\begin{aligned} Z_n(X) &= \oplus_a Z_n(X_a) \\ B_n(X) &= \oplus_a B_n(X_a) \\ H_n(X) &= \oplus_a H_n(X_a) \end{aligned}$$

Suppose next that X is path connected. $Z_0(X) = S_0(X)$ is free on the constant maps $\sigma : \Delta^0 \to X$ which may be identified with the points $x \in X$. However, given two points, $x_1, x_2 \in X$, there is a path $\sigma : \Delta^1 \to X$ from x_1 to x_2.

It follows that $\partial_1 \sigma = x_2 - x_1$ whence any two cycles in $Z_0(X)$ differ by a boundary. Fix any $x_0 \in X$. It follows that $H_0(X)$ is generated by the coset of x_0. We leave it to the student to show that this coset is of infinite order in $H_0(X) = Z_0(X)/B_0(X)$. $\square$

It should be noted that in the above proof, the direct sum decomposition is related to the geometry as follows. Let $i_a : X_a \to X$ be the inclusion of X_a in X. This induces $i_{a,n} : H_n(X_a) \to H_n(X)$ for each n.

The collection of these homomophisms induce

$$\oplus_a i_{a,n} : \oplus_a H_n(X_a) \to H_n(X).$$

Namely, any element of the left hand side can be expressed $(h_a)_{a\in A}$ where $h_a \in H_n(X_a)$ and $h_a = 0$ for all but a finite number of a. The map is

$$(h_a)_{a\in A} \mapsto \sum_{a\in A} i_{a,n}(h_a).$$

This homomorphism is in fact one of the two inverse isomorphisms of the Proposition.

3.2. Homotopies. We want to show that homotopic maps $f, g : X \to Y$ induce the same homomorphisms $f_n = g_n : H_n(X) \to H_n(Y)$ of singular homology groups. This requires introducing a certain kind of algebraic construction called a chain homotopy which arises out of the geometry.

To see how this comes about, let $F : X \times I \to Y$ be a homotopy from f to g, i.e., $F(x, 0) = f(x), F(x, 1) = g(x)$ for $x \in X$. Let $\sigma : \Delta^n \to X$ be a singular n-simplex in X. This induces a map $\sigma \times \mathrm{Id} : \Delta^n \times I \to X \times I$ which is called appropriately a *singular* $(n+1)$*-prism* in $X \times I$. F on the 'bottom' of this prism is basically $f_n(\sigma)$ and on the 'top' is $g_n(\sigma)$. The top and bottom are part of the boundary (in the naive sense) of the prism. The rest of the boundary might be described loosely as '$\partial_n\sigma \times I$' which is a 'sum' of prisms of one lower dimension based on the faces of σ.

The basic algebra we need reflects the geometry of the prism. We shall describe it for the standard prism $\Delta^n \times I$ where we can make do with affine maps. Our first difficulty is that a prism is not a simplex, so we have to subdivide it into simplices. We view $\Delta^n \times I$ as a subset of $\mathbf{R}^{n+2}$ with standard basis $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$. Then Δ^n may be viewed as the affine simplex spanned by the first $n+1$ basis vectors (the bottom of the prism), i.e., by setting $t = 0$. Let $\overline{\Delta}_n$ denote the affine simplex spanned by $\overline{\mathbf{e}}_0 = \mathbf{e}_0 + \mathbf{e}_{n+1}, \ldots, \overline{\mathbf{e}}_n = \mathbf{e}_n + \mathbf{e}_{n+1}$ (the top

of the prism), i.e., by setting $t = 1$. Define

$$p_n = \sum_{i=0}^{n} (-1)^i [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n].$$

As a sum of affine simplices in the space $\Delta_n \times I$, p_n may be viewed as an $(n+1)$-chain in $S_{n+1}(\Delta^n \times I)$. It corresponds to a decomposition of the prism into simplices with signs to account for orientations. The diagrams below show pictures for $n = 1, 2$.

More generally, if $\alpha = [\mathbf{x}_0, \ldots, \mathbf{x}_n]$ is any affine n-simplex (in some Euclidean space), define

$$p_n(\alpha) = \sum_{i=0}^{n} (-1)^i [\mathbf{x}_0, \ldots, \mathbf{x}_i, \overline{\mathbf{x}}_i, \ldots, \overline{\mathbf{x}}_n].$$

Viewing $\alpha : \Delta^n \to \operatorname{Im} \alpha$ as a map, we have have a corresponding map of prisms $\alpha \times \mathrm{Id} : \Delta^n \times I \to \operatorname{Im} \alpha \times I$ which induces

$$(\alpha \times \mathrm{Id})_{n+1} : S_{n+1}(\Delta^n \times I) \to S_{n+1}(\operatorname{Im} \alpha \times I)$$

and it is not hard to verify the formula

$$(\alpha \times \mathrm{Id})_{n+1}(p_n) = p_n(\alpha).$$

Furthermore, since $p_n(\alpha)$ has been defined for any affine n-simplex, we may define it by linearity on any linear combination of affine n-simplices. Such a linear combination may appear as an element of $S_n(X)$ for an appropriate subspace X of some Euclidean space. In particular,

$$p_n(\partial_{n+1} \Delta^{n+1}) = \sum_{i=0}^{n+1} (-1)^i p_n(\epsilon_i^{n+1}) \in S_{n+1}(\Delta^{n+1} \times I).$$

Include negative n under this rubric by setting such $p_n = 0$.

Proposition 6.7 (Standard Prism Lemma). *In $S_n(\Delta^n \times I)$, we have*

$$\partial_{n+1} p_n = \overline{\Delta}^n - \Delta^n - p_{n-1}(\partial_n \Delta^n).$$

This basically asserts that the boundary of the standard prism is what we expect it to be, but it includes the signs necessary to get the orientations right.

PROOF. For $n \geq 1$, we have

$$\begin{aligned}\partial_{n+1} p_n &= \sum_{i=0}^{n} (-1)^i \partial_{n+1}[\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n] \\ &= \sum_{i=0}^{n} (-1)^i \Big(\sum_{j=0}^{i} (-1)^j [\mathbf{e}_0, \ldots, \hat{\mathbf{e}}_j, \ldots \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n] \\ &\qquad + \sum_{j=i}^{n} (-1)^{j+1} [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \hat{\overline{\mathbf{e}}}_j, \ldots, \overline{\mathbf{e}}_n] \Big).\end{aligned}$$

First separate out the terms where $i = j$. These form the collapsing sum

$$\begin{aligned}\sum_{i=0}^{n} (-1)^{2i} ([\mathbf{e}_0, \ldots, \mathbf{e}_{i-1}, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n] - [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_{i+1}, \ldots, \overline{\mathbf{e}}_n]) \\ = [\overline{\mathbf{e}}_0, \ldots, \overline{\mathbf{e}}_n] - [\mathbf{e}_0, \ldots, \mathbf{e}_n] = \overline{\Delta}^n - \Delta^n.\end{aligned}$$

In the remaining sum, fix a j and consider all terms with that j and $i \neq j$. Write these in the order

$$\begin{aligned}(-1)^{j+1} \sum_{i=0}^{j-1} (-1)^i [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \hat{\overline{\mathbf{e}}}_j, \ldots, \overline{\mathbf{e}}_n]) \\ +(-1)^{j+1} \sum_{i=j+1}^{n} (-1)^{i-1} [\mathbf{e}_0, \ldots, \hat{\mathbf{e}}_j, \ldots \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n].\end{aligned}$$

(We have also reorganized the placement of various signs.) This adds up to

$$(-1)^{j+1} p_{n-1}(\epsilon_j^n) = -(-1)^j p_{n-1}(\epsilon_j^n).$$

Now add up for all $j = 0, \ldots, n$ to get

$$-\sum_{j=0}^{n} (-1)^j p_{n-1}(\epsilon_j^n) = -p_{n-1}(\sum_{j=0}^{n} (-1)^j \epsilon_j^n) = -p_{n-1}(\partial_n \Delta^n)$$

as required.

We leave it for the student to check this explicitly for $n = 0$. □

We may now carry this over to the spaces X and Y as follows. For a singular n-simplex $\sigma : \Delta^n \to X$, define

$$T_n(\sigma) = (F \circ (\sigma \times \mathrm{Id}))_{n+1}(p_n) \in S_{n+1}(Y).$$

Extending by linearity yields a homomorphism $T_n : S_n(X) \to S_{n+1}(Y)$. Also extend this to $n < 0$ by letting it be zero.

Proposition 6.8 (Chain Homotopy Lemma). *We have the following equality of homomorphisms $S_n(X) \to S_{n+1}(Y)$*

$$\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n = g_n - f_n.$$

This may also be written as a single equation encompassing all n

$$\partial \circ T_\sharp + T_\sharp \circ \partial = g_\sharp - f_\sharp.$$

The proof is given below, but first notice that we now have a proof of the following important result.

Theorem 6.9 (Homotopy Axiom). *Let $f, g : X \to Y$ be homotopic maps. Then for each n, $H_n(f) = H_n(g) : H_n(X) \to H_n(Y)$.*

This may also be written $f_* = g_* : H_*(X) \to H_*(Y)$.

Proof. Let c_n be a singular n-cycle representing some element of $H_n(X)$. By the above Lemma, since $\partial_n c_n = 0$, we have

$$\partial_{n+1}(T_n(c_n)) = g_n(c_n) - f_n(c_n).$$

It follow that the two terms on the right represent the same element of $H_n(Y)$. From the definition of the induced homomorphisms $H_n(g)$ and $H_n(f)$, it follows that they are equal. $\square$

Proof of Lemma 6.8. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. We have

$$\begin{aligned}
\partial_{n+1}(T_n(\sigma)) &= \partial_{n+1}(F_{n+1}((\sigma \times \mathrm{Id})_{n+1}(p_n))) \\
&= F_n((\sigma \times \mathrm{Id})_n(\partial_{n+1}(p_n))) \\
&= F_n((\sigma \times \mathrm{Id})_n(\overline{\Delta}^n - \Delta^n - p_{n-1}(\partial_n \Delta^n))).
\end{aligned}$$

Expand this out and consider each of the three terms in succession. In doing this, recall that each argument on which the functions are evaluated is in fact a chain in some chain group and as such is a linear combination of maps. For example, the first term is really the map $F \circ (\sigma \times \mathrm{Id}) \circ \overline{\Delta}^n$ from the *space* Δ^n to Y. However, $\overline{\Delta}^n : \Delta^n \to \Delta^n \times I$ is given by $\mathbf{x} \mapsto (\mathbf{x}, 1)$, so the map describing the first term is $g \circ \sigma$ or

$g_n(\sigma)$. Similarly, the second term is $f_n(\sigma)$. For the third term, we have

$$\begin{aligned}
F_n((\sigma \times \mathrm{Id})_n(p_{n-1}(\partial_n \Delta^n))) &= \sum_{i=0}^{n} (-1)^i F_n((\sigma \times \mathrm{Id})_n(p_{n-1}(\epsilon_i^n))) \\
&= \sum_{i=0}^{n} (-1)^i F_n((\sigma \times \mathrm{Id})_n((\epsilon_i^n \times \mathrm{Id})_n(p_{n-1})))) \\
&= \sum_{i=0}^{n} (-1)^i F_n((\sigma \circ \epsilon_i^n \times \mathrm{Id})_n((p_{n-1}))) \\
&= \sum_{i=0}^{n} (-1)^i T_{n-1}(\sigma \circ \epsilon_i^n) \\
&= T_{n-1}(\sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^n) = T_{n-1}(\partial_n \sigma).
\end{aligned}$$

(Make sure you understand the reasons for each step!) □

Corollary 6.10. *If $f : X \to Y$ is a homotopy equivalence, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

(We may abbreviate this by saying simply $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.)

Proof. Choose $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identities. Then $g_* \circ f_*$ and $f_* \circ g_*$ are the respective identities of homology groups and g_* is the inverse of f_*. □

Corollary 6.11. *Let X be a contractible space. Then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbf{Z}$.*

Proof. X is homotopy equivalent to a point space, and every point space has the indicated homology. □

3.3. More Homological Algebra. Let $f, g : C \to C'$ be morphisms of chain complexes. A *chain homotopy* from f to g is a collection of homomorphisms $T_n : C_n \to C'_{n+1}$ such that

$$d_{n+1} \circ T_n + T_{n-1} \circ d_n = g_n - f_n$$

for each n. From the above discussion, we see that homotopic maps of spaces induce chain homotopic homomorphisms of singular chain complexes. The following algebraic analogue of the homotopy axiom is easy to prove.

Proposition 6.12. *If $f, g : C \to C'$ are chain homotopic morphisms of chain complexes, then they induce the same homomorphisms $H_n(C) \to H_n(C')$ of homology groups of these chain complexes.*

PROOF. Look at the proof in the case of the singular complex of a space. $\square$

Let X be a contractible space; in particular suppose $\mathrm{Id}_X \sim \epsilon$ where $\epsilon : X \to X$ maps everything to a single point p. Then the theory above shows there is a chain homotopy consisting of homomorphisms $T_n : S_n(X) \to S_{n+1}(X)$ such that $\partial_{n+1}T_n + T_{n-1}\partial_n = \epsilon_n - \mathrm{Id}_n$ for each n. Unfortunately, even though ϵ is constant, the homomorphism ϵ_n induced by ϵ in each dimension won't generally be the trivial homomorphism. However, it is possible to remedy this situation by choosing homomorphisms $D_n : S_n(X) \to S_{n+1}(X)$ such that $\partial_{n+1}D_n + D_{n-1}\partial_n = -\mathrm{Id}_n$ for $n > 0$ and $\partial_1 D_0 = \epsilon_0 - \mathrm{Id}_0$ where ϵ_0 sends every 0-simplex to the 0-simplex at p. Such a collection of D_n is called a *contracting chain homotopy*. The existence of such a contracting chain homotopy implies directly that $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbf{Z}$.

4. The Exact Homology Sequence– the Jill Clayburgh Lemma

4.1. Some Homological Algebra. Let C be a chain complex and C' a subcomplex, i.e., C'_n is a subgroup of C_n for each n and $d_n(C'_n) \subseteq C'_{n-1}$ for each n. (The last condition can be stated simply $d(C') \subseteq C'$.) We form a chain complex C/C' called the *quotient chain complex* as follows. Put $(C/C')_n = C_n/C'_n$ and define $d''_n(\bar{c}_n) = \overline{d_n c_n}$ where, as usual, $\bar{c}_n = c_n + C'_n$ denotes the coset of c_n. It is not hard to check that the definition of d'' does not depend on the choice of coset representative. Generally, we shall say that a sequence of chain complexes

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$$

is exact if

$$0 \to C'_n \to C_n \to C''_n \to 0$$

is an exact sequence of groups for each n. According to this definition, if C' is a subcomplex of C, then

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' = C/C' \to 0,$$

where i is the inclusion monomorphism and j is the epimorphism on to the quotient complex, is an exact sequence of chain complexes.

PROPOSITION 6.13. *Given an exact sequence of chain complexes, we have the induced homomorphisms*

$$H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(j)} H_n(C'')$$

for all n. The above sequence is exact, i.e., $\operatorname{Ker} H_n(j) = \operatorname{Im} H_n(i)$.

PROOF. Since $j_n \circ i_n = 0$, it follows that $H_n(j) \circ H_n(i) = 0$ and the left hand side of the proposed equality contains the right hand side. For the reverse inclusion, let c be a cycle which represents an element of $\gamma \in H_n(C) = Z_n(C)/B_n(C)$. Supposing $H_n(j)(\gamma) = 0$, it follows that $j_n(c) \in B_n(C'')$, i.e.,

$$j_n(c) = d_{n+1}c'' = d_{n+1}j_{n+1}(x) = j_n(d_{n+1}x)$$

for some $x \in C_{n+1}$. Hence, $j_n(c - d_{n+1}x) = 0$. It follows that $c - d_{n+1}x = i_n c'$ for some $c' \in C'_n$. However, c' is a cycle, since i_n is a monomorphism and $i_n(d_n c') = d_n(i_n c') = d_n(c - d_{n+1}x) = 0$. This says that the homology class of c' maps to the homology class of $c - dx$ which is the same as the homology class of c. □

More importantly, we imbed the above sequence in a *long exact sequence*

$$\cdots \to H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(j)} H_n(C'')$$
$$\xrightarrow{\partial_n} H_{n-1}(C') \xrightarrow{H_{n-1}(i)} H_{n-1}(C) \xrightarrow{H_{n-1}(j)} H_{n-1}(C'') \to \dots$$

where $\partial_n : H_n(C'') \to H_n(C')$ is a collection of homomorphisms called *connecting homomorphims* which we now define. (The perceptive reader will notice a potential conflict of notation when we apply this to the singular chain complex of a space where the boundary homomorphisms have also been denoted by '∂'. Such conflicts are inevitable if one wants a notation which will act as a spur to the memory in involved situations. Usually they don't cause any problem if one keeps the context straight.) Let $c'' \in Z_n(C'')$ represent an element $\gamma'' \in H_n(C'')$. Since j is an epimorphism, we have $c'' = j(c)$ for some $c \in C_n$. Moreover, $j_{n-1}(d_n c) = d_n(j_n(c)) = d_n c'' = 0$. Hence, $d_n c = i_{n-1}c'$ for some $c' \in C'_{n-1}$. Since i_{n-2} is a monomorphism and $i_{n-2}(d_{n-1}c') = d_{n-1}i_{n-1}c' = d_{n=1}(d_n c) = 0$, it follows that c' is a cycle and it represents some element $\gamma' \in H_{n-1}(C')$. Let $\partial_n(\gamma'') = \gamma'$. We leave it to the student to verify that if c'' is changed to a homologous cycle $c'' + d_{n+1}x$, then the above process gives a cycle in C'_{n-1} which is homologous to c'. Hence, the definition of $\partial_n \gamma''$ does not depend on the choice of the cycle c. We also leave it to the student to prove that ∂_n is a homomorphism for each n.

PROPOSITION 6.14. *The above sequence is exact at every place.*

PROOF. In doing the calculations, we shall drop subscripts when dealing with elements and homomorphisms at the chain level.

$\operatorname{Ker} H_{n-1}(i) = \operatorname{Im} \partial_n$:

Let $c'' \in Z_n(C'')$ represent an element $\gamma'' \in H_n(C'')$. Then $\partial_n\gamma''$ is represented by $c' \in Z_{n-1}(C')$ where $i(c') = dc, j(c) = c''$. Then $i_{n-1}(\partial_n\gamma'')$ is represented by $i(c')$ which is a boundary. Hence, $H_{n-1}(i) \circ \partial_n = 0$.

We leave it as an exercise for the student to show that every element satisfying $i_{n-1}\gamma' = 0$ is of the form $d_n\gamma''$.

$\operatorname{Ker} \partial_n = \operatorname{Im} H_n(j)$:

We leave it as an exercise for the student to show that $\partial_n \circ j_n = 0$.

Suppose that $d_n\gamma'' = 0$ and $c'' \in Z_n(C'')$ represents γ''. That means if we choose $c \in C_n$ such that $j(c) = c''$, then $dc = i(dx')$ for some $x \in C'_{n-1}$. Then $d(c - ix') = 0$, so $c - ix' \in Z_n(C)$. Also, $j(c - ix') = j(c) = c''$. Let $c - ix'$ represent $\gamma \in H_n(C)$. We have $j_n(\gamma) = \gamma''$. $\square$

A little more formalism provides another way to think of the long exact sequence. We define a *graded abelian group* A to be a collection of abelian groups $A_n, n \in \mathbf{Z}$. (Thus a graded abelian group may be viewed as a complex in which the 'd' homomorphisms are trivial.) A morphism $f : A \to B$ from one graded group to another *of degree* k consists of a collection of homomorphisms $f_n : A_n \to B_{n+k}$. Thus, a chain complex C is a graded group together with a morphism d of degree -1. Similarly, a chain homotopy is a morphism of graded groups of degree $+1$.

Using this language, we may think of the symbols $H_*(C'), H_*(C)$, and $H_*(C'')$ as denoting graded groups with morphisms i_*, j_*, ∂_* of degrees $0, 0, -1$ respectively. Then, the long exact sequence may be summarized by the exact triangular diagram

$$\begin{tikzcd}
H_*(C') \arrow[rr, "H_*(i)"] & & H_*(C) \arrow[dl, "H_*(j)"] \\
& H_*(C'') \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

Note that the linear long exact sequence relating the component groups may be thought of as covering this diagram just as the real line covers the circle as its universal covering space.

We now apply the above algebra to singular homology. Let X be a space and A a subspace. Let $i : A \to X$ denote the inclusion. Clearly, we may identify $S_*(A)$ with the subcomplex $i_\sharp(S_*(A))$ of $S_*(X)$. Define the quotient chain complex

$$S_*(X, A) = S_*(X)/S_*(A),$$

i.e., $S_n(X, A) = S_n(X)/S_n(A)$ and $\partial_n : S_n(X, A) \to S_{n-1}(X, A)$ is the quotient of $d_n : S_n(X) \to S_{n-1}(X)$. We call $S_*(X, A)$ the *relative singular chain complex*. Its homology groups are called the *relative singular homology groups* and denoted

$$H_n(X, A).$$

The geometric significance of the relative homology groups is a bit murky. It has something to do with the homology of the quotient space X/A, but they are not exactly the same. $S_n(X, A)$ has as a basis the set of cosets of all singular n-simplices which do not have images in A. (Why?) However, two singular n-simplices $\sigma \neq \sigma'$ in X could project to the same singular n-simplex in X/A. Thus, there is no simple relation between $S_n(X, A)$ and $S_n(X/A)$. We shall see later that in certain cases, $H_n(X, A)$ is indeed the same as $H_n(X/A)$.

The short exact sequence of chain complexes

$$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$$

induces a long exact homology sequence

$$\begin{aligned} \cdots \to H_n(A) \xrightarrow{i_n} H_n(X) \to H_n(X, A) \\ \xrightarrow{\partial_n} H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X, A) \to \dots \end{aligned}$$

The homomorphism $\partial_n : H_n(X, A) \to H_{n-1}(A)$ has a simple naive geometric interpretation. A cycle in $S_n(X, A)$ may be viewed as a n-chain in X with boundary in A. Such a boundary defines an $(n-1)$-cycle in A.

The relative singular homology groups may be thought of as generalizations of the (absolute) singular homology groups. Indeed, if we take $A = \emptyset$ to be the empty subspace, then $S_n(A)$ is the trivial subgroup of $S_n(X)$ for every n, and $H_n(X, A) = H_n(X)$. In addition, all the properties we have discussed for absolute groups also hold for relative groups, but there are some minor changes that need to be made. For example, if X is a point space, then $H_n(X, A) = 0$ for $n > 0$ for either of the two possible subspaces $A = X$ or $A = \emptyset$, but $H_0(X, X) = 0$. Similarly, if X is a disjoint union of path connected subspaces X_a and A is a subspace of X, then it is not too hard to see that $H_n(X, A) \cong \oplus_a H_n(X_a, X_a \cap A)$, but if X is path connected, we have $H_0(X, A) = 0$ for $A \neq \emptyset$. (Use the long exact sequence and the fact that $H_0(A) \to H_0(X)$ is an isomorphism in that case.) The functoriality holds without qualification. Namely, if $f : (X, A) \to (Y, B)$ is a map of pairs, then $f_\sharp(S_*(A)) \subseteq S_*(B)$, so $f_\sharp$ induces a chain morphism $S_*(X, A) \to S_*(Y, B)$ which in turn induces a homomomorphism

$f_* : H_*(X, A) \to H_*(Y, B)$. It is not hard to check that this provides a functors H_n from the category of pairs of topological spaces to the category of abelian groups (or if you prefer a single functor to the category of graded groups).

The student should verify at least some of the assertions made in the above paragraph, although none of them is particularly startling.

The homotopy axiom for relative singular homology requires a bit more discussion. Let $f, g : (X, A) \to (Y, B)$ be maps of pairs of spaces. A *relative homotopy* from f to g is a map $F : X \times I \to Y$ which is a homotopy from f to g in the ordinary sense and which also satisfies the condition $F(A \times I) \subseteq B$. Note that this implies that the restriction $F' : A \times I \to B$ of F is a homotopy of the restrictions $f', g' : A \to B$ of f, g. It is not generally true, however, that if $f \sim g$ and $f' \sim g'$ that there is a relative homotopy from f to g. (Try to find a counterexample and insert it here in your notes!)

PROPOSITION 6.15 (Relative Homotopy Axiom). *Let $f, g : (X, A) \to (Y, B)$ be relatively homotopic. Then $f_* = g_* : H_*(X, A) \to H_*(Y, B)$.*

PROOF. We just have to follow through the proof in the absolute case and see that everything relativizes properly. By definition, $T_n(\sigma) = (F \circ (\sigma \times \mathrm{Id}))_{n+1}(p_n)$ for σ a singular n-simplex in X. If σ is in fact a singular n-simplex in A, then by the hypothesis, the image of $F \circ (\sigma \times \mathrm{Id})$ is in B. It follows that $T_n(\sigma)$ lies in $S_{n+1}(B)$. Thus, T_n induces a homomorphism of quotients $\overline{T}_n : S_n(X, A) \to S_{n+1}(Y, B)$, and the formula

$$\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n = g_n - f_n$$

projects onto the corresponding formula for $\overline{T}_n$. It then follows as before that f and g induce the same homomorphism of relative homology. □

The *existence of the long exact homology sequence* (which relates absolute and relative homology) is one of the *axioms* we shall use to derive properties of a homology theory. However, there is one important aspect of this sequence we haven't mentioned. Namely, the connecting homomorphisms $\partial_n : H_n(X, A) \to H_n(A)$ are 'natural' in the following sense.

PROPOSITION 6.16 (Naturality of the Connecting Homomorphism). *Let $f : (X, A) \to (Y, B)$ be a map of pairs. Let $f' : A \to B$ be the*

restriction of f. Then the following diagram commutes

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f'_n} \\ H_n(Y,B) & \xrightarrow{\partial_n} & H_{n-1}(B) \end{array}$$

commutes.

Proof. This follows from the homological algebra below. ☐

4.2. More Homological Algebra.

Proposition 6.17. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & D' & \longrightarrow & D & \longrightarrow & D'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain complexes. Then the induced diagrams

$$\begin{array}{ccc} H_n(C'') & \xrightarrow{\partial_n} & H_{n-1}(C') \\ \downarrow{\scriptstyle f''_n} & & \downarrow{\scriptstyle f'_{n-1}} \\ H_n(D'') & \xrightarrow{\partial_n} & H_{n-1}(D') \end{array}$$

commute.

Proof. Let $\gamma'' \in H_n(C'')$, and let $c'' \in Z_n(C'')$ represent it. Choose $c \in C_n$ which maps to c'' and $c' \in Z_{n-1}(C')$ which maps to dc. $f'(c')$ represents $f'_{n-1}(\partial_n(\gamma''))$. Because the diagram of chain complexes is commutative, $f(c) \in D_n$ maps to $f''(c'')$ and $f'(c') \in Z_{n-1}(D')$ maps to $df(c) = f(dc)$. This says that $f'(c')$ also represents $\partial_n(f''_n(\gamma''))$. ☐

The *long exact sequence axiom* should now be taken to assert the existence of connecting homomophisms with the above naturailty property and such that the long homology sequence is exact.

The structure of singular homology theory can be made a bit cleaner by treating it entirely as a functor on the category of pairs (X, A) of spaces and maps of such. As noted above, the absolute singular homology groups are included in this theory by considering pairs of the form $(X, \emptyset)$. To make this theory look a bit more symmetric, one may extend the the long exact sequence to pairs as follows. Suppose

we have spaces $A \subseteq B \subseteq X$. Then, it is not too hard to derive an exact homology sequence

$$
\begin{tikzcd}
H_*(B,A) \arrow[rr] & & H_*(X,A) \arrow[dl] \\
& H_*(X,B) \arrow[ul, "\partial_*"] &
\end{tikzcd}
$$

where $\partial_* : H_*(X,B) \to H_*(B,A)$ is a morphism of degree -1 and natural in an obvious sense. We leave this derivation as an exercise for the student.

As we saw above in our discussion of the dimension axiom, the dimension $n = 0$ tends to create exceptions and technical difficulties. Another example of this is the comparison between $H_n(X)$ and $H_n(X, \{P\})$ where P is a point of X. Since $H_n(\{P\}) = 0$ for $n > 0$, the exact homology seqeunce

$$\to H_n(\{P\}) = 0 \to H_n(X) \to H_n(X, \{P\}) \to H_{n-1}(\{P\}) = 0 \to$$

shows that $H_n(X) \cong H_n(X, \{P\})$ for $n > 1$. It is in fact true that this is true also for $n = 1$. This follows from the fact that $H_0(\{P\}) \to H_0(X)$ is a monomorphism, which we leave as an exercise for the student. However, another approach to this issue to to introduce the *reduced homology groups*. These are defined as follows. Consider the unique map $X \to \{P\}$ to any space consisting of a single point. Define $\tilde{H}_*(X) = \operatorname{Ker} H_*(X) \to H_*(\{P\})$. It is not too hard to see that this group does not depend on which particular single point space you use. Also, it is consistent with induced homomorphisms, so it provides a functor on the category of spaces. Note that $\tilde{H}_n(X) = H_n(X)$ for $n \neq 0$, so only in dimension zero is there is a difference. The reduced homology groups give us a way to avoid exceptions in dimension zero. Indeed, $\tilde{H}_0(\{P\}) = 0$ so the dimension axiom may be rephrased by asserting that all the reduced homology groups of a point space are trivial. Similarly, all the reduced homology groups of a contractible space are trivial. Finally, it is possible to show that the exact homology sequence for a pair remains valid if we replace $H_*(X)$ and $H_*(A)$ by the corresponding reduced groups. Of course, this only makes a difference at the tail end of the sequence

$$\to H_1(X,A) \xrightarrow{\partial_1} \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X,A) \to 0.$$

(This includes the assertion that $\operatorname{Im} \partial_1 \subseteq \tilde{H}_0(A) = \operatorname{Ker} H_0(A) \to H_0(\{P\})$.) Using this exact sequence and the fact that $\tilde{H}_0(\{P\}) = 0$, we see that we have quite generally $\tilde{H}_n(X) \cong H_n(X, \{P\})$ for any non-empty space X.

5. Excision and Applications

We now come to an important property called the *excision axiom*. This is quite powerful and it will allow us finally to calculate some interesting homology groups, but it is somewhat technical. Its significance will become clear as we use it in a variety of circumstances. The excision axiom says that given a space X and subspace A, we can 'cut out' subsets of A which are not too large without changing relative homology. We would like to be able to 'cut out', all of A, but in general that is not possible.

THEOREM 6.18 (Excision Axiom). *Let $A \subseteq X$ be spaces. Suppose U is a subspace of A with the property that $\overline{U}$, the closure of U, is contained in A°, the interior of A. Then the inclusion $(X-U, A-U) \to (X, A)$ induces an isomorphism*

$$H_n(X-U, A-U) \cong H_n(X, A)$$

of relative homology groups for every n.

The proof of this theorem is quite difficult, so we shall defer it until we have derived some important consequences.

Using excision, we can finally calculate some homology groups.

LEMMA 6.19. *Let $A^n = \{\mathbf{x} \in D^n \mid 1/2 \le |\mathbf{x}| \le 1\}$. Then for each i and each $n > 0$,*

$$H_i(D^n, A^n) \cong \tilde{H}_{i-1}(S^{n-1}).$$

PROOF. The exact sequence

$$\tilde{H}_i(D^n) = 0 \to H_i(D^n, A^n) \to \tilde{H}_{i-1}(A^n) \to \tilde{H}_{i-1}(D^n) = 0$$

shows that $H_i(D^n, A^n) \cong \tilde{H}_{i-1}(A^n)$. On the other hand, $A^n = S^{n-1} \times I$, so S^{n-1} is a deformation retract of A^n and $\tilde{H}_{i-1}(A^n) \cong \tilde{H}_{i-1}(S^{n-1})$. □

THEOREM 6.20. *We have*

$$H_0(S^n) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{if } n = 0 \\ \mathbf{Z} & \text{if } n > 0. \end{cases}$$

For $i > 0$,

$$H_i(S^n) = \begin{cases} 0 & \text{if } 0 < i < n \text{ or } i > n \\ \mathbf{Z} & \text{if } i = n. \end{cases}$$

Proof. The zero sphere S^0 consists of two points and the conclusion is clear for it. Note that $\tilde{H}_0(S^0) \cong \mathbf{Z}$ and in fact it is easy to see that it consists of all elements of the form $(n, -n)$ in $\mathbf{Z} \oplus \mathbf{Z}$. (Just consider the kernel of $H_0(S^0) \to H_0(\{P\}) = \mathbf{Z}$.)

The remaining statements amount to the assertion that for $n > 0$, $\tilde{H}_i(S^n) = 0$ except for $i = n$ in which case we get $\mathbf{Z}$. To prove this latter assertion, proceed as follows. Let $U^n = \{\mathbf{x} \in S^n \mid -1 \leq x_{n+1} < 0\}$ and let $B^n_+ = \{\mathbf{x} \in S^n \mid -1 \leq x_{n+1} \leq 1/2\}$. By excision, we have $H_i(S^n - U^n, B^n - U^n) \cong H_i(S^n, B^n)$. However, the pair $(D^n_+ = S^n - U^n, A^n_+ = B^n - U^n)$ is clearly homeomorphic to the pair (D^n, A^n). It follows from this and the above lemma that $H_i(S^n, B_n) \cong H_{i-1}(S^{n-1})$. On the other hand, since B^n is contractible, the long exact sequence for the pair (S^n, B^n) shows that $\tilde{H}_i(S^n) \cong H_i(S^n, B^n)$ for all i. Putting these facts together, we get

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}) \cong \ldots \cong \tilde{H}_{i-n}(S^0).$$

Since the last term is zero if $i \neq n$ and $\mathbf{Z}$ if $i = n$, we are done. □

Important Note on the Proof. There is one fact which can be squeezed out of this proof which we shall need later. Namely, consider the map $r^n : S^n \to S^n$ which sends $x_0 \to -x_0$ and fixes all other coordinates. (This is the reflection in the hyperplane perpendicular to the x_0 axis.) If you carefully check all the isomorphisms in the above proof, you will see that they may be chosen to be consistent with the maps r^n, i.e., so that the diagram

$$\begin{tikzcd}
\tilde{H}_i(S^n) \arrow[r, "\cong"] \arrow[d, "r^n_*"'] & H_{i-1}(S^{n-1}) \arrow[d, "r^{n-1}_*"] \\
H_i(S^n) \arrow[r, "\cong"'] & H_{i-1}(S^{n-1})
\end{tikzcd}$$

commutes. You should go through all the steps and check this for yourself. It is a good exercise in understanding where all the isomorphisms used in the proof come from.

Once we have calculated the homology groups of spheres, we may derive a whole lot of consequences.

Theorem 6.21. *(a) S^n and S^m for $n \neq m$ do not have the same homotopy type.*

(b) R^n and R^m for $n \neq m$ are not homeomorphic.

(c) S^n for $n > 1$ is a space which is simply connected but not contractible.

Proof. You should be able to figure these out. ☐

The Brouwer Fixed Point Theorem now follows exactly as proposed in the first introduction.

Theorem 6.22 (Brouwer). *Any map $f : D^n \to D^n, n > 0$, has a fixed point.*

Proof. Go back and look at the introduction. The nonexistence of a fixed point allowed us to construct a retraction of D^n onto S^{n-1} whence $H_n(S^n) = \mathbf{Z}$ is a direct summand of $H_n(D^n) = 0$. ☐

5.1. Degree and Vector Fields on Spheres. Let $f : S^n \to S^n$. We call such a map a *self–map.* It induces a homomorphism $f_* : H_n(S^n) = \mathbf{Z} \to H_n(S^n) = \mathbf{Z}$. (For completeness, use $\tilde{H}_0$ for $n = 0$.) Such a homomorphism is necessarily multiplication by some integer d which is called the *degree* of the map f and denoted $d(f)$. As in our discussion of the fundamental group, the degree measures in some sense how many times the image of the map covers the sphere.

Proposition 6.23. *Degree has the following properties.*

(a) Given self–maps, f, g of S^n, we have $d(g \circ f) = d(g)d(f)$. Also, the degree of the identity is 1 and the degree of a constant self–map is zero.

(b) Homotopic self maps have the same degree.

(c) Reflection in a hyperplane through the origin has degree -1.

(d) The antipode map has degree $(-1)^{n+1}$.

The converse of (b) is also true, i.e., self–maps with the same degree are homotopic. (This follows from an important theorem called Hopf's Theorem, which you will see later.)

Proof. Everything in (a) follows by functorality, i.e., $\mathrm{Id}_* = \mathrm{Id}$ and $(g \circ f)_* = g_* \circ f_*$. To see that the degree of a constant map is zero, factor it through a point space.

(b) follows from the homotopy axiom.

(d) follows from (a) and (c). Namely, the antipode map $\mathbf{x} \mapsto -\mathbf{x}$ may be obtained by composing the $n + 1$ component reflections

$$r_k : x_i \mapsto \begin{cases} x_i & i \neq k \\ \mapsto -x_i & i = k. \end{cases}$$

Finally, to prove (c), choose a coordinate system so that the hyperplane is given by $x_0 = 0$ and the associated reflection is $x_0 \mapsto -x_0$, with the other coordinates fixed. We noted above that the calculation $H_n(S^n) \cong \tilde{H}_0(S^0) \cong \mathbf{Z}$ is consistent with the reflection (by induction).

However, $\tilde{H}_0(S^0)$ can be identified with the subgroup of $\mathbf{Z} \oplus \mathbf{Z}$ consisting of all $(n, -n)$ where a basis for $\mathbf{Z} \oplus \mathbf{Z}$ may be taken to be the two points $x_0 = 1$ and $x_0 = -1$ in S^0. Clearly, the reflection swtiches these points and so sends $(n, -n)$ to $(-n, n) = -(n, -n)$. $\square$

Proposition 6.24. *Let f, g be self–maps of S^n. If $f(x) \neq g(x)$ for all $x \in S^n$, then $g \sim a \circ f$ where a is the antipode map.*

Proof. Since $g(x)$ is never antipodal to $-f(x)$, the line connecting them never passes through the origin. Define $G(x,t) = t(-f(x)) + (1-t)g(x)$, so $G(x,t) \neq 0$ for $0 \leq t \leq 1$. Define $F(x,t) = G(x,t)/|G(x,t)|$. $F(x,0) = g(x)$ and $F(x,1) = -f(x) = a(f(x))$. $\square$

A *tangent vector field* T on a sphere S^n is a function $T : S^n \to \mathbf{R}^{n+1}$ such that for each $x \in S^n$, $T(x) \cdot x = 0$. Thus, we may view the vector $T(x)$ sitting at the end of the vector x on S^n and either it is zero or it is tangent to the sphere there. (This is a special case of a much more general concept which may be defined for any differentiable manifold.)

Theorem 6.25. *There do not exist non-vanishing vector fields T defined on an even dimensional sphere S^{2n}.*

For the case S^2, this is sometimes interpreted as saying something about the possibility of combing hair growing on a billiard ball.

Proof. Assume T is a nonvanishing tangent vector field on S^{2n}. Let $t(x) = T(x)/|T(x)|$. Since $t(x) \perp x$, we certainly never have $t(x) = x$. It follows from the propostion that $t \sim a \circ \mathrm{Id} = a$. Hence, $d(t) = (-1)^{2n+1} = -1$. However, it is also true that $t(x) \neq -x$ for the same reason, so $t \sim a \circ a = \mathrm{Id}$. Thus, $d(t) = 1$. Thus, we have a contradiction to the assumption that T never vanishes. $\square$

Odd dimensional spheres do have non-vanishing vector fields. For example, for $n = 1$, we have $T(x_0, x_1) = (-x_1, x_0)$.

Similarly for odd $n > 1$, define

$$T(x_0, \dots, x_n) = (-x_1, x_0, -x_3, x_2, \dots, -x_n, x_{n-1}).$$

One can ask how many *linearly independent* vector fields there are on S^{2n+1} as a function of n. A lower bound was established by Hurwitz and Radon in the 1920s, but the proof that this number is also an upper bound was done by J. F. Adams in 1962 using K-theory.

6. Proof of the Excision Axiom

Suppose $\overline{U} \subseteq A^\circ$ as required by the excision axiom. Consider the inclusion monomorphism $i_\sharp : S_*(X - U) \to S_*(X)$. Since a singular simplex has image both in $X - U$ and in A if and only if its image is in $A - U$, it follows that $S_*(X - U) \cap S_*(A) = S_*(A - U)$. By basic group theory, this tells us that $i_\sharp$ induces a monomomorphism of quotients

$$i_\sharp : S_*(X - U, A - U) \to S_*(X, A).$$

The image of this monomorphism is

$$S'_*(X, A) = (S_*(X - U) + S_*(A))/S_*(A)$$

which is a subcomplex of $S_*(X)/S_*(A) = S_*(X, A)$. We shall show below that the inclusion $S'_*(X, A) \to S_*(X, A)$ induces an isomorphism in homology. Putting this together with $i_\sharp$, we see that the induced homomorphism $H_*(X - U, A - U) \to H_*(X, A)$ is an isomorphism.

We shall accomplish the desired task by proving something considerably more general. Let $\mathcal{U}$ denote a collection of subspaces $\{U\}$ of X such that the interiors $\{U^\circ\}$ *cover* X. In the above application, the collection consists of two sets A and $X - U$. (Under the excision hypothesis, we have

$$X = (X - \overline{U}) \cup A^\circ$$

so the interiors cover.) Consider the subcomplex $S^{\mathcal{U}}_*(X)$ with basis all singular simplices with images in some subspace U in $\mathcal{U}$. Such singular chains are called *$\mathcal{U}$-small*. Similarly, let $S^{\mathcal{U}}_*(A)$ be the corresponding subcomplex for A and $S^{\mathcal{U}}_*(X, A)$ the resulting quotient complex. Since $S^{\mathcal{U}}_*(A) = S^{\mathcal{U}}_*(X) \cap S_*(A)$, it follows that $S^{\mathcal{U}}_*(X, A)$ may be identified with a subcomplex of $S_*(X, A)$. We shall show in all these cases that the inclusion $S^{\mathcal{U}}_*(-) \to S_*(-)$ induces an isomorphism in homology. The student should verify that in the above application

$$\begin{aligned} S^{\mathcal{U}}_*(X) &= S_*(X - U) + S_*(A) \\ S^{\mathcal{U}}_*(A) &= S_*(A) \\ S^{\mathcal{U}}_*(X, A) &= S'_*(X, A). \end{aligned}$$

In order to prove the one isomorphism we want, that in the relative case, it suffices to prove the isomomorphisms for X and A. For, the

diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^{\mathcal{U}}_*(A) & \longrightarrow & S^{\mathcal{U}}_*(X) & \longrightarrow & S^{\mathcal{U}}_*(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X,A) & \longrightarrow & 0
\end{array}$$

induces a diagram in homology

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{\mathcal{U}}_n(A) & \longrightarrow & H^{\mathcal{U}}_n(X) & \longrightarrow & H^{\mathcal{U}}_n(X,A) & \longrightarrow & H^{\mathcal{U}}_{n-1}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \cdots
\end{array}$$

Suppose we have shown that the vertical arrows between the absolute groups are isomorphisms. Then it follows from the following result that the vertical arrows between the relative groups are isomorphisms as well.

Proposition 6.26 (Five Lemma). *Suppose we have a commutative diagram of abelian groups*

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

with exact rows. If f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism.

Proof. You will have to do this yourself. It is just a tedious diagram chase. See the Exercises where you will be asked to prove something slightly more general, the so-called 'Four Lemma'. $\square$

It now follows that we need only prove the desired isomorphism in the absolute case and then apply it separately to X and to A. The rest of this section will be concerned with that task.

6.1. Barycentric Subdivision. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Our approach will be to subdivide Δ^n into affine sub-simplices of small enough diameter such that σ carries each into at least one set in the covering. We know this is possible if the sets are small enough by the Lebesgue Covering Lemma since $\sigma(\Delta^n)$ is compact. Since the

Lebesgue Covering Lemma requires an open covering, we need to assume the interiors of the sets cover X.

The subdivision accomplished by iterating the process of *barycentric subdivision* which we now describe. We define this in general for affine n-simplices by induction. First, we need some notation Given an affine n-simplex $\alpha = [\mathbf{x}_0, \ldots, \mathbf{x}_n]$, let

$$\mathbf{b}(\alpha) = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{x}_i$$

denote the *barycenter* of the vertices. Generally, let $a = \sum_j \alpha_j$ be an affine p-chain, i.e., a linear combination of affine p-simplices. If $\mathbf{b}$ is any point not in the affine subspaces spanned by the vertices of any α_j in a, then let $[\mathbf{b}, a] = \sum_j [\mathbf{b}, \alpha_j]$.

We define an operator Sd which associates with each affine n-chain a another affine n-chain $Sd_n(a)$. We do this by defining it on affine simplices and extending by linearity. For a zero-simplex $\alpha = [x_0]$, let $Sd_0(\alpha) = \alpha$. Assume Sd_{n-1} has been defined, and let α be an affine n-simplex. Define

$$Sd_n(\alpha) = [\mathbf{b}(\alpha), Sd_{n-1}(\partial_n \alpha)].$$

The n-simplices occuring in $Sd_n\alpha$ constitute the barycentric subdivision of α. We claim that this subdivision operator is consistent with the boundary operator for affine simplices. Namely, if $\alpha = [\mathbf{x}_0, \ldots, \mathbf{x}_n]$, then abbreviating $\mathbf{b} = \mathbf{b}(\alpha)$ and $\alpha_i = \alpha \circ \epsilon_i^n$ (the ith face of α), we have

$$\partial_n(Sd_n(\alpha)) = \partial_n[\mathbf{b}, Sd_{n-1}(\partial_n \alpha)].$$

However, it is not hard to check that for any affine p-chain a we have

$$\begin{aligned} \partial_{p+1}[\mathbf{b}, a] &= a - [\mathbf{b}, \partial_p a] \qquad p > 0 \\ \partial_1[\mathbf{b}, a] &= a - (\tilde{\partial}_0(a)\mathbf{b} \qquad p = 0, \end{aligned}$$

where $\tilde{\partial}_0$ denotes the sum of the coefficients operator. (Just prove the formula for simplices and extend by linearity.) Hence, it follows that for $n > 1$,

$$\partial_n(Sd_n(\alpha)) = \partial_n[\mathbf{b}, Sd_{n-1}(\partial_n \alpha)] = Sd_{n-1}(\partial_n \alpha) - [\mathbf{b}, \partial_{n-1}(Sd_{n-1}(\partial_n \alpha))].$$

By induction, $\partial_{n-1} Sd_{n-1}\partial_n = Sd_{n-2}\partial_{n-1}\partial_n = 0$, so we conclude $\partial_n Sd_n = Sd_{n-1}\partial_n$ as required. The proof for $n = 1$ is the same except we use that $\tilde{\partial}_0\partial_1 = 0$.

We now extend the operator Sd to singular chains in any space by defining for a singular n-simplex $\sigma : \Delta^n \to X$,

$$Sd_n^X(\sigma) = \sigma_\sharp(Sd_n\Delta^n).$$

As usual, define Sd_n to be zero in negative dimensions because that is the only possible definition.

Note first that there is a slight subtlety in the definition of Sd. For an affine n-simplex viewed as a map from the standard simplex, there are two possible definitions of $Sd_n(\alpha)$, either the original definition or $\alpha_\sharp(\Delta^n)$. If you examine the arguments we shall use carefully, we must know that these definitions are the same. We leave that for the student to verify.

Note next that Sd^X has an important naturality property, i.e., if $f : X \to Y$ is a map, then

$$\begin{aligned} f_\sharp(Sd_n^X(\sigma) &= f_\sharp(\sigma_\sharp(Sd_n\Delta^n) \\ &= (f\circ\sigma)_\sharp(Sd_n\Delta^n) \\ &= Sd_n^Y(f\circ\sigma) = Sd_n^Y(f_\sharp(\sigma)) \end{aligned}$$

so

$$f_\sharp \circ Sd^X = Sd^Y \circ f_\sharp.$$

Thirdly, we note that Sd^X commutes with the boundary operator on $S_*(X)$. We leave that as an exercise for the student.

LEMMA 6.27. *Let X be a space. There exist natural homomorphisms $T_n^X : S_n(X) \to S_{n+1}(X)$ such that*

$$\partial_{n+1}T_n^X + T_{n-1}^X\partial_n = Sd_n^X - \mathrm{Id}_n$$

for each n.

In other words, the subdivision operator is chain homotopic to the identity.

PROOF. Rather than trying to define T_n explicitly, we use a more abstract approach. A vast generalization of this method called the *method of acyclic models* allows us to proceed in a similar manner in many different circumstances. We shall go into this method in greater detail later, but for the moment concentrate on how we use the contractibility of certain standard spaces, i.e., affine simplices, to define T_n.

We proceed by induction. Let $T_n^X = 0$ for $n < 0$. For $n = 0$, it is not hard to see that $Sd_0^X = \mathrm{Id}_0$ for any space X, so we may take $T_0^X = 0$. Suppose $n > 0$ and T_p^X has been defined for every $p < n$ and every space X, that it is natural in the sense that if $f : X \to Y$ is a map then $f_\sharp \circ T^X = T^Y \circ f_\sharp$, and that it satifies the desired formula. Let $T = T^{\Delta^n}$, and consider

$$u = Sd_n(\Delta^n) - \Delta^n - T_{n-1}(\partial_n \Delta^n).$$

We have

$$\begin{aligned}
\partial_n u &= \partial_n Sd_n(\Delta^n) - \partial_n \Delta^n - \partial_n T_{n-1}(\partial_n \Delta^n) \\
&= Sd_{n-1}(\partial_n \Delta^n) - \partial_n \Delta^n - Sd_{n-1}(\partial_n \Delta^n) + \partial_n \Delta_n + T_{n-2}(\partial_{n-1}\partial_n \Delta^n) \\
&= 0.
\end{aligned}$$

Since Δ^n is contractible, its homology in all dimensions $n > 0$ is trivial. Hence, $u = \partial_{n+1} t_n$ for some singular $(n+1)$-chain t_n in $S_{n+1}(\Delta^n)$. By construction,

$$\partial_{n+1} t_n + T_{n-1}(\partial_n \Delta^n) = Sd_n(\Delta^n) - \Delta^n.$$

Now define $T_n^X : S_n(X) \to S_{n+1}(X)$ by

$$T_n^X(\sigma) = \sigma_\sharp(t_n)$$

for singular n-simplices σ in X. As in the case of Sd_n^X, the naturality of T_n^X follows immediately from its definition. (Check this yourself!) The desired formula is verified as follows.

$$\begin{aligned}
\partial_{n+1} T_n^X(\sigma) &= \partial_{n+1} \sigma_\sharp t_n = \sigma_\sharp \partial_{n+1} t_n \\
&= \sigma_\sharp Sd_n(\Delta^n) - \sigma_\sharp \Delta^n - \sigma_\sharp T_{n-1}(\partial_n \Delta^n) \\
&= Sd_n^X \sigma_\sharp(\Delta^n) - \sigma_\sharp \Delta^n - T_{n-1}^X \sigma_\sharp(\partial_n \Delta^n) \\
&= Sd_n^X(\sigma) - \sigma - T_{n-1}^X(\partial_n \sigma)
\end{aligned}$$

as required. □

Note that it follows easily from this that $(Sd^X)^q$ is chain homotopic to the identity for each q. (See the Exercises.)

We are now ready to prove

THEOREM 6.28. *Let $\mathcal{U}$ be a collection of subspaces with interiors covering X. Then $H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.*

PROOF. In order to reduce to consideration of $\mathcal{U}$-small chains, we iterate the process of barycentric subdivision. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Denote by $|\sigma|$ its image, also called its *support*.

(Similarly, the support $|c|$ of any chain is the union of the supports of the simplices occuring in it.)

We shall show first that for q sufficiently large, $(Sd^X)^q(\sigma)$ is $\mathcal{U}$-small, i.e., the support $|\tau|$ of every singular simplex τ in $(Sd^X)^q(\sigma)$ is contained in some set of the covering. Since any singular chain involves at most finitely many singular simplices, the same asserstion will then apply to each singular chain. By naturality,

$$(Sd^X)^q(\sigma) = \sigma_\sharp(Sd^q(\Delta^n)).$$

(As usual, for α an affine simplex $Sd(\alpha)$ may be viewed as an affine chain in any convenient subspace of $\mathbf{R}^N$, where N is a convenient integer. In this case, the subspace is Δ^n.) Hence, it suffices to show that for q sufficiently large, the diameter of the support of each affine simplex in $Sd^q(\Delta^n)$ is small enough for the conclusions to be valid by the Lebesgue Covering Lemma. For an affine chain a, this maximum diameter is called the *mesh* of the chain and we shall denote it $m(a)$.

LEMMA 6.29. *If α is an affine n-simplex, then* $m(Sd(\alpha)) \leq \dfrac{n}{n+1} m(\alpha)$.

PROOF. Let $\mathbf{b} = \mathbf{b}(\alpha)$ be the barycenter of $\alpha = [\mathbf{x}_0, \ldots, \mathbf{x}_n]$. Then

$$m(Sd(\alpha)) \leq \max_{i=1,\ldots,n} |\mathbf{b} - \mathbf{x}_i|.$$

(See the above diagram for an indication of the argument for this, in particular why only the lengths of the edges from $\mathbf{b}$ to the vertices of α need to be considered.) Hence,

$$\begin{aligned} |\mathbf{b} - \mathbf{x}_i| &= |\frac{1}{n+1} \sum_j \mathbf{x}_j - \mathbf{x}_i| = \frac{1}{n+1} |\sum_{j \neq i} (\mathbf{x}_j - x_i)| \\ &\leq \frac{1}{n+1} \sum_{j \neq i} |\mathbf{x}_j - \mathbf{x}_i| \leq \frac{n}{n+1} m(\alpha). \end{aligned}$$

□

It follows that $m(Sd^q(\Delta^n)) \leq (n/n+1)^q m(\Delta^n) \to 0$ as $q \to \infty$. Thus as mentioned above, and immediate consequence of this is that for any singular n-cycle c in X, there is a q such that $(Sd^X)^q c$ is $\mathcal{U}$-small. Since $(Sd^X)^q c$ is homologous to c (using the fact that $(Sd^X)^q$ is chain

homotopic to Id), it follows that any singular n-cycle is homologous to one in $S_*^{\mathcal{U}}(X)$.

Note also that by reasoning similar to that above, it is easy to see that if c is a $\mathcal{U}$-small singular chain, then so is $(Sd^X)^q c$ for any q.

Suppose c is $\mathcal{U}$-small, and $c = \partial y$ where $y \in S_*(X)$. (y need note be $\mathcal{U}$-small. Then $(Sd^X)^q c = \partial((Sd^X)^q y)$, and $(Sd^X)^q y$ will be $\mathcal{U}$-small if q is large enough by the above Lemma. Let T be the natural chain homotopy introduced above, i.e., $T_n(\sigma) = \sigma_\sharp(t_n)$ where $t_n \in S_{n+1}(\Delta^n)$. We shall show that $(Sd^X)^q c$ is homologous to c by means of the boundary of a $\mathcal{U}$-small chain. To this end, we write

$$Sd^q - \mathrm{Id} = Sd^{q-1}(Sd - \mathrm{Id}) + Sd^{q-2}(Sd - \mathrm{Id}) + \cdots + Sd - \mathrm{Id},$$

and we see that it suffices to show that $Sd^X c - c$ is the boundary of a $\mathcal{U}$-small chain. Let σ be a $\mathcal{U}$-small simplex in c. We have

$$Sd^X\sigma - \sigma = \partial T^X\sigma + T^X\partial\sigma.$$

When we add up the singular simplices in c, the second terms on the right will add up to zero. Hence, it suffices to show that $T^X\sigma$ is $\mathcal{U}$-small if σ is. However, as above, $T^X(\sigma) = \sigma_\sharp(t_n)$ so the support of every singular simplex appearing in this chain is contained in the support of σ, and we are done. □

7. Relation between π_1 and H_1

THEOREM 6.30. *Let X be a path connected space and x_0 a base point. Then there is a natural isomorphism*

$$h : \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \to H_1(X_0).$$

Here 'naturality' means that either for change of basepoint or for maps of spaces with basepoint, the appropriate diagrams are commutative. Think about what that should mean in each case.

The rest of this section is devoted to the proof.

Let σ be any path in X. σ is also a singular 1-simplex. If σ is a loop, it is clear that the corresponding 1-simplex is a cycle. This provides us a map from the loops at x_0 to $Z_1(X)$. Suppose $\sigma \sim_{\dot{I}} \tau$. Then σ is homologous in $S_1(X)$ to τ. For, let $F : I^2 \to X$ be a homotopy from σ to τ which is constant on both vertical edges. The diagram below shows an affine 2-chain a in I^2 with $\partial F_\sharp(a) = F_\sharp(\partial a) = \sigma + \epsilon_{x_2} - \tau - \epsilon_{x_1}$, where x_1, x_2 are the common endpoints of the paths. If σ, τ are loops at x_0, then $x_1 = x_2 = x_0$ and it follows that σ is homologous to τ. It follows that we get a map $\pi_1(X, x_0) \to H_1(X)$. We claim next that this map is a homomorphism. Indeed, the diagram below shows that $\sigma * \tau$ is homologous to $\sigma + \tau$ for any two paths σ, τ for which the

$*$ composition is defined. Since $H_1(X)$ is abelian, we get finally an induced homomorphism $\pi_1/[\pi_1,\pi_1] \to H_1$. Call this homomorphism h.

To show h is an isomorphism, we shall define an inverse homomorphism j. For each point $x \in X$, let ϕ_x be a fixed path from x_0 to x, and assume $\phi_{x_0} = \epsilon_{x_0}$. For σ an singular 1-simplex in X, let

$$\hat{\sigma} = \phi_{x_1} * \sigma * \overline{\phi}_{x_2}$$

where x_1, x_2 are the endpoints of σ. The map $\sigma \mapsto \hat{\sigma}$ defines a homomorphism $S_1(X) \to \pi_1/[\pi_1,\pi_1]$ since the latter group is abelian and the singular 1-simplices form a basis of the former group. Under this homomorphism, boundaries map to the identity. For, let ρ be a singular 2-simplex. Then if $\partial\rho = \sigma_0 - \sigma_1 + \sigma_2$ as indicated below, where σ_0 has vertices x_1, x_2, σ_1 has vertices x_3, x_2, and σ_2 has vertices x_3, x_1.

It follows that $\partial\rho$ maps to the homotopy class of the path

$$\phi_{x_1} * \sigma_0 * \overline{\phi}_{x_2} * \phi_{x_2} * \overline{\sigma}_1 * \overline{\phi}_{x_3} * \phi_{x_3} * \sigma_2 * \overline{\phi}_{x_1} \sim_{\dot{I}} \phi_{x_1} * (\sigma_0 * \overline{\sigma}_1 * \sigma_2) * \overline{\phi}_{x_1}.$$

However, the expression in parentheses is clearly ρ_* of a loop γ in Δ^2 based at e_1. Since Δ^1 is simply connected, it follows that

$$\phi_{x_1} * (\sigma_0 * \overline{\sigma}_1 * \sigma_2) * \overline{\phi}_{x_1} \sim_{\dot{I}} \phi_{x_1} * (\epsilon_{x_1}) * \overline{\phi}_{x_1} \sim_{\dot{I}} \epsilon_{x_0}.$$

If we restrict the homomorphism defined by $\sigma \mapsto \hat{\sigma}$ to the subgroup $Z_1(X)$, since $B_1(X)$ maps to the identity, we get a homomorphism $j : H_1(X) \to \pi_1/[\pi_1,\pi_1]$. We shall show that h and j are inverse homomorphisms.

If σ is a loop at x_0 in X, then $\hat{\sigma} = \sigma$, so it follows that $\sigma \mapsto \sigma \mapsto \hat{\sigma}$ is the identity, i.e., $j \circ h = \mathrm{Id}$.

Suppose $c = \sum_i n_i\sigma_i$ is a singular 1-cycle. Let σ_i have vertices p_i and q_i. Then under $h \circ j$, σ_i maps to $\hat{\sigma}_i$ which by the above is homologous to $\phi_{p_i} + \sigma_i + \overline{\phi}_{q_i}$. (In general, $\alpha * \beta$ is homologous to $\alpha + \beta$.) By the same reasoning, $\overline{\phi}_{q_i}$ is homologous to $-\phi_{q_i}$, so σ_i maps to $\phi_{p_i} + \sigma_i - \phi_{q_i}$. Hence, $c = \sum_i n_i\sigma_i$ maps to

$$\sum_i n_i\sigma_i + \sum_i n_i(\phi_{p_i} - \phi_{q_i}) = c + \sum_i n_i(\phi_{p_i} - \phi_{q_i}).$$

However, $\sum_i n_i(p_i - q_i) = \partial c = 0$, and this implies that $\sum_i n_i(\phi_{p_i} - \phi_{q_i}) = 0$. For, the first sum may be reduced to a linear combination of a set of distinct singular 0-simplices in which the coefficients must

necessarily be zero, and the second sum is the same linear combination of a set of distinct singular 1-simplices in one to one correlation with the former set, so it also must be zero.

8. The Mayer-Vietoris Sequence

The Mayer-Vietoris sequence is the analogue for homology of the Seifert–VanKampen Theorem for the fundamental group.

Let $X = X_1 \cup X_2$ where the interiors of X_1 and X_2 also cover X. Let $i_1 : X_1 \cap X_2 \to X_1, i_2 : X_1 \cap X_2 \to X_2$ and $j_1 : X_1 \to X, j_2 : X_2 \to X$ be the inclusion maps. The first pair induces the homomorphism

$$i_* : H_*(X_1 \cap X_2) \to H_*(X_1) \oplus H_*(X_2)$$

defined by $i_*(\gamma) = (i_{1*}(\gamma), -i_{2*}(\gamma))$. The second pair induce the homomorphism

$$j_* : H_*(X_1) \oplus H_*(X_2) \to H_*(X)$$

defined by $j_*(\gamma_1, \gamma_2) = j_{1*}(\gamma_1) + j_{2*}(\gamma_2)$. It is not hard to check that $j_* i_* = 0$. In fact, we shall see that $\operatorname{Ker} j_* = \operatorname{Im} i_*$ and that this is part of a long exact sequence for homology. To this end, we need to invent a connecting homomorphism

$$\partial_* : H_*(X) \to H_*(X_1 \cap X_2)$$

which reduces degree by 1.

There are two ways to do this. First, the covering $X = X_1 \cup X_2$ satisfies the conditions necessary for $H_*^{\mathcal{U}}(X) \to H_*(X)$ to be an isomorphism. In this case, it is not hard to check that

$$S^{\mathcal{U}}(X) = S_*(X_1) + S_*(X_2).$$

Also, there is a homomorphism

$$S_*(X_1) \oplus S_*(X_2) \to S_*(X_1) + S_*(X_2).$$

defined by $(c_1, c_2) \mapsto c_1 + c_2$. The kernel of this homomorphism is easily seen to be the image of

$$S_*(X_1 \cap X_2) \to S_*(X_1) \oplus S_*(X_2)$$

defined by $c \mapsto (i_{1\sharp}(c), -i_{2\sharp}(c))$ so it follows that we have a short exact sequence of chain complexes

$$0 \to S_*(X_1 \cap X_2) \to S_*(X_1) \oplus S_*(X_2) \to S^{\mathcal{U}}(X) \to 0.$$

The long exact homology seqeunce of this sequence is the Mayer–Vietoris sequence.

Another approach is to derive the Mayer-Vietoris sequence from the excision axiom. This has the advantage that we don't need to add

Mayer-Vietoris as an additional 'axiom' in any more abstract approach to homology theories.

Let $U = X_1 - X_1 \cap X_2$ and let $A = X_1$. Then, $\overline{U} \subseteq X_1{}^\circ$. For, let $x \in \overline{U}$. If $x \in X_2{}^\circ$, then there is an open neighborhood W of x contained in X_2, so W cannot intersect $U = X_1 - X_1 \cap X_2$. This contradicts the assertion that $x \in \overline{U}$, so it must be true that $x \notin X_2{}^\circ$. Hence, $x \in X_1{}^\circ \subseteq X_1$ as required.

We have $X - U = X - (X_1 - X_1 \cap X_2) = X_2$ and $X_1 - U = X_1 \cap X_2$. Hence, by excision, the inclusion $h : (X_2, X_1 \cap X_2) \to (X, X_1)$ induces an isomorphsim

$$h_* : H_*(X_2, X_1 \cap X_2) \to H_*(X, X_1).$$

Consider the commutative diagram of long exact sequences

$$\begin{tikzcd}
H_n(X_1 \cap X_2) \arrow[r, "i_{2*}"] \arrow[d, "i_{1*}"'] & H_n(X_2) \arrow[r, "k_*"] \arrow[d, "j_{2*}"'] & H_n(X_2, X_1 \cap X_2) \arrow[r, "\partial"] \arrow[d, "h_*"'] & H_{n-1}(X_1 \cap X_2) \arrow[d, "i_{1*}"'] \\
H_n(X_1) \arrow[r, "j_{1*}"] & H_n(X) \arrow[r, "l_*"] & H_n(X, X_1) \arrow[r] & H_{n-1}(X_1)
\end{tikzcd}$$

Define the homomorphisms i_* and j_* as above and define the desired connecting homomorphism $H_n(X) \to H_{n-1}(X_1 \cap X_2)$ as $\partial_* \circ (h_*)^{-1} \circ l_*$. It is not hard to check that this homomorphism is natural in the obvious sense. We leave it to the student to check this. That the Mayer–Vietoris sequence is exact follows from the following Lemma

Lemma 6.31 (Barratt-Whitehead). *Suppose we have a commutative diagram of abelian groups*

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"'] & B_n \arrow[r, "j_n"] \arrow[d, "g_n"'] & C_n \arrow[r, "k_n"] \arrow[d, "h_n"'] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & {} \\
\cdots \arrow[r] & A'_n \arrow[r, "i'_n"] & B'_n \arrow[r, "j'_n"] & C'_n \arrow[r, "k'_n"] & A'_{n-1} \arrow[r, "i'_{n-1}"] & \cdots \arrow[r] & {}
\end{tikzcd}$$

with exact rows. Suppose h_n is an isomorphism for every n. Then

$$\cdots \longrightarrow A_n \xrightarrow{(f_n, -i_n)} A'_n \oplus B_n \xrightarrow{i'_n + g_n} B'_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \cdots$$

is exact where $\partial = k_n \circ (h_n)^{-1} \circ j'_n$.

Proof. This is just a diagram chase, which we leave to the student. $\square$

Note that the Mayer–Vietoris sequence works also for reduced homology provided $X_1 \cap X_2 \neq \emptyset$.

8.1. Applications of the Mayer–Vietoris Sequence.

PROPOSITION 6.32. *For the 2-torus T^2, we have*

$$\begin{aligned} H_i(T^2) &= \mathbf{Z} \qquad i = 0 \\ &= \mathbf{Z} \oplus \mathbf{Z} \qquad i = 1 \\ &= \mathbf{Z} \qquad i = 2 \\ &= 0 \qquad \textit{otherwise.} \end{aligned}$$

PROOF. We give two arguments. (i) Identify the torus as the unit square with opposite edges identified as below. Define open sets U and V as indicated in the diagram. Then $T = U \cup V$, U and V each have a circle as a deformation retract, and $U \cap V$ has two connected components, each of which has a circle as a deformation retract. Hence, the Mayer–Vietoris sequence gives

$$H_2(U) \oplus H_2(V) = 0 \to H_2(T) \to H_1(U \cap V) = \mathbf{Z} \oplus \mathbf{Z} \to H_1(U) \oplus H_1(V) = \mathbf{Z} \oplus \mathbf{Z} \to$$
$$H_1(T) \to \tilde{H}_0(U \cap V) = \mathbf{Z} \to \tilde{H}_0(U) \oplus \tilde{H}_0(V) = 0.$$

Hence, we need to identify the middle homomorphism $\mathbf{Z}^2 \to \mathbf{Z}^2$. In the map $H_1(U \cap V) \to H_1(U)$, each basis element b_i of $H_1(U \cap V)$ maps to a generator b' of $H_1(U)$. Similarly, under $H_1(U \cap V) \to H_1(V)$, each b_i maps to a generator b'' of $H_1(V)$. Since $i_* = i_{1*} \oplus -i_{2*}$, the middle homomorphism may be described by the two by two integral matrix

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Writing elements of $\mathbf{Z}^2$ as column vectors, we see that a basis for the kernel of this homomorphism is

$$b_1 - b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and it is free of rank one. It follows from this that $H_2(T) = \mathbf{Z}$. Similarly, since the the image of the homomorphism is spanned by the columns of the matrix, $b' - b''$ is also a basis for the image. Since $\mathbf{Z}^2 = \mathbf{Z}(b_1 - b_2) \oplus \mathbf{Z}b_2$, it follows that the cokernel of the homomorphism is also free of rank one. Hence, we have a short exact sequence

$$0 \to \mathbf{Z} \to H_1(T) \to \mathbf{Z} \to 0$$

which necessarily splits. Hence, $H_1(T) \cong \mathbf{Z} \oplus \mathbf{Z}$ as claimed.

(ii) Argue instead as we did when applying the Seifert-VanKampen Theorem to calculate the fundamental group of T. Choose U to be the square with its boundary eliminated, and V to be the punctured torus as indicated below. Then U is contractible, V has a wedge of two circles as a deformation retract, and $U \cap V$ has a circle as a deformation retract. By a simple argument using the Mayer–Vietoris sequence, it is possible to see that the wedge of two circles has trivial reduced homology in all dimensions except one, and $H_1 = \mathbf{Z} \oplus \mathbf{Z}$. Each loop of the wedge provides a basis element. (H_1 may also be calculated by using the isomorphism with $\pi_1/[\pi_1, \pi_1]$. (See the Exercises.) Now apply the Mayer-Vietoris sequence to get

$$0 \to H_2(T) \to H_1(U \cap V) = \mathbf{Z} \to H_1(U) \oplus H_1(V) = 0 \oplus (\mathbf{Z} \oplus \mathbf{Z}) \to H_1(T) \to 0.$$

However, the diagram below shows that a generator of $H_1(U \cap V)$ maps to zero in $H_1(V)$, so we obtain the desired results.

□

Note that in the homomorphism $j_* : H_1(U) \oplus H_1(V) \to H_1(T)$ a generator of either factor goes onto the summand $\mathbf{Z}$ on the left. In either case, it is not hard to see that this can be identified with a singular 1-simplex consisting of a loop going around the torus in one direction. Clearly, the other loop should provide the other generator, but this is not clear from the above argument. However, if we use the isomorphism

$$\pi_1(T)/[\pi_1(T), \pi_1(T)] \cong H_1(T)$$

discussed previously, it is clear that we can identify two loops for the torus as a basis for $H_1(T)$ since we already know they form a basis for the fundamental group, which in this case is free abelian of rank two. This is clearly a quicker and more effective way to compute H_1, but we need the Mayer–Vietoris sequence to compute H_2.

9. Some Important Applications

In this section, we shall prove certain important classical theorems. The Jordan Curve Theorem asserts that any simple closed curve C in $\mathbf{R}^2$ divides the plane into two regions each of which has C as its boundary. We shall prove a generalization of this to higher dimensions.

First we need some preliminaries.

A closed r-cell in S^n is any subspace e^r which is homemorphic to the standard r-cell $I^r = I \times I \times \cdots \times I$. Note that since S^n is not contractible, a closed r-cell cannot be all of S^n.

Theorem 6.33. *Let e^r be a closed r-cell in S^n, $n, r \geq 0$. Then $\tilde{H}_q(S^n - e^r) = 0$ for all q.*

Proof. The result is clear for $n = 0$. Suppose $n > 0$. We shall proceed by induction on r. The result is true for $r = 0$, since a 0-cell is a point and in that case $S^n - e^r$ is contractible. Suppose it has been proven for $0, 1, \dots, r-1$. We may decompose

$$I^r = (I^{r-1} \times [0, 1/2]) \cup (I^{r-1} \times [1/2, 1])$$

and corresponding to this we get a decomposition of $e^r = e' \cup e''$. compact sets, they are closed in S^n, so $S^n - e'$ and $S^n - e''$ are open sets in S^n. Their intersection is $S^n - e^r$. Their union on the other hand is $S^n - e' \cap e''$, but $e' \cap e''$ is homemorphic to

$$I^{r-1} \times [0, 1/2] \cap I^{r-1} \times [1/2, 1] = I^{r-1}$$

so it is an $(r-1)$-cell. Hence, by induction $\tilde{H}_q(S^n - e' \cap e'') = 0$ for all q. From the Mayer–Vietoris sequence this implies that

$$\tilde{H}_q(S^n - e^r) \cong \tilde{H}_q(S^n - e') \oplus \tilde{H}_1(S^n - e'')$$

for all q. Suppose for some q that the left hand side is non-trivial. Then one of the groups on the right is non-trivial; say $H_q(S^n - e') \neq 0$. However, e' is homemomorphic to $I^{r-1} \times [0, 1/2] \simeq I^r$, so it is also a closed r-cell. Clearly, we may iterate this argument to obtain a sequence of closed r-cells $E_1 = e^r \supseteq E_2 \supseteq E_3 \supseteq \dots$ with $e^{r-1} = \cap_i E_i$ a closed $(r-1)$-cell. Also, for each i, $\tilde{H}_q(S^n - E_i) \to \tilde{H}_1(S^n - E_{i+1})$ is a non-trivial monomorphism onto a direct summand.

Lemma 6.34. *Let $X_1 \subseteq X_2 \subseteq X_3 \dots$ be an ascending chain of spaces each of which is open in the union $X = \cup_i X_i$.*

(a) Suppose $\gamma \in \tilde{H}_q(X_i)$ maps to zero in $\tilde{H}_q(X)$. Then there is a $j > i$ such that γ maps to zero in $\tilde{H}_q(X_j)$.

(b) For each $\rho \in \tilde{H}_q(X)$, there exists an i such that ρ is the image of some $\gamma \in \tilde{H}_q(X_i)$.

Note that part (a) of Lemma 6.34 proves the theorem. For, we may take $X_i = S^n - E_i$. Then, if $\gamma \in H_q(X_1)$ is non-trival, it must map to zero in $H_q(X) = H_q(S^n - e^{r-1}) = 0$. Hence, it must map to zero in some $\tilde{H}_q(X_j)$ with $j > 1$, but that can't happen by the construction. □

Proof of Lemma 6.34. We shall prove the Lemma for ordinary homology. The case of reduced homology for $q = 0$ is similar, and we leave it for the student.

(a) Let c be a cycle representing $\gamma \in H_q(X_i)$. The support c is the union of a finite number of supports of singular simplices, so it is compact. Let $c = \partial x$ for some singular chain x in X. The support of x is also compact. Let A be the union of the supports of c and x. Since A is compact, and since the X_j form an open covering, it follows that A

is covered by finitely many X_j. Since the subspaces form an ascending chain, it follows that there is a j such that $A \subseteq X_j$. It is clear that the homology class of c is zero in $H_q(X_j)$, and we may certainly take $j > i$ if we choose.

(b) Let c be a cycle representing $\rho \in H_q(X)$. As above, the support of c is compact and contained in some X_i. □

THEOREM 6.35. *Let S be a proper subspace of S^n which is homeomorphic to S^r with $n > 0$, $r \geq 0$. Then*

$$\tilde{H}_q(S^n - S) = \begin{cases} \mathbf{Z} & \text{if } q = n - r - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This Theorem has several interesting consequences.

COROLLARY 6.36. *Suppose S is a proper subspace of S^n which is homeomorphic to S^r with $n > 0$, $r \geq 0$. Then we must have $r < n$. If $r \neq n - 1$, then $S^n - S$ is path connected. In particular if K is a knot in S^3, then $S^3 - K$ is path connected. If $r = n - 1$, then $S^n - S$ has two components.*

PROOF OF COROLLARY 6.36. If $r \geq n$, then we would have a nonzero homology group in a negative dimension, which is impossible. Since S is compact, it is closed in S^n, so its complement is open and is necessarily locally path connected. Hence, the path components are the same as the components. Moreover, the rank of $\tilde{H}_0$ is one less than the number of path components, and $\tilde{H}_0(S^n - S) = 0$ unless $r = n - 1$ in which case it is $\mathbf{Z}$. □

PROOF OF THEOREM 6.35. We proceed by induction on r. For $r = 0$, S consists of two points, and $S^n - S$ is homeomorphic to $\mathbf{R}^n$ less a point, so it has S^{n-1} as a deformation retract. Since

$$\begin{aligned} \tilde{H}_q(S^{n-1}) &= 0 \qquad q \neq n - 1 = n - r - 1 \\ \tilde{H}_{n-1}(S^{n-1}) &= \mathbf{Z}, \end{aligned}$$

the result follows for $r = 0$.

Suppose the theorem has been proved for $0, 1, \ldots, r-1$. Decompose $S^r = D^r_+ \cup D^r_-$ where $D^r_+ = \{x \in S^r \mid x_r \geq 0\}$ and $D^r_- = \{x \in S^r \mid x_r \leq 0\}$. Let $S = e' \cup e''$ be the corresponding decomposition of S. Note that e' and e'' are closed r-cells and $S' = e' \cap e''$ is homeomorphic to S^{r-1}. Now apply the reduced Mayer–Vietoris sequence to the open covering

$$S^n - S' = S^n - e' \cap e'' = (S^n - e') \cup (S^n - e''),$$

where $(S^n - e') \cap (S^n - e'') = S^n - e' \cup e'' = S^n - S$. (Note that this requires $S^n - S \neq \emptyset$.) Since $S^n - e'$ and $S^n - e''$ are acyclic, i.e., they

have trivial reduced homology in every dimension, it follows that

$$\tilde{H}_{q+1}(S^n - S') \cong \tilde{H}_q(S^n - S)$$

for every q. However, by induction, the left hand side is zero unless $q+1 = n-(r-1)-1 = n-r$ in which case it is $\mathbf{Z}$. But $q+1 = n-r$ if and only if $q = n-r-1$, so we get what we need for the right hand side. □

We have now done most of the work for proving the following theorem.

THEOREM 6.37 (Jordan-Brouwer Separation Theorem). *Let S be a subspace of S^n homemorphic to S^{n-1} for $n > 0$. Then $S^n - S$ has two connected components each of which has S as its boundary.*

PROOF. We already know that $S^n - S$ has two components U and V. All that remains is to show that both have S as boundary. Consider $\overline{U} - U$. Since U and V are disjoint open sets, no point of V is in $\overline{U} - U$. Hence, $\overline{U} - U \subseteq S$.

Conversely, let $x \in S$. Let W be an open neighborhood of x. We may choose a decomposition of $S \simeq S^{n-1}$ into two closed $(n-1)$-cells e', e'' with a common boundary and we may assume one of these $e' \subset W$. Since $S^n - e''$ is path connected ($\tilde{H}_0(S^n - e'') = 0$), we may choose a path α in S^n connecting some point of U to some point in V which does not pass thru e''. Let s be the least upper bound of all t such that $\alpha([0,t]) \subseteq U$. Certainly, $\alpha(s) \in \overline{U}$, so it is not in V, but since U is open $\alpha(s)$ is also not in U. Hence, $\alpha(s) \in S$. Since it is not in e'', $\alpha(s) \in e' \subseteq W$. Thus, it follows that every open neighborhood W of x contains a point of $\overline{U}$. Hence, x is in the closure of $\overline{U}$, i.e., it is in $\overline{U}$. This shows that $S \subseteq \overline{U}$, but since $S \cap U = \emptyset$, we conclude finally that $S \subseteq \overline{U} - U$. □

COROLLARY 6.38 (Jordan-Brouwer). *Let $n \geq 2$ and let S be a subspace of $\mathbf{R}^n$ which is homeomorphic to S^{n-1}. Then S separates $\mathbf{R}^n$ into two components, one of which is bounded and one of which is unbounded.*

PROOF. Exercise. □

You should think about what happens for $n = 1$.

In $\mathbf{R}^2$, by a theorem of Schoenfles, the bounded component is homemorphic to an open disk. However, this result does not extend to higher dimensions unless one makes extra assumptions about the imbedding of S^{n-1} in $\mathbf{R}^n$. There is an example, called the *Alexander horned sphere*,

of a homemomorph of S^2 in $\mathbf{R}^3$ where the components of the complement are not even simply connected. It is constructed as follows.

Start with a two torus, and cut out a cylindrical section as indicated in the above diagram to the left. Put a cap on each end, so the result is homeomorphic to S^2. On each cap, erect a handle, cut out a cylindrical section, cap it, etc. Continue this process indefinitely. At any finite stage, you still have a space homeomorphic to S^2. Now take the union of all these surfaces and all limit points of that set. (In fact, the set of limit points forms a Cantor set.) It is possible to see that the resulting subpace of $\mathbf{R}^3$ is also homemorphic to S^2. Next modify the process as follows. When adding the pair of handles to each opposing pair of caps, arrange for them to link as indicated in the above diagram to the right. Otherwise, do the same as before. It is intuitively clear that the outer complement of the resulting subspace is not simply connected, although it might not be so easy to prove. Also, the space is still homeomorphic to S^2, since the linking should not affect that. Now modify this process at any stage (more than once if desired), as on the right of the above diagram, by deforming two of these cylinders and linking them as indicated. This deformation does not change the fact that the space is homeomorphic to a sphere, but it is clear that the complement is not simply connected.

Theorem 6.39 (Invariance of Domain). *Let U be an open set in $\mathbf{R}^n$ (alternately S^n) and let $h : U \to \mathbf{R}^n\,(S^n)$ be a one-to-one continuous map. Then $h(U)$ is open, and h provides a homeomorphism from U to $h(U)$.*

This theorem is important because it insures that various pathological situations cannot arise. For example, in the definition of a manifold, we required that every point have a neighborhood homeomophic to an open ball in $\mathbf{R}^n$ for some fixed n. Suppose we allowed different n for different points. If two such neighborhoods intersect, the theorem on invariance of domain may be used to show that the 'n' for those neighborhoods must be the same. Hence, the worst that could happen would be that the dimension n would be different for different components of the space.

Note that 'invariance of domain' is not necessarily true for spaces other than $\mathbf{R}^n$ or S^n. For example, take $U = [0, 1/2)$ as an open subset of $[0, 1]$ and map it into $[0, 1]$ by $t \mapsto t + 1/2$. The image $[1/2, 1)$ is not open in $[0, 1]$.

PROOF. It suffices to consider the alternate case for S^n. (Why?) Let $x \in U$. Choose a closed ball D centered at x and contained in U. Since D is compact, $h(D)$ is in fact homemomorphic to D, so it is a closed n-cell. It follows that $S^n - h(D)$ is connected. Hence, $S^n - h(\partial D) = (S^n - h(D)) \cup h(D^\circ)$ is a decomposition of $S^n - h(\partial D)$ into two disjoint connected sets. Since the Jordan-Brouwer Theorem tells us that $S^n - h(\partial D)$ has precisely two components, both or which are open, it follows that one of these sets is $h(D^\circ)$, so that set is an open neighborhood of $h(x)$. It follows from this that $h(U)$ is open. $\square$